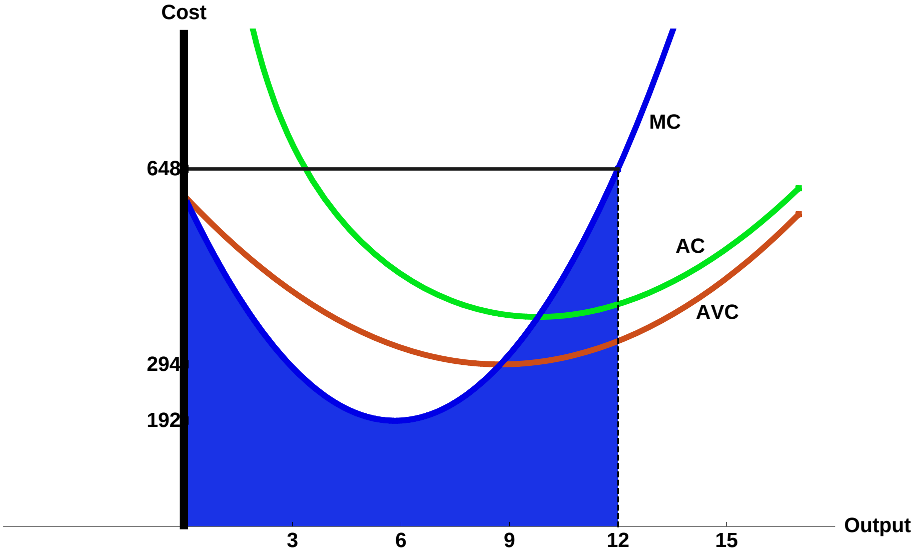
ECONOMICS 207
SPRING 2008
PROBLEM SET 13
KEY

Problem 1. The cost function for a firm is a rule or mapping that tells the minimum total cost of production of any output level produced by the firm for a fixed level of input prices. If the variable y represents the output of the firm, then the cost function is given by c(y,w) or c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output, i.e., Marginal Cost(MC) $= \frac{dc(y)}{dy}$. A graph of price and marginal cost is given in figure 1.

FIGURE 1. Profit Maximization Given Marginal Cost



Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$\begin{aligned} price \; &= \; p \; = \$648 \\ cost \; &= \; c(y) \; = 800 \; + \; 600y \; - \; 70y^2 \; + \; 4y^3 \end{aligned}$$

Date: August 25, 2008.

a. Write down an equation for profit for this firm and find the potential levels of output that maximize profit.

The profit is given by

$$\begin{aligned} Profit &= Revenue - Cost = py - c(y) \\ &= 648y - \left(800 + 600y - 70y^2 + 4y^3\right) \\ &= -800 + 48y + 70y^2 - 4y^3 \end{aligned}$$

Then the derivatives of profit are given by

$$\frac{d\,Profit}{d\,y} = 48 + 140y - 12y^2 \tag{1}$$

$$\frac{d^2\,Profit}{d\,y^2} = 140 - 24y \tag{2}$$

Set the first derivative, equation (1), to be zero.

$$\begin{aligned} & \frac{d\,Profit}{d\,y} = 48 + 140y - 12y^2 &= 0 \\ \Rightarrow \quad & 12 + 35y - 3y^2 &= 0 \\ \Rightarrow \quad & (12 - y)(1 + 3y) &= 0 \\ \Rightarrow \quad & y = 12 \quad \text{or} \quad y = -1/3 \end{aligned}$$

b. Using the second order conditions from profit maximization, determine which of the levels of output from part a that maximize profit?

Check the second derivative, equation (2), for $y = 12$ and $y = -1/3$ respectively.

$$\left.\frac{d^2\,Profit}{d\,y^2}\right|_{y=12} = 140 - 24 \times 12 = -148 < 0$$

$$\left.\frac{d^2\,Profit}{d\,y^2}\right|_{y=-1/3} = 140 - 24 \times (-1/3) = 148 > 0$$

So the optimal level is that $y = 12$.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$\begin{aligned} Profit =& py - c(y) \\ Profit =& 648y - \left[800 \ + \ 600y \ - \ 70 \ y^2 \ + \ 4y^3\right] \end{aligned}$$

Finding the first derivative of profit with respect to y and setting it to zero are given by

$$\frac{d\,Profit}{d\,y} = p - c'(y) = 0,$$

which is equivalent to

$$p = c'(y) \tag{3}$$

The left hand side of equation (3) is the price while the right hand side of equation (3) is the marginal cost. Therefore, setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y.

d. Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$\begin{aligned} price \ = \ p \ &= \$648 \\ cost \ = \ c(y) \ &= 800 \ + \ 600y \ - \ 70y^2 \ + \ 4y^3 \end{aligned}$$

Without writing down an equation for profit, find the levels of output which potentially maximize profit using what you have learned in general about profit maximization for a competitive firm.

Set the price equal to the marginal cost—the first derivative of cost—and solve for the optimal level.

$$\begin{aligned} p \ &= \ c'(y) \\ \Rightarrow \qquad 648 \ &= \ 600 - 140y + 12y^2 \\ \Rightarrow \quad 48 + 140y - 12y^2 \ &= \ 0 \\ \Rightarrow \quad 4(12 - y)(1 + 3y) \ &= \ 0 \\ \Rightarrow \qquad y = 12 \qquad &\text{or} \qquad y = -1/3 \end{aligned}$$

e. Given that you have no second order conditions from profit maximization per se, make a coherent argument about which of the two potential output levels you found in part d maximizes profit.

When the price is higher than the marginal cost, the firm can profit more from increasing the output. And when the price is less than the marginal cost, the firm can profit more from decreasing the output. As a result, the firm attains its profit maximization when the price is equal to the marginal cost. Also, at the optimal level, the marginal cost should be upward. That is, the second derivative of cost should be bigger then zero at the optimal level.

And the scold derivative of cost is given by

$$c''(y) = -140 + 24y$$

When $y = -1/3$, $c''(-1/3) = -140 + 24 \times (-1/3) = -148 < 0$.
When $y = 12$, $c''(12) = -140 + 24 \times 12 = 148 > 0$.

So it is level $y = 12$ that maximizes the profit.

f. Integrate the marginal cost function you found in part d to obtain the variable cost function and then use it to show that the level of variable cost for this firm when it maximizes profit with a price of \$648 is \$4032.

The variable cost is given by

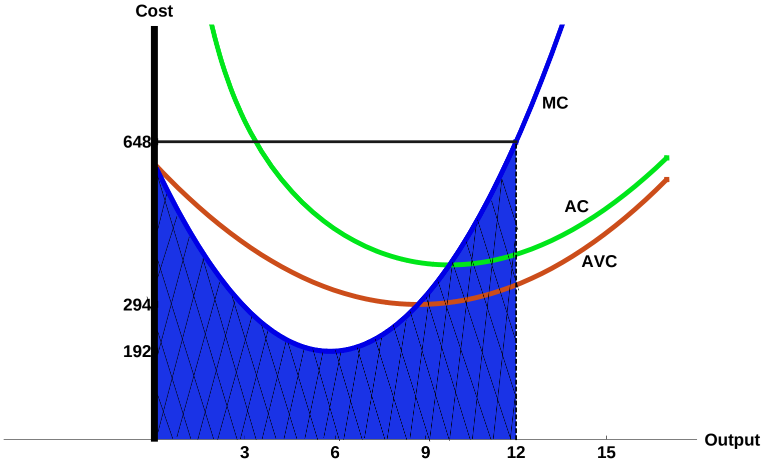
$$\begin{aligned} VC(y) &= \int_0^y c'(y)dy = \int_0^y \left(600 - 140y + 12y^2\right) dy \\ &= 600y - 70y^2 + 4y^3 \end{aligned}$$

When the price is \$648, the optimal level for y is 12 and thus the variable cost is given by

$$VC(12) = 600 \times 12 - 70 \times 12^2 + 4 \times 12^3 = 4032$$

g. Cross-hatch the area represented by variable cost in part f and explain why the integral of the marginal cost function or the area under the marginal cost curve represents variable cost.

Figure 2. Variable Cost for Profit Maximization



h. What is revenue for this profit maximizing firm?

When $y = 12$, the profit is given by

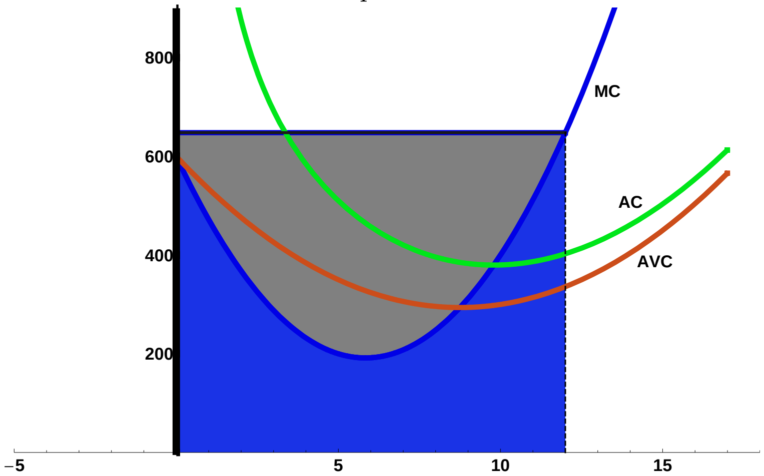
$$Revenue = py = 648 \times 12 = 7776$$

i. What is producer surplus for this profit maximizing firm?
When $y = 12$, the producer surplus is given by

$$\begin{aligned} Producer\ surplus &= Revenue - variable\ cost \\ &= 7776 - 4032 = 3744 \end{aligned}$$

j. Shade this level of producer surplus in Figure 1.

Figure 3. Producer surplus for Profit Maximization



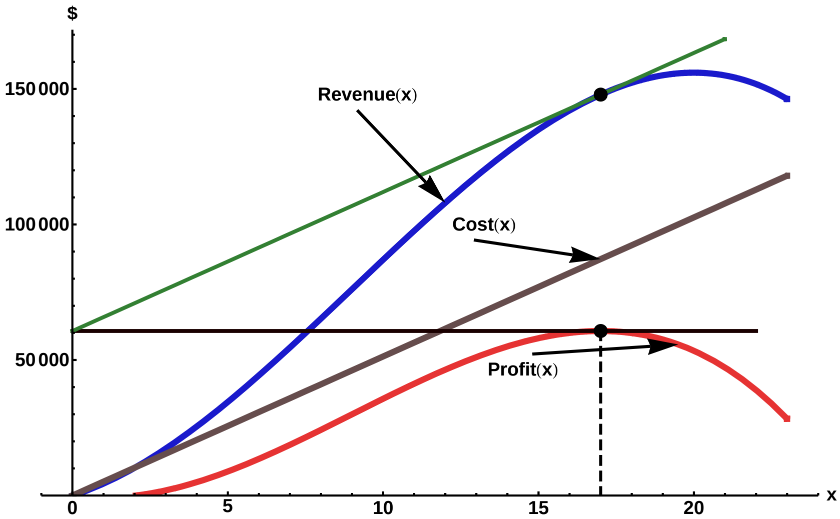
Problem 2. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 10$$

$$input\ price = w = 5130$$

$$y = output = f(x) = 360x + 81x^2 - 3x^3$$

FIGURE 4. Revenue, Cost and Profit



a. Find values of x that potentially maximize **output** for this firm.

Set the first derivative of output with respect to x to zero.

$$\begin{aligned} \frac{dy}{dx} = 360 + 162x - 9x^2 &= 0 \\ \Rightarrow \quad 9(20-x)(2+x) &= 0 \\ \Rightarrow \quad x = 20 \quad \text{or} \quad & x = -2 \end{aligned}$$

b. Show which values of x in part a actually maximize output.

Check the second derivative of output with respect to x for $x = 20$ and $x = -2$ respectively.

$$\left.\frac{d^2y}{dx^2}\right|_{x=20} = 162 - 18x|_{x=20} = 162 - 18 \times 20 = -198 < 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 162 - 18x|_{x=20} = 162 - 18 \times (-2) = 198 > 0$$

So it is $x = 20$ that maximizes output.

c. Write down an equation that represents profit for the firm.

The profit is given by

$$\begin{aligned} Profit &= Revenue - Cost = py - wx \\ &= 10\left(360x + 81x^2 - 3x^3\right) - 5130x = 10\left(-153x + 81x^2 - 3x^3\right) \end{aligned}$$

d. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

Set the first derivative of profit with respect to x and solve for x.

$$\begin{aligned} \frac{d\,Profit}{d\,x} = 10\left(-153 + 162x - 9x^2\right) &= 0 \\ \Rightarrow \qquad (x-17)(x-1) &= 0 \\ \Rightarrow \qquad x = 17 \quad \text{or} \quad x &= 1 \end{aligned}$$

e. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

Check the second derivative of profit with respect to x for $x = 17$ and $x = 1$ respectively.

$$\left.\frac{d^2\,Profit}{d\,x^2}\right|_{x=17} = 10\,(162 - 18x)|_{x=17} = 10 \times (162 - 18 \times 17) = -1440 < 0$$

$$\left.\frac{d^2\,Profit}{d\,x^2}\right|_{x=1} = 10\,(162 - 18x)|_{x=1} = 10 \times (162 - 18 \times 1) = 1440 > 0$$

So it is $x = 17$ that maximizes the profit.

f. Explain in words why the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the following information in explaining this phenomenon. Say something about the benefits of using an input not being less than the cost of the input.

$$\begin{aligned} Output =& y = f(x) \\ MP =& \text{Marginal Product} = \frac{d\,f(x)}{dx} = f'(x) = \frac{\Delta y}{\Delta x} \\ Revenue =& pf(x) \\ Cost =& wx \\ Profit = \pi =& Revenue - Cost = pf(x) - wx \\ \frac{d\pi}{dx} =& \end{aligned}$$

For profit maximization, the first derivative of profit with respect to y is set to zero. That is,

$$\frac{d\,Profit}{d\,x} = pf'(x) - w = 0,$$

which is equivalent to

$$pf'(x) = w \tag{4}$$

The left hand side of equation (4) is the price multiplied by marginal product, i.e., the value of the marginal product while the right hand side of equation (4) is price of the single variable input. As a result, the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use.

In other words, when the value of the marginal product is higher than the price of single variable input, the firm can profit more by increasing the output. However, when the value of the marginal product is less than the price of single variable input, the firm can profit more by decreasing the output. So the firm attains its profit maximization when the value of the marginal product is equal to the price of single variable input.

g. Using the information from part 2f, explain why the slope of total revenue in figure 4 is equal to the slope of total cost at the profit maximizing level of input use.

The left hand side of equation (4) is also the slope of total revenue while the right hand side is the slope of total cost. Therefore, they are the same at the profit maximizing level of input use.

Problem 3. For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2.

Write it in the form π = $pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function. Then set the partial derivatives with respect to x_1 and x_2 equal to zero and solve the equations for the levels of x_1 and x_2 that maximize profit. Then show that the level you found actually maximizes profit.

a.

$$f(x_1, x_2) = 20x_1 + 25x_2 - x_1^2 + x_1x_2 - x_2^2$$

$$p = 4$$

$$w_1 = 40, \quad w_2 = 24$$

$$x_1 = 13, \quad x_2 = 16, \quad y = 443$$

$$\pi = 4\left(20x_1 + 25x_2 - x_1^2 + x_1x_2 - x_2^2\right) - 40x_1 - 24x_2$$
$$= 4\left(10x_1 + 19x_2 - x_1^2 + x_1x_2 - x_2^2\right)$$

$\frac{\partial \pi}{\partial x_1} = 4(10 - 2x_1 + x_2)$	$\frac{\partial \pi}{\partial x_2} = 4(19 + x_1 - 2x_2)$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -8$

Find potential profit maximizing levels of x_1 and x_2.

By setting the first derivative to zero, we obtain

$$\frac{\partial \pi}{\partial x_1} = 4(10 - 2x_1 + x_2) = 0$$

$$\frac{\partial \pi}{\partial x_2} = 4(19 + x_1 - 2x_2) = 0$$

That is,

$$10 - 2x_1 + x_2 = 0 \tag{5}$$

$$19 + x_1 - 2x_2 = 0 \tag{6}$$

Add equation (5) multiplied by 2 to the second equation.

$$\begin{aligned} 19 + x_1 - 2x_2 + 2\,(10 - 2x_1 + x_2) &= 0 \\ \Rightarrow \quad 39 - 3x_1 &= 0 \\ \Rightarrow \quad x_1 &= 13 \end{aligned}$$

Substitute $x_1 = 13$ into $10 - 2x_1 + x_2 = 0$.

$$\begin{aligned} 10 - 2x_1 + x_2 &= 0 \\ \Rightarrow \quad 10 - 26 + x_2 &= 0 \\ \Rightarrow \quad x_2 &= 16 \end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \dfrac{\partial^2 \pi}{\partial x_1^2} & \dfrac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 \pi}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \pi}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} -8 & 4 \\ 4 & -8 \end{vmatrix} = 64 - 16 = 48$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 13$, $x_2 = 16$ represents a point of profit maximization.

b.

$$\begin{aligned} f(x_1, x_2) &= 100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2 \\ p &= 4 \\ w_1 &= 90, \qquad w_2 = 60 \\ x_1 &= 23, \qquad x_2 = \frac{29}{2}, \qquad y = 1880 \end{aligned}$$

$$\begin{aligned} \pi &= 4\left(100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2\right) - 90x_1 - 60x_2 \\ &= 310x_1 + 140x_2 - 8x_1^2 + 4x_1x_2 - 8x_2^2 \end{aligned}$$

$\frac{\partial \pi}{\partial x_1} = 310 - 16x_1 + 4x_2$	$\frac{\partial \pi}{\partial x_2} = 140 + 4x_1 - 16x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -16$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -16$

Find potential profit maximizing levels of x_1 and x_2.

By setting the first derivative of profit to zero, we obtain

$$\frac{\partial \pi}{\partial x_1} = 310 - 16x_1 + 4x_2 = 0 \tag{7}$$

$$\frac{\partial \pi}{\partial x_2} = 140 + 4x_1 - 16x_2 = 0 \tag{8}$$

Add equation (8) multiplied by 4 to equation (7).

$$\begin{aligned} 310 - 16x_1 + 4x_2 + 4\left(140 + 4x_1 - 16x_2\right) &= 0 \\ \Rightarrow \quad 870 - 60x_2 &= 0 \\ \Rightarrow \quad 870 - 60x_2 &= 0 \\ \Rightarrow \quad x_2 &= 29/2 \end{aligned}$$

Substitute $x_2 = 29/2$ into equation (8).

$$\begin{aligned} 140 + 4x_1 - 16x_2 &= 0 \\ \Rightarrow \quad 140 + 4x_1 - 16 \times (29/2) &= 0 \\ \Rightarrow \quad x_1 &= 23 \end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} -16 & 4 \\ 4 & -16 \end{vmatrix} = 256 - 16 = 240$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 23$, $x_2 = 29/2$ represents a point of profit maximization.

c.

$$
\begin{aligned}
f(x_1, x_2) &= x_1^{1/4} x_2^{1/2} \\
p &= 8 \\
w_1 &= 2, \qquad w_2 = 1 \\
x_1 &= 16, \qquad x_2 = 64, \qquad y = 16
\end{aligned}
$$

$$\pi = 8x_1^{1/4}x_2^{1/2} - 2x_1 - x_2$$

$\frac{\partial \pi}{\partial x_1} = 2x_1^{-3/4}x_2^{1/2} - 2$	$\frac{\partial \pi}{\partial x_2} = 4x_1^{1/4}x_2^{-1/2} - 1$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{3}{2}x_1^{-7/4}x_2^{1/2}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = x_1^{-3/4}x_2^{-1/2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = x_1^{-3/4}x_2^{-1/2}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -2x_1^{1/4}x_2^{-3/2}$

Find potential profit maximizing levels of x_1 and x_2.
By setting the first profit with respect to x_1 and x_2, we obtain

$$\frac{\partial \pi}{\partial x_1} = 2x_1^{-3/4}x_2^{1/2} - 2 = 0 \tag{9}$$

$$\frac{\partial \pi}{\partial x_2} = 4x_1^{1/4}x_2^{-1/2} - 1 = 0 \tag{10}$$

Rearrange equation (9) .

$$\begin{aligned} 2x_1^{-3/4}x_2^{1/2} - 2 &= 0 \\ \Rightarrow \quad x_2^{1/2} &= x_1^{3/4} \\ \Rightarrow \quad x_2^{-1/2} &= x_1^{-3/4} \end{aligned}$$

Substitute $x_2^{-1/2} = x_1^{-3/4}$ into equation (10).

$$\begin{aligned} 4x_1^{1/4}x_2^{-1/2} - 1 &= 0 \\ \Rightarrow \quad x_1^{1/4}x_1^{-3/4} &= 1/4 \\ \Rightarrow \quad x_1^{-1/2} &= 1/4 \\ \Rightarrow \quad x_1 &= 16 \end{aligned}$$

Substitute $x_1 = 16$ into $x_2^{1/2} = x_1^{3/4}$.

$$\begin{aligned} x_2^{1/2} &= x_1^{3/4} \\ \Rightarrow \quad x_2^{1/2} &= 16^{3/4} = 8 \\ \Rightarrow \quad x_2 &= 64 \end{aligned}$$

In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} = 0$	$\frac{\partial \pi}{\partial x_2} = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{3}{2} \times 16^{-7/4} \times 64^{1/2} = -\frac{3}{32}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 16^{-3/4} \times 64^{-1/2} = \frac{1}{64}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{64}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -2 \times 16^{1/4} \times 64^{-3/2} = -\frac{1}{128}$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} -\frac{3}{32} & \frac{1}{64} \\ \frac{1}{64} & -\frac{1}{128} \end{vmatrix} = \frac{3}{4096} - \frac{1}{4096} = \frac{1}{2048}$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 16$, $x_2 = 64$ represents a point of profit maximization.

d.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/2} x_2^{1/4} \\ p &= 640 \\ w_1 &= 128, \qquad w_2 = 25 \\ x_1 &= 100, \qquad x_2 = 256, \qquad y = 40 \end{aligned}$$

$$\pi = 640 x_1^{1/2} x_2^{1/4} - 128 x_1 - 25 x_2$$

$\frac{\partial \pi}{\partial x_1} = 320 x_1^{-1/2} x_2^{1/4} - 128$	$\frac{\partial \pi}{\partial x_2} = 160 x_1^{1/2} x_2^{-3/4} - 25$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -160 x_1^{-3/2} x_2^{1/4}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 80 x_1^{-1/2} x_2^{-3/4}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 80 x_1^{-1/2} x_2^{-3/4}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -120 x_1^{1/2} x_2^{-7/4}$

Find potential profit maximizing levels of x_1 and x_2.
By setting the first profit with respect to x_1 and x_2, we obtain

$$\frac{\partial \pi}{\partial x_1} = 320 x_1^{-1/2} x_2^{1/4} - 128 = 0 \qquad (11)$$

$$\frac{\partial \pi}{\partial x_2} = 160 x_1^{1/2} x_2^{-3/4} - 25 = 0 \qquad (12)$$

Rearrange equation (11) .

$$\begin{aligned} 320 x_1^{-1/2} x_2^{1/4} - 128 &= 0 \\ \Rightarrow \quad 320 x_2^{1/4} &= 128 x_1^{1/2} \\ \Rightarrow \quad x_1^{1/2} &= \tfrac{5}{2} x_2^{1/4} \end{aligned}$$

Substitute $x_1^{1/2} = \frac{5}{2} x_1^{1/4}$ into equation (12).

$$\begin{aligned} 160 x_1^{1/2} x_2^{-3/4} - 25 &= 0 \\ \Rightarrow \quad 160 \left(\tfrac{5}{2} x_2^{1/4} \right) x_2^{-3/4} - 25 &= 0 \\ \Rightarrow \quad 400 x_2^{-1/2} &= 25 \\ \Rightarrow \quad x_2^{1/2} &= 400/25 = 16 \\ \Rightarrow \quad x_2 &= 256 \end{aligned}$$

Substitute $x_2 = 256$ into $x_1^{1/2} = \frac{5}{2} x_2^{1/4}$.

$$\begin{aligned} x_1^{1/2} &= \tfrac{5}{2} x_2^{1/4} \\ \Rightarrow \quad x_1^{1/2} &= \tfrac{5}{2} \times 256^{1/4} = 10 \\ \Rightarrow \quad x_1 &= 100 \end{aligned}$$

In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} = 0$	$\frac{\partial \pi}{\partial x_2} = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -160 \times 100^{-3/2} \times 256^{1/4} = -\frac{16}{25}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 80 \times 100^{-1/2} \times 256^{-3/4} = \frac{1}{8}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{8}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -120 \times 100^{1/2} \times 256^{-7/4} = -\frac{75}{1024}$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} -\frac{16}{25} & \frac{1}{8} \\ \frac{1}{8} & -\frac{75}{1024} \end{vmatrix} = \frac{3}{64} - \frac{1}{64} = \frac{1}{32}$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 100$, $x_2 = 256$ represents a point of profit maximization.

Problem 4. Find the listed partial derivatives of each of the following functions.

a. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 40x_1 + 24x_2 - \lambda\left(20x_1 + 25x_2 - x_1^2 + x_1x_2 - x_2^2 - 443\right)$

$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 40 - \lambda(20 - 2x_1 + x_2)$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 24 - \lambda(25 + x_1 - 2x_2)$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -\left(20x_1 + 25x_2 - x_1^2 + x_1x_2 - x_2^2 - 443\right)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = 2\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\lambda$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 20 - 2x_1 + x_2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = 2\lambda$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 25 + x_1 - 2x_2$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 20 - 2x_1 + x_2$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 25 + x_1 - 2x_2$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

b. $\mathcal{L}(x_1, x_2, \lambda) = \left(100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2\right) - \lambda\,(90x_1 + 60x_2 - 1980)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 100 - 4x_1 + x_2 - 90\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 50 + x_1 - 4x_2 - 60\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\,(90x_1 + 60x_2 - 1980)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 1$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 90$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 1$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -4$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = -60$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 90$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 60$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

c. $\mathcal{L}(x_1, x_2, \lambda) \; = \; 2x_1 + x_2 - \lambda\left(x_1^{1/4} x_2^{1/2} - 16\right)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 2 - \frac{1}{4}\lambda x_1^{-3/4} x_2^{1/2}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 1 - \frac{1}{2}\lambda x_1^{1/4} x_2^{-1/2}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\left(x_1^{1/4} x_2^{1/2} - 16\right)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{3}{16}\lambda x_1^{-7/4} x_2^{1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{1}{8}\lambda x_1^{-3/4} x_2^{-1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{1}{4} x_1^{-3/4} x_2^{1/2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{1}{8}\lambda x_1^{-3/4} x_2^{-1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{1}{4}\lambda x_1^{1/4} x_2^{-3/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \frac{1}{2} x_1^{1/4} x_2^{-1/2}$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = \frac{1}{4} x_1^{-3/4} x_2^{1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = \frac{1}{2} x_1^{1/4} x_2^{-1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

d. $\mathcal{L}(x_1, x_2, \lambda) \ = \ \left(x_1^{1/3} x_2^{1/4}\right) - \lambda\left(128x_1 + 25x_2 - 19200\right)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{1/4} - 128\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{1}{4} x_1^{1/3} x_2^{-3/4} - 25\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\left(128x_1 + 25x_2 - 19200\right)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{2}{9} x_1^{-5/3} x_2^{1/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{12} x_1^{-2/3} x_2^{-3/4}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 128$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{12} x_1^{-2/3} x_2^{-3/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{3}{16} x_1^{1/3} x_2^{-7/4}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 25$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 128$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 25$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Problem 5. Consider the following matrix and vector.

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix},$$

$$v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix},$$

a. Find the determinant of the matrix V.

The value of the determinant of V is unchanged if the first column is subtracted from the second column and the first column multiplied by 2 is subtracted from the third column. So the determinant of matrix V can be found as follows.

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix} = 1$$

b. Find the inverse of the matrix V using the adjoint method.

The adjoint of matrix V is given by

$$\operatorname{adj}(V) = \begin{pmatrix} \begin{vmatrix} 4 & 6 \\ -3 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 6 \\ -2 & -3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ -3 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -6 & -3 & -1 \\ -3 & 1 & 1 \\ -2 & 0 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 6 & -3 & -2 \\ -3 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Then the inverse of matrix V is given by

$$V^{-1} = \frac{\operatorname{adj}(V)}{\det[V]} = \operatorname{adj}(V)$$

$$= \begin{pmatrix} 6 & -3 & -2 \\ -3 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

c. Using the inverse from part b, solve the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

The solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = V^{-1}v = \begin{pmatrix} 6 & -3 & -2 \\ -3 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \times (-1) + (-3) \times (-2) + (-2) \times (-1) \\ -3 \times (-1) + 1 \times (-2) + 0 \times (-1) \\ -1 \times (-1) + 1 \times (-2) + 1 \times (-1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

d. Using Cramer's rule, solve the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

By Cramer's rule, the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\begin{pmatrix} \begin{vmatrix} -1 & 1 & 2 \\ -2 & 4 & 6 \\ -1 & -3 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 & 2 \\ 3 & -2 & 6 \\ -2 & -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 & -1 \\ 3 & 4 & -2 \\ -2 & -3 & -1 \end{vmatrix} \end{pmatrix}}{\det[V]}$$

$$= \begin{pmatrix} \begin{vmatrix} -1 & 0 & 0 \\ -2 & 2 & 2 \\ -1 & -4 & -5 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 & 0 \\ 1 & -2 & 2 \\ -3 & -1 & -5 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 & -1 \\ 1 & 2 & -2 \\ -3 & -4 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$