ECONOMICS 207
SPRING 2008
PROBLEM SET 14

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;-w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \frac{\partial^2 \pi(x_1,x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi(x_1,x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x_1,x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x_1,x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1,x_2,\lambda) \;=\; f(x_1,x_2) - \lambda g(x_1,x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda \partial x_2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_2} & \frac{\partial g(x_1,x_2)}{\partial x_1} \\ \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_2} & \frac{\partial g(x_1,x_2)}{\partial x_2} \\ \frac{\partial g(x_1,x_2)}{\partial x_1} & \frac{\partial g(x_1,x_2)}{\partial x_2} & 0 \end{bmatrix}$$

where we use the equivalencies

$$\frac{\partial g(x_1,x_2)}{\partial x_1} = -\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial \lambda}$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} = -\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial \lambda}.$$

Date: April 22, 2008.

Problem 1. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2) - w_1x_1 - w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$f(x_1, x_2) = 90x_1 + 45x_2 - 3x_1^2 + 3x_1x_2 - 2x_2^2$$

$$p = 4$$

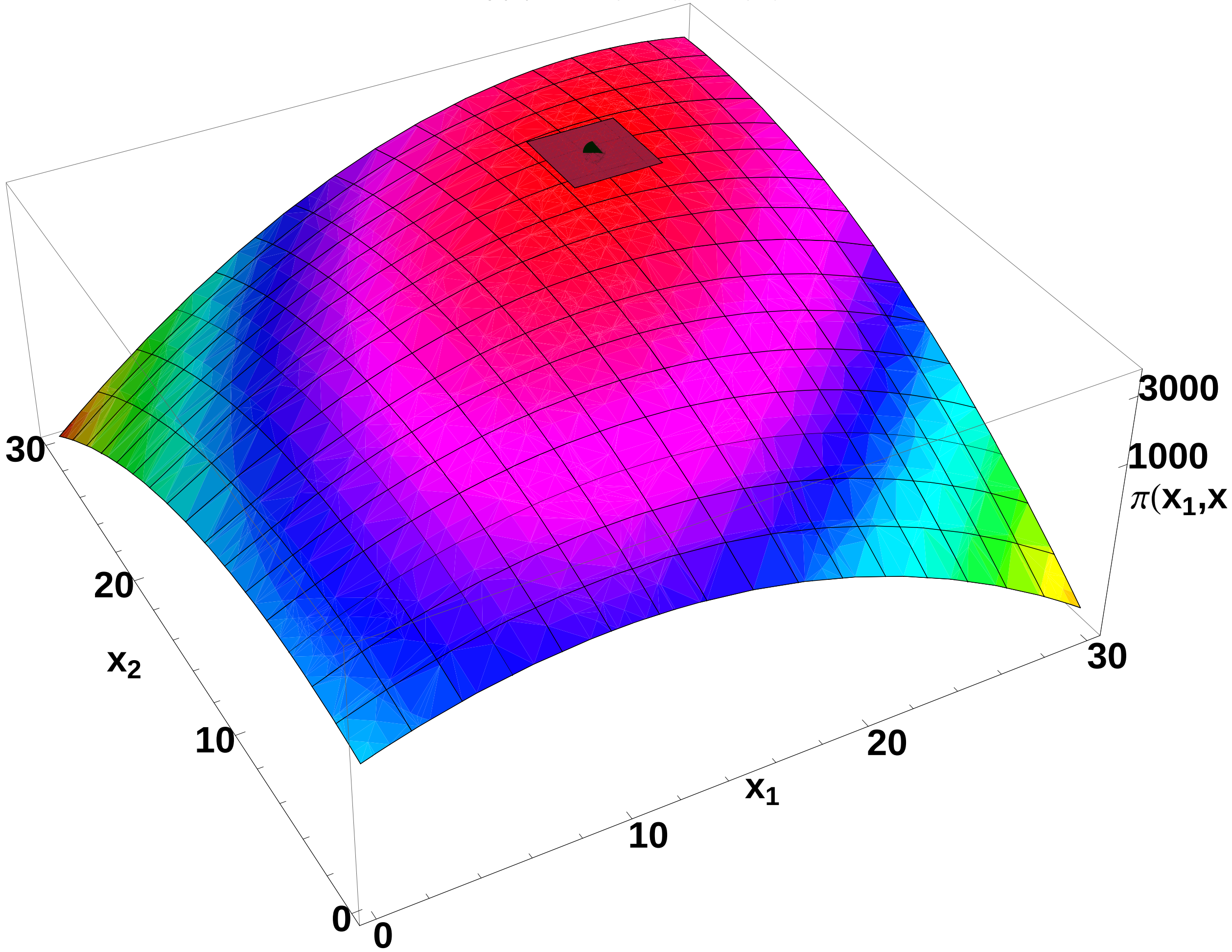
$$w_1 = 120, \qquad w_2 = 80$$

$$\pi =$$

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

Find potential profit maximizing levels of x_1 and x_2.

Figure 1. Maximum Profit



By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \\ \\ \end{vmatrix} =$$

Problem 2. a. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form π = $pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

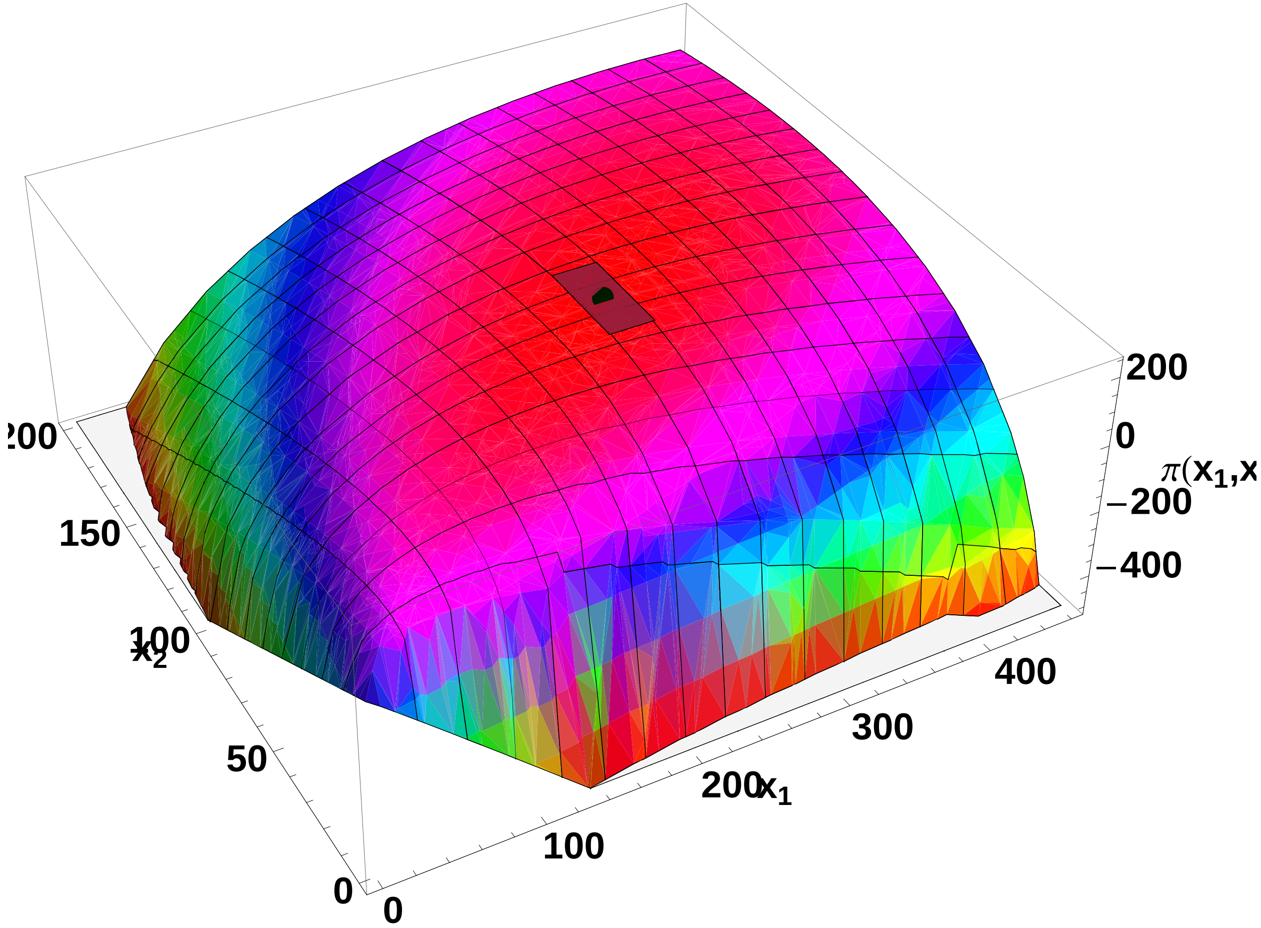
$$\begin{aligned} f(x_1, x_2) &= x_1^{3/5} x_2^{1/4} \\ p &= 20 \\ w_1 &= 4, \qquad w_2 \;=\; 5 \end{aligned}$$

$\pi \;=$

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

b. Show that the profit maximizing levels of x_1 and x_2 are 243 and 81.

Figure 2. Maximum Profit



c. In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

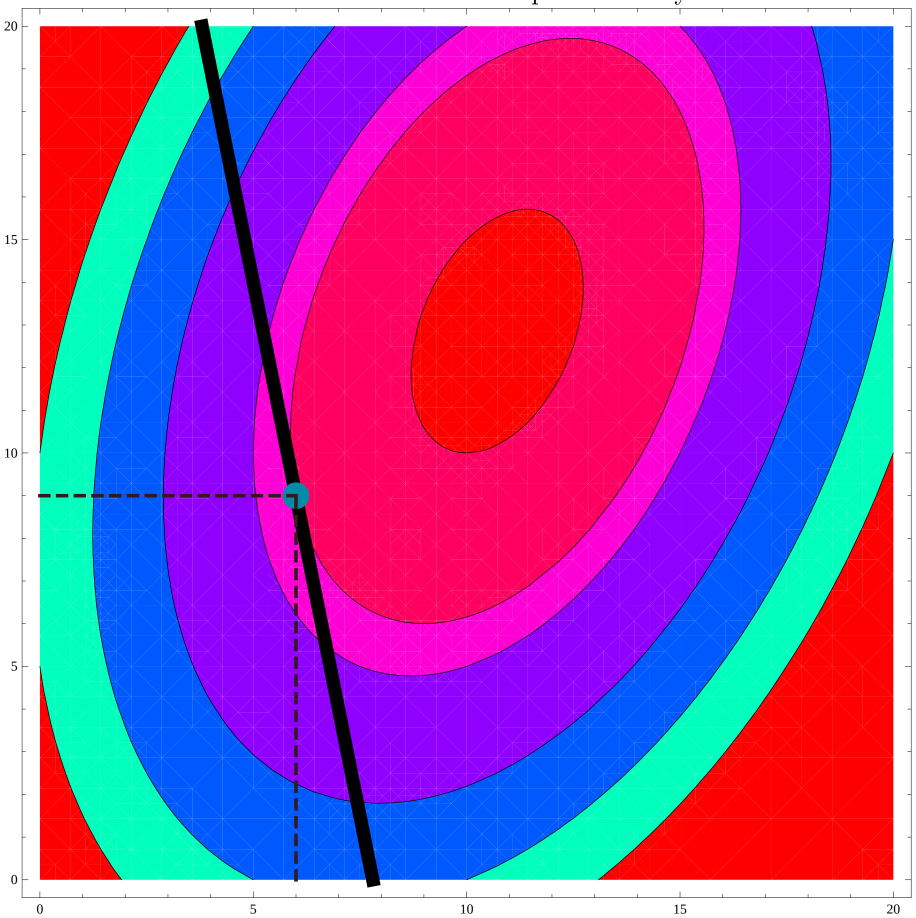
$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} =$$

Problem 3. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 30x_1 + 15x_2 - 2x_1^2 + x_1x_2 - x_2^2 - \lambda\,(60x_1 + 12x_2 - 468)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 30 - 4x_1 + x_2 - 60\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 15 + x_1 - 2x_2 - 12\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(60x_1 + 12x_2 - 468)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 1$	$\frac{\partial g(x_1, x_2)}{\partial x_1} = 60$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$-\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} =$
$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} =$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Figure 3. Maximize Output or Utility



b. Show that three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $\mathrm{x}_1 = 6$, $\mathrm{x}_2 = 9$, and $\lambda = \frac{1}{4}$.

c. Substitute the x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 9216.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

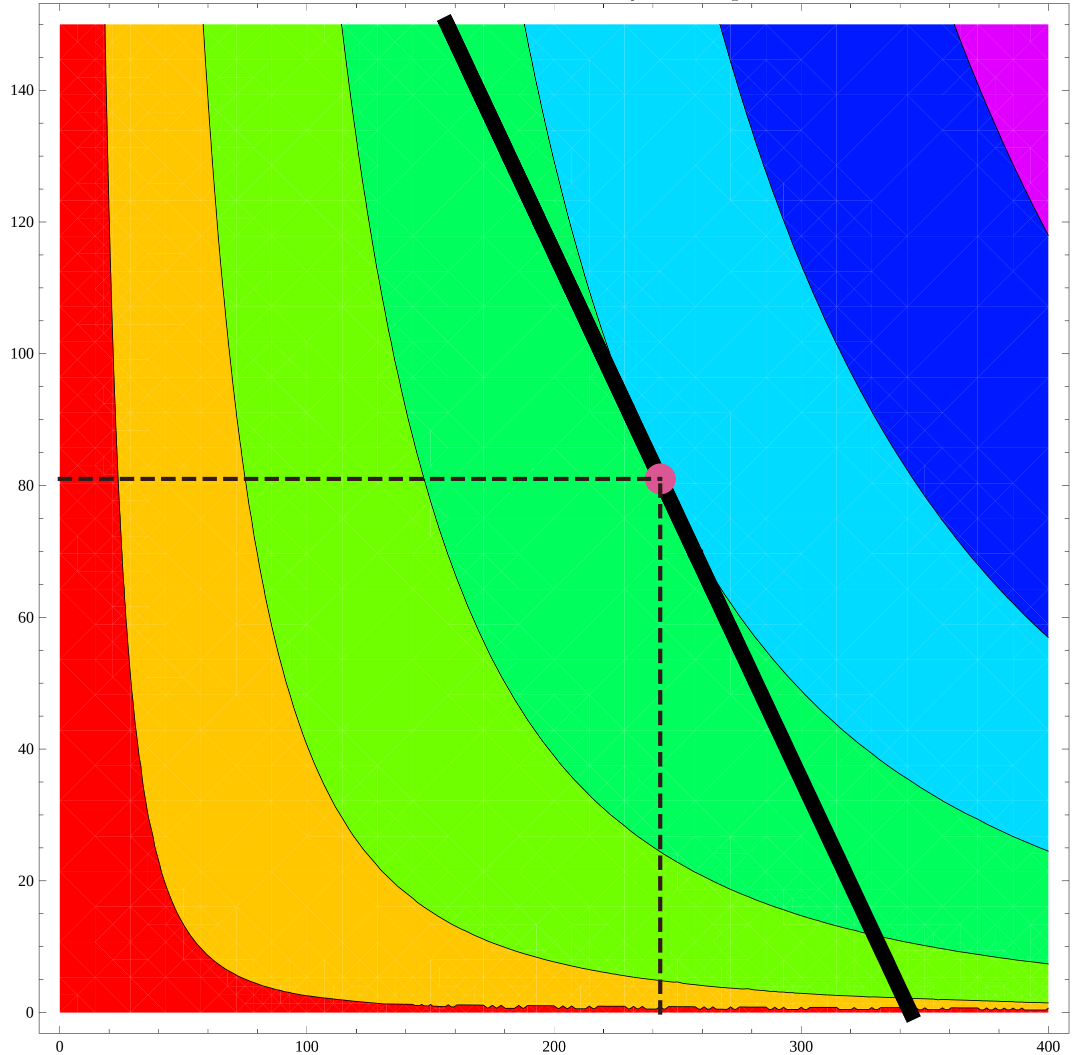
A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 4. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; x_1^{3/5} x_2^{1/4} - \lambda\,(4x_1 + 5x_2 - 1377)$$

$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} =$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} =$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$\frac{\partial g(x_1,x_2)}{\partial x_1} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$\frac{\partial g(x_1,x_2)}{\partial x_2} =$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} =$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} =$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} =$

Figure 4. Maximize Utility or Output



b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $\mathrm{x}_1 = 243$, $\mathrm{x}_2 = 81$, and $\lambda = \dfrac{1}{20}$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $\frac{17}{243}$.

$$\left| \begin{array}{ccc} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{2}{6075} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{1620} & \frac{\partial g(x_1,x_2)}{\partial x_1} = 4 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = & \frac{\partial g(x_1,x_2)}{\partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{array} \right| =$$

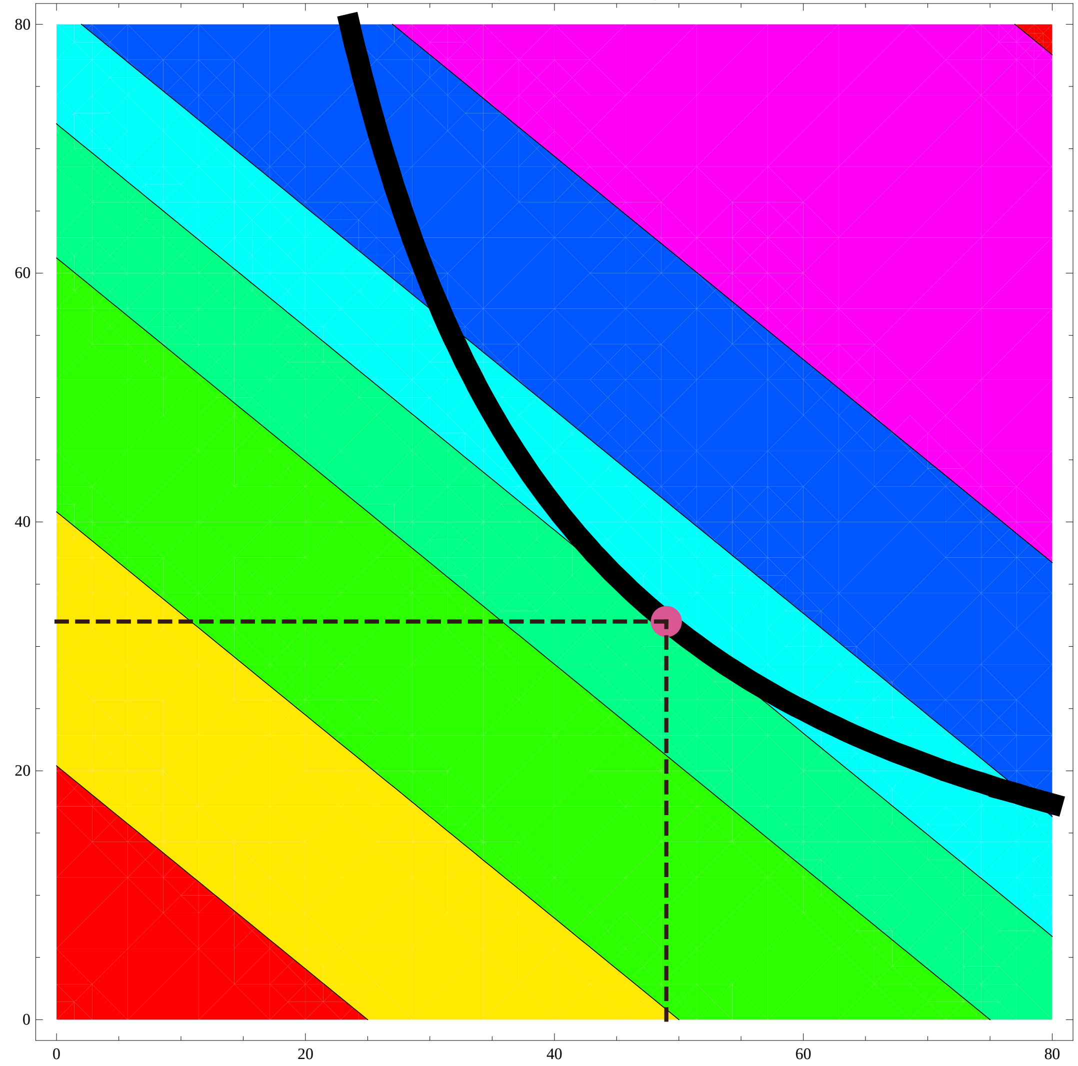
A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 5. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 40x_1 + 49x_2 - \lambda\left(x_1^{1/2}x_2^{2/5} - 28\right)$$

$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 40 - \frac{1}{2}\lambda x_1^{-1/2} x_2^{2/5}$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 49 - \frac{2}{5}\lambda x_1^{1/2} x_2^{-3/5}$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$\frac{\partial g(x_1,x_2)}{\partial x_1} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$\frac{\partial g(x_1,x_2)}{\partial x_2} =$
$-\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Figure 5. Minimize Cost



b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $\mathrm{x}_1 = 49$, $\mathrm{x}_2 = 32$, and $\lambda = 140$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{9}{40}$

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{20}{49} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{1}{2} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.