ECONOMICS 207
SPRING 2008
PROBLEM SET 14
KEY

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;-w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_1} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_2} \\ \frac{\partial g(x_1, x_2)}{\partial x_1} & \frac{\partial g(x_1, x_2)}{\partial x_2} & 0 \end{bmatrix}$$

where we use the equivalencies

$$\frac{\partial g(x_1, x_2)}{\partial x_1} = -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda}$$

$$\frac{\partial g(x_1, x_2)}{\partial x_2} = -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda}.$$

Date: August 25, 2008.

Problem 1. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form π = $pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$f(x_1,x_2) = 90x_1 + 45x_2 - 3x_1^2 + 3x_1x_2 - 2x_2^2$$
$$p = 4$$
$$w_1 = 120, \qquad w_2 = 80$$

$$\pi = 4\left(90x_1 + 45x_2 - 3x_1^2 + 3x_1x_2 - 2x_2^2\right) - 120x_1 - 80x_2$$
$$= 240x_1 + 100x_2 - 12x_1^2 + 12x_1x_2 - 8x_2^2$$

$\frac{\partial \pi}{\partial x_1} = 240 - 24x_1 + 12x_2$	$\frac{\partial \pi}{\partial x_2} = 100 + 12x_1 - 16x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -24$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 12$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 12$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -16$

Find potential profit maximizing levels of x_1 and x_2.
By setting the first derivative to zero, we obtain

$$\frac{\partial \pi}{\partial x_1} = 240 - 24x_1 + 12x_2 = 0 \tag{1}$$

$$\frac{\partial \pi}{\partial x_2} = 100 + 12x_1 - 16x_2 = 0 \tag{2}$$

Add the equation (2) multiplied by 2 to equation (1).

$$\begin{aligned} 240 - 24x_1 + 12x_2 + 2\,(100 + 12x_1 - 16x_2) &= 0 \\ \Rightarrow \quad 440 - 20x_2 &= 0 \\ \Rightarrow \quad x_2 &= 22 \end{aligned}$$

Substitute $x_2 = 22$ into equation (1).

$$\begin{aligned} 240 - 24x_1 + 12x_2 &= 0 \\ \Rightarrow \quad 240 - 24x_1 + 12 \times 22 &= 0 \\ \Rightarrow \quad x_1 &= 21 \end{aligned}$$

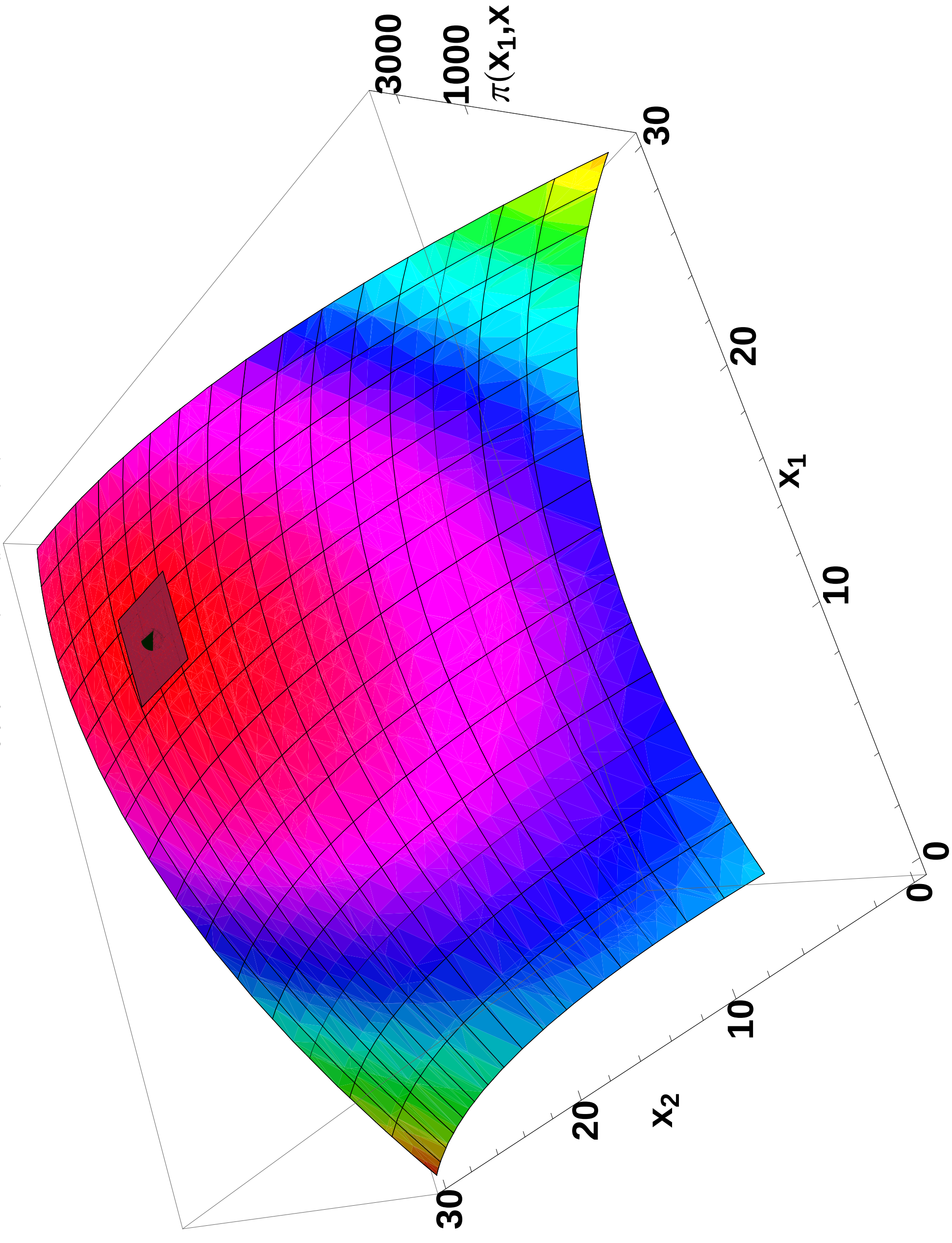


FIGURE 1. Maximum Profit

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -24 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 12 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 12 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -16 \end{vmatrix} = -24 \times (-16) - 12 \times 12 = 240 > 0$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 21$, $x_2 = 22$ represent a point of profit maximization.

Problem 2. a. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

$$f(x_1,x_2) = x_1^{3/5}x_2^{1/4}$$
$$p = 20$$
$$w_1 = 4, \qquad w_2 = 5$$

$$\pi = 20x_1^{3/5}x_2^{1/4} - 4x_1 - 5x_2$$

$\frac{\partial\pi}{\partial x_1} = 12x_1^{-2/5}x_2^{1/4} - 4$	$\frac{\partial\pi}{\partial x_2} = 5x_1^{3/5}x_2^{-3/4} - 5$
$\frac{\partial^2\pi}{\partial x_1\partial x_1} = -\frac{24}{5}x_1^{-7/5}x_2^{1/4}$	$\frac{\partial^2\pi}{\partial x_1\partial x_2} = 3x_1^{-2/5}x_2^{-3/4}$
$\frac{\partial^2\pi}{\partial x_2\partial x_1} = 3x_1^{-2/5}x_2^{-3/4}$	$\frac{\partial^2\pi}{\partial x_2\partial x_2} = -\frac{15}{4}x_1^{3/5}x_2^{-7/4}$

b. Show that the profit maximizing levels of x_1 and x_2 are 243 and 81.

By setting the first derivative to zero, we obtain

$$\frac{\partial \pi}{\partial x_1} = 12x_1^{-2/5}x_2^{1/4} - 4 = 0 \tag{3}$$

$$\frac{\partial \pi}{\partial x_2} = 5x_1^{3/5}x_2^{-3/4} - 5 = 0 \tag{4}$$

Rearrange equation (3).

$$\begin{aligned} 12x_1^{-2/5}x_2^{1/4} - 4 &= 0 \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{1}{3}x_1^{2/5} \\ \Rightarrow \quad x_2^{-3/4} &= 27x_1^{-6/5} \end{aligned}$$

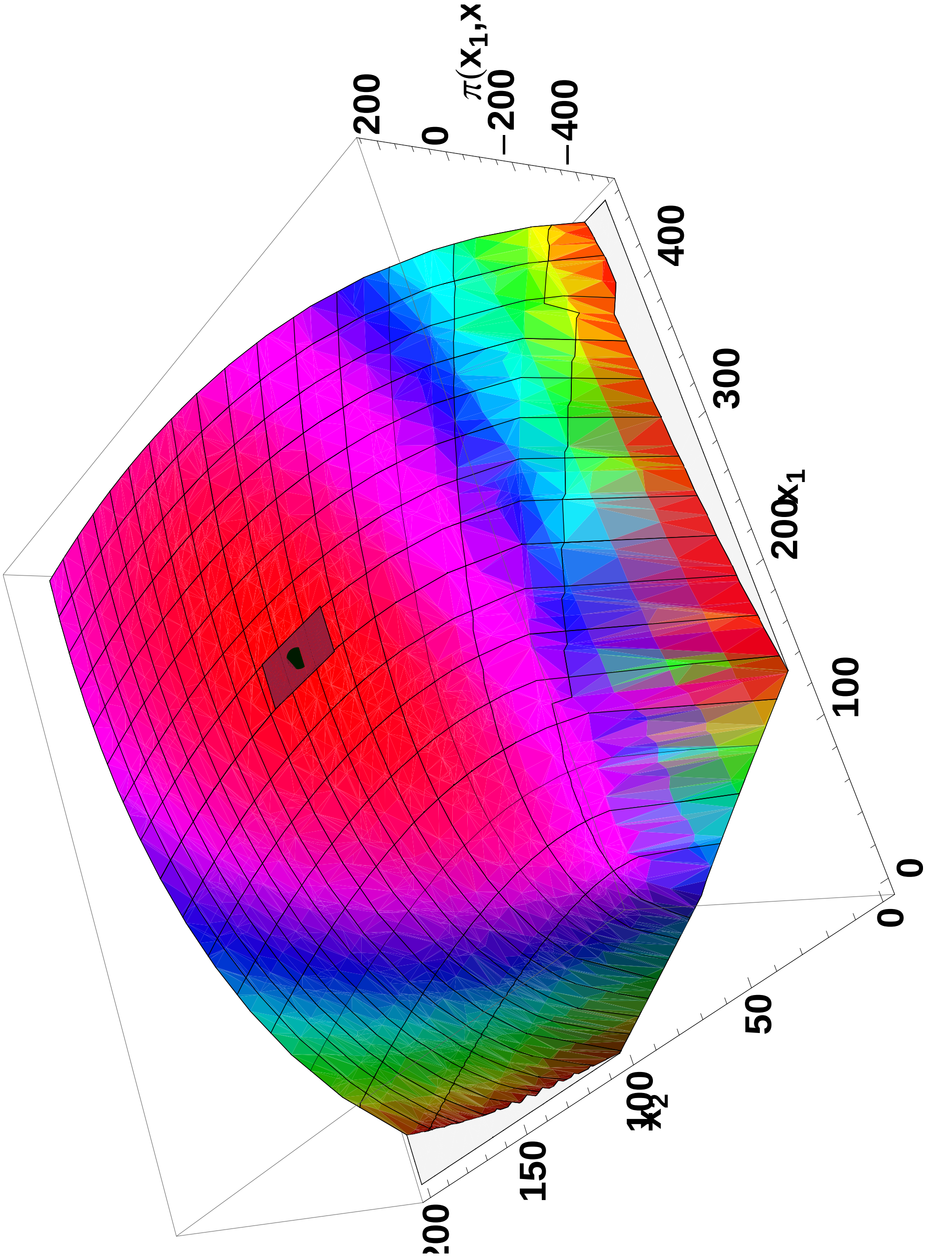
Substitute $x_2^{-3/4} = 27x_1^{-6/5}$ into equation (4).

$$\begin{aligned} 5x_1^{3/5}x_2^{-3/4} - 5 &= 0 \\ \Rightarrow \quad 5x_1^{3/5}\left(27x_1^{-6/5}\right) - 5 &= 0 \\ \Rightarrow \quad x_1^{-3/5} &= 3^{-3} \\ \Rightarrow \quad x_1 &= 3^5 = 243 \end{aligned}$$

Then substitute $x_1 = 243$ into $x_2^{1/4} = \frac{1}{3}x_1^{2/5}$.

$$\begin{aligned} x_2^{1/4} &= \tfrac{1}{3}x_1^{2/5} \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{1}{3}243^{2/5} = 3 \\ \Rightarrow \quad x_2 &= 81 \end{aligned}$$

FIGURE 2. Maximum Profit



c. In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} = 0$	$\frac{\partial \pi}{\partial x_2} = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{24}{5} \times 243^{-7/5} \times 81^{1/4} = -\frac{8}{1215}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 3 \times 243^{-2/5} \times 81^{-3/4} = \frac{1}{81}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{81}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{15}{4} \times 243^{3/5} \times 81^{-7/4} = -\frac{5}{108}$

d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{8}{1215} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \frac{1}{81} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{81} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{5}{108} \end{vmatrix} = -\frac{8}{1215} \times \left(-\frac{5}{108}\right) - \frac{1}{81} \times \frac{1}{81} = \frac{2}{6561} - \frac{1}{6561} = \frac{1}{6561} > 0$$

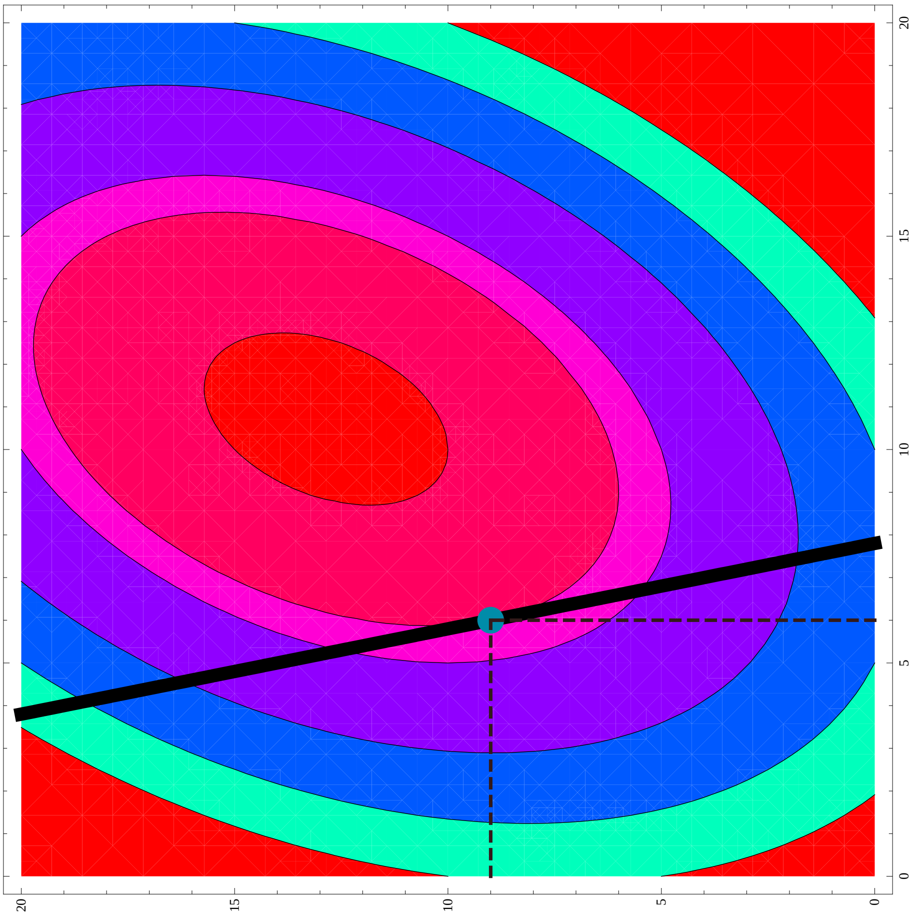
Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 243$, $x_2 = 81$ represent a point of profit maximization.

Problem 3. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 30x_1 + 15x_2 - 2x_1^2 + x_1x_2 - x_2^2 - \lambda\,(60x_1 + 12x_2 - 468)$$

$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 30 - 4x_1 + x_2 - 60\lambda$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 15 + x_1 - 2x_2 - 12\lambda$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -(60x_1 + 12x_2 - 468)$
$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_1} = -4$	$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} = 1$	$\frac{\partial g(x_1,x_2)}{\partial x_1} = 60$
$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1} = 1$	$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_2} = -2$	$-\frac{\partial^2\mathcal{L}}{\partial x_2\partial \lambda} = 12$
$\frac{\partial g(x_1,x_2)}{\partial x_1} = 60$	$-\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_2} = 12$	$\frac{\partial^2\mathcal{L}}{\partial \lambda\partial \lambda} = 0$

FIGURE 3. Maximize Output or Utility



b. Show that three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 6$, $x_2 = 9$, and $\lambda = \frac{1}{4}$.

By setting the first derivative to zero, we obtain

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 30 - 4x_1 + x_2 - 60\lambda = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 15 + x_1 - 2x_2 - 12\lambda = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(60x_1 + 12x_2 - 468) = 0 \tag{7}$$

Subtract equation (6) multiplied by 5 from equation (5).

$$30 - 4x_1 + x_2 - 60\lambda - 5\,(15 + x_1 - 2x_2 - 12\lambda) = 0$$

$$\Rightarrow \quad -45 - 9x_1 + 11x_2 = 0 \tag{8}$$

Add equation (8) multiplied by -10 to equation (7) multiplied by 3/2.

$$\begin{aligned} 450 + 90x_1 - 110x_2 - (90x_1 + 18x_2 - 702) &= 0 \\ \Rightarrow \quad -128x_2 + 1152 &= 0 \\ \Rightarrow \quad x_2 &= 9 \end{aligned}$$

Substitute $x_2 = 9$ into equation (8).

$$\begin{aligned} -45 - 9x_1 + 11x_2 &= 0 \\ \Rightarrow \quad -45 - 9x_1 + 11 \times 9 &= 0 \\ \Rightarrow \quad -9x_1 &= -54 \\ \Rightarrow \quad x_1 &= 6 \end{aligned}$$

Substitute $x_1 = 6$ and $x_2 = 9$ into equation (5).

$$\begin{aligned} 30 - 4x_1 + x_2 - 60\lambda &= 0 \\ \Rightarrow \quad 30 - 4 \times 6 + 9 - 60\lambda &= 0 \\ \Rightarrow \quad -60\lambda &= -15 \\ \Rightarrow \quad \lambda &= \tfrac{1}{4} \end{aligned}$$

c. Substitute the x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 9216.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 1 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 60 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 1 & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2 & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 12 \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = 60 & \frac{\partial g(x_1,x_2)}{\partial x_2} = 12 & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix}$$

$$= 60 \begin{vmatrix} 1 & 60 \\ -2 & 12 \end{vmatrix} - 12 \begin{vmatrix} -4 & 60 \\ 1 & 12 \end{vmatrix}$$

$$= 60 \times 132 - 12 \times (-108) = 9216$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 4. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; x_1^{3/5} x_2^{1/4} - \lambda\,(4x_1 + 5x_2 - 1377)$$

$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = \frac{3}{5} x_1^{-2/5} x_2^{1/4} - 4\lambda$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = \frac{1}{4} x_1^{3/5} x_2^{-3/4} - 5\lambda$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -\,(4x_1 + 5x_2 - 1377)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{6}{25} x_1^{-7/5} x_1^{1/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{3}{20} x_1^{-2/5} x_2^{-3/4}$	$\frac{\partial g(x_1,x_2)}{\partial x_1} = 4$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{3}{20} x_1^{-2/5} x_2^{-3/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{3}{16} x_1^{3/5} x_2^{-7/4}$	$\frac{\partial g(x_1,x_2)}{\partial x_2} = 5$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 4$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 5$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

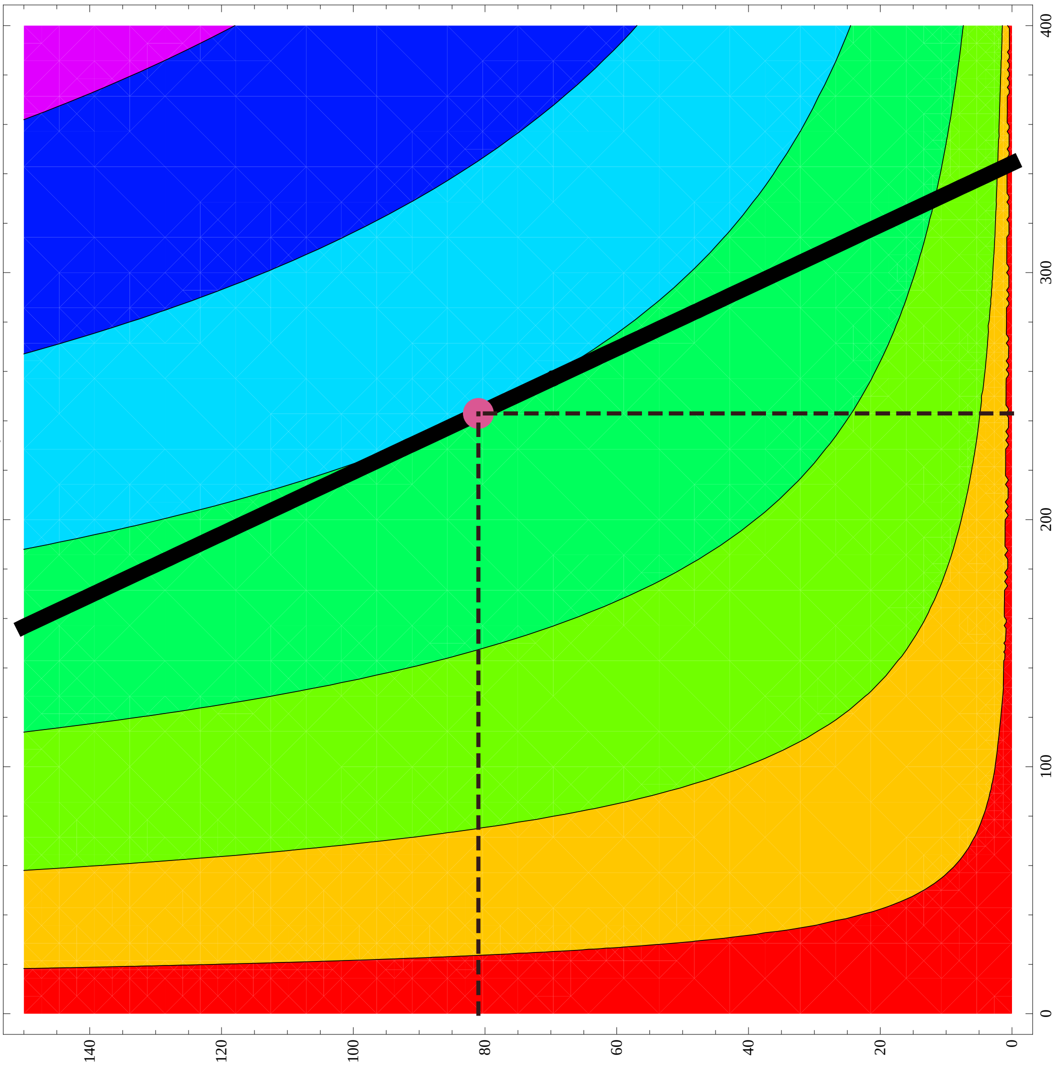


FIGURE 4. Maximize Utility or Output

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 243$, $x_2 = 81$, and $\lambda = \frac{1}{20}$.

By setting the first derivative to zero, we obtain

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{3}{5} x_1^{-2/5} x_2^{1/4} - 4\lambda = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{1}{4} x_1^{3/5} x_2^{-3/4} - 5\lambda = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(4x_1 + 5x_2 - 1377) = 0 \tag{11}$$

Rearrange equation (9) and equation (10).

$$\frac{3}{5} x_1^{-2/5} x_2^{1/4} = 4\lambda \tag{12}$$

$$\frac{1}{4} x_1^{3/5} x_2^{-3/4} = 5\lambda \tag{13}$$

Divide the left side of equation (12) by the left side of equation (13); and divide the right side of equation (12) by the right side of equation (13).

$$\begin{aligned} \frac{\frac{3}{5} x_1^{-2/5} x_2^{1/4}}{\frac{1}{4} x_1^{3/5} x_2^{-3/4}} &= \frac{4\lambda}{5\lambda} \\ \Rightarrow \quad x_1 &= 3x_2 \end{aligned}$$

Substitute $x_1 = 3x_2$ into equation (11).

$$\begin{aligned} -(4x_1 + 5x_2 - 1377) &= 0 \\ \Rightarrow \quad 12x_2 + 5x_2 - 1377 &= 0 \\ \Rightarrow \quad x_2 &= 81 \end{aligned}$$

Then $x_1 = 3x_2 = 243$.
Substitute $x_1 = 243$, $x_2 = 81$ into equation (9).

$$\begin{aligned} \tfrac{3}{5} x_1^{-2/5} x_2^{1/4} - 4\lambda &= 0 \\ \Rightarrow \quad \tfrac{3}{5} \times 243^{-2/5} \times 81^{1/4} - 4\lambda &= 0 \\ \Rightarrow \quad \tfrac{1}{5} - 4\lambda &= 0 \\ \Rightarrow \quad \lambda &= \tfrac{1}{20} \end{aligned}$$

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $\frac{17}{243}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{2}{6075} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{1620} & \frac{\partial g(x_1,x_2)}{\partial x_1} = 4 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{1620} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{3}{16} \times 243^{3/5} \times 81^{-7/4} = -\frac{1}{432} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 5 \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = 4 & \frac{\partial g(x_1,x_2)}{\partial x_2} = 5 & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix}$$

$$= 4 \begin{vmatrix} \frac{1}{1620} & 4 \\ -\frac{1}{432} & 5 \end{vmatrix} - 5 \begin{vmatrix} -\frac{2}{6075} & 4 \\ \frac{1}{1620} & 5 \end{vmatrix}$$

$$= \frac{17}{243}$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 5. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 40x_1 + 49x_2 - \lambda\left(x_1^{1/2}x_2^{2/5} - 28\right)$$

$\dfrac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 40 - \frac{1}{2}\lambda x_1^{-1/2}x_2^{2/5}$	$\dfrac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 49 - \frac{2}{5}\lambda x_1^{1/2}x_2^{-3/5}$	$\dfrac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -\left(x_1^{1/2}x_2^{2/5} - 28\right)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{1}{4}\lambda x_1^{-3/2}x_2^{2/5}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{1}{5}\lambda x_1^{-1/2}x_2^{-3/5}$	$\frac{\partial g(x_1,x_2)}{\partial x_1} = \frac{1}{2}x_1^{-1/2}x_2^{2/5}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{1}{5}\lambda x_1^{-1/2}x_2^{-3/5}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{6}{25}\lambda x_1^{1/2}x_2^{-8/5}$	$\frac{\partial g(x_1,x_2)}{\partial x_2} = \frac{2}{5}x_1^{1/2}x_2^{-3/5}$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = \frac{1}{2}x_1^{-1/2}x_2^{2/5}$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = \frac{2}{5}x_1^{1/2}x_2^{-3/5}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

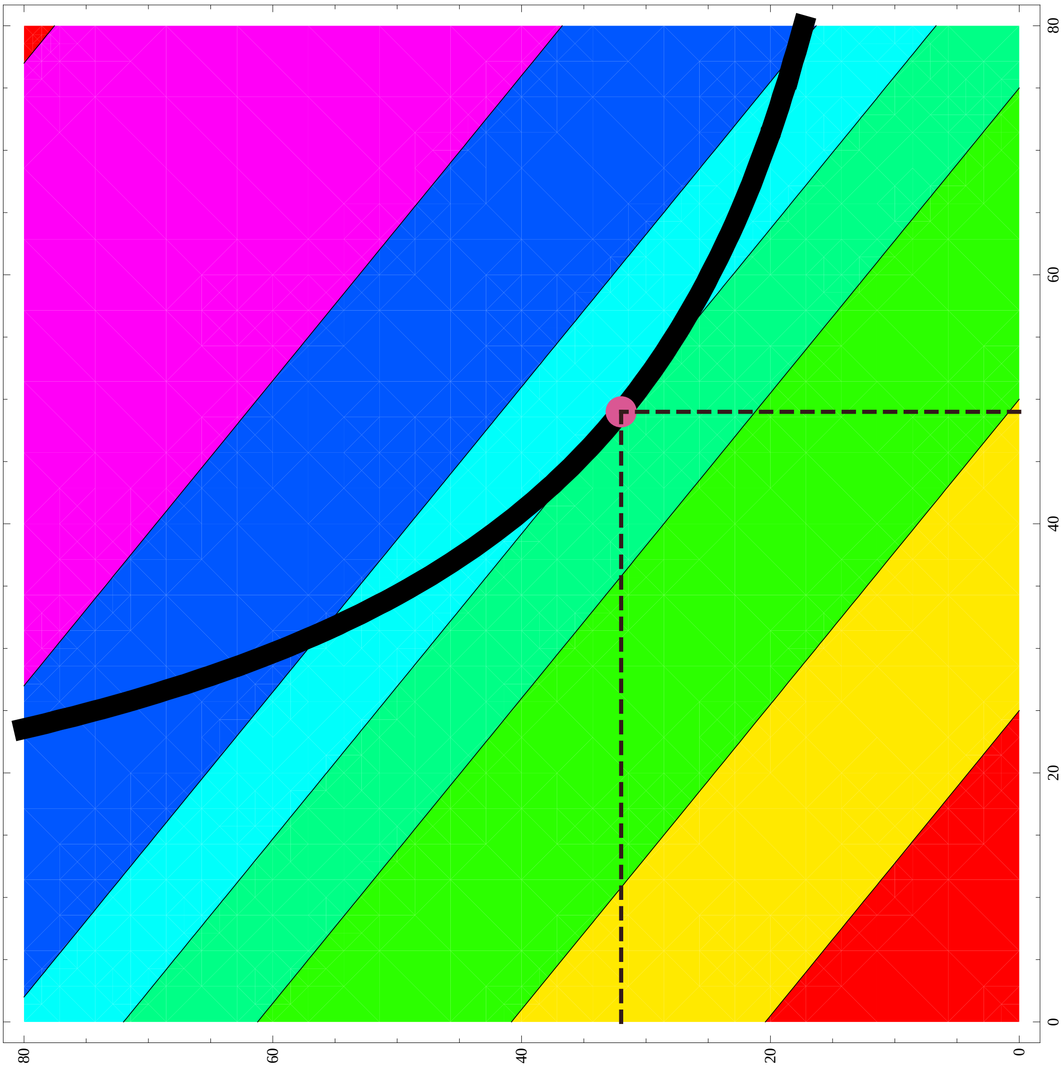


FIGURE 5. Minimize Cost

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 49$, $x_2 = 32$, and $\lambda = 140$.

By setting the first derivative to zero, we obtain

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 40 - \frac{1}{2}\lambda x_1^{-1/2} x_2^{2/5} = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 49 - \frac{2}{5}\lambda x_1^{1/2} x_2^{-3/5} = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\left(x_1^{1/2} x_2^{2/5} - 28\right) = 0 \tag{16}$$

Rearrange equation (14) and equation (15).

$$x_1^{-1/2} x_2^{2/5} = \frac{80}{\lambda} \tag{17}$$

$$x_1^{1/2} x_2^{-3/5} = \frac{245}{2\lambda} \tag{18}$$

Divide the left side of equation (17) by the left side of equation (18) and divide the right side of equation (17) by the right side of equation (18).

$$\frac{x_1^{-1/2} x_2^{2/5}}{x_1^{1/2} x_2^{-3/5}} = \frac{\frac{80}{\lambda}}{\frac{245}{2\lambda}}$$

$$\Rightarrow \quad \frac{x_2}{x_1} = \frac{32}{49}$$

Substitute $x_2 = \frac{32}{49}x_1$ into equation (16).

$$\begin{aligned}
& -\left(x_1^{1/2} x_2^{2/5} - 28\right) &=& \quad 0 \\
\Rightarrow \quad & x_1^{1/2}\left(\tfrac{32}{49}x_1\right)^{2/5} &=& \quad 28 \\
\Rightarrow \quad & x_1^{9/10} &=& \quad 28\left(\tfrac{49}{32}\right)^{2/5} = 2^2 \times 7 \times 7^{4/5} \times \left(\left(\tfrac{1}{2}\right)^5\right)^{2/5} = 7^{9/5} \\
\Rightarrow \quad & x_1 = & & 7^2 = 49
\end{aligned}$$

Thus, $x_2 = \frac{32}{49}x_1 = 32$.

Substitute $x_1 = 49$, $x_2 = 32$ into equation (14).

$$\begin{aligned} 40 - \tfrac{1}{2}\lambda x_1^{-1/2} x_2^{2/5} &= 0 \\ \Rightarrow \quad 40 - \tfrac{1}{2}\lambda 49^{-1/2} \times 32^{2/5} &= 0 \\ \Rightarrow \quad 40 - \tfrac{2}{7}\lambda &= 0 \\ \Rightarrow \quad \lambda &= 140 \end{aligned}$$

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{9}{40}$

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{20}{49} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{1}{2} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{1}{2} \times 49^{-1/2} \times 32^{2/5} = \frac{2}{7} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{1}{5} \times 140 \times 49^{-1/2} \times 32^{-3/5} = -\frac{1}{2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{6}{25} \times 140 \times 49^{1/2} \times 32^{-8/5} = \frac{147}{160} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \frac{2}{5} \times 49^{1/2} \times 32^{-3/5} = \frac{7}{20} \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = \frac{2}{7} & \frac{\partial g(x_1,x_2)}{\partial x_2} = \frac{7}{20} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix}$$

$$= \frac{2}{7} \begin{vmatrix} -\frac{1}{2} & \frac{2}{7} \\ \frac{147}{160} & \frac{7}{20} \end{vmatrix} - \frac{7}{20} \begin{vmatrix} \frac{20}{49} & \frac{2}{7} \\ -\frac{1}{2} & \frac{7}{20} \end{vmatrix} = -\frac{9}{40} < 0$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.