ECONOMICS 207
SPRING 2008
PROBLEM SET 15

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;-w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \dfrac{\partial^2 \pi(x_1,x_2)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 \pi(x_1,x_2)}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 \pi(x_1,x_2)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \pi(x_1,x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1,x_2,\lambda) \;=\; f(x_1,x_2) - \lambda g(x_1,x_2)$$

is given by

$$H_B = \begin{bmatrix} \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_2} & -\dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial \lambda} \\ \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_2} & -\dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial \lambda} \\ -\dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda \partial x_1} & -\dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda \partial x_2} & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial x_2} & \dfrac{\partial g(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial x_2} & \dfrac{\partial g(x_1,x_2)}{\partial x_2} \\ \dfrac{\partial g(x_1,x_2)}{\partial x_1} & \dfrac{\partial g(x_1,x_2)}{\partial x_2} & 0 \end{bmatrix}$$

where we use the equivalencies

$$\frac{\partial g(x_1,x_2)}{\partial x_1} = -\,\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1 \partial \lambda}$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} = -\,\frac{\partial^2 \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2 \partial \lambda}.$$

Date: 28 April 2008.

Problem 1. Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2.

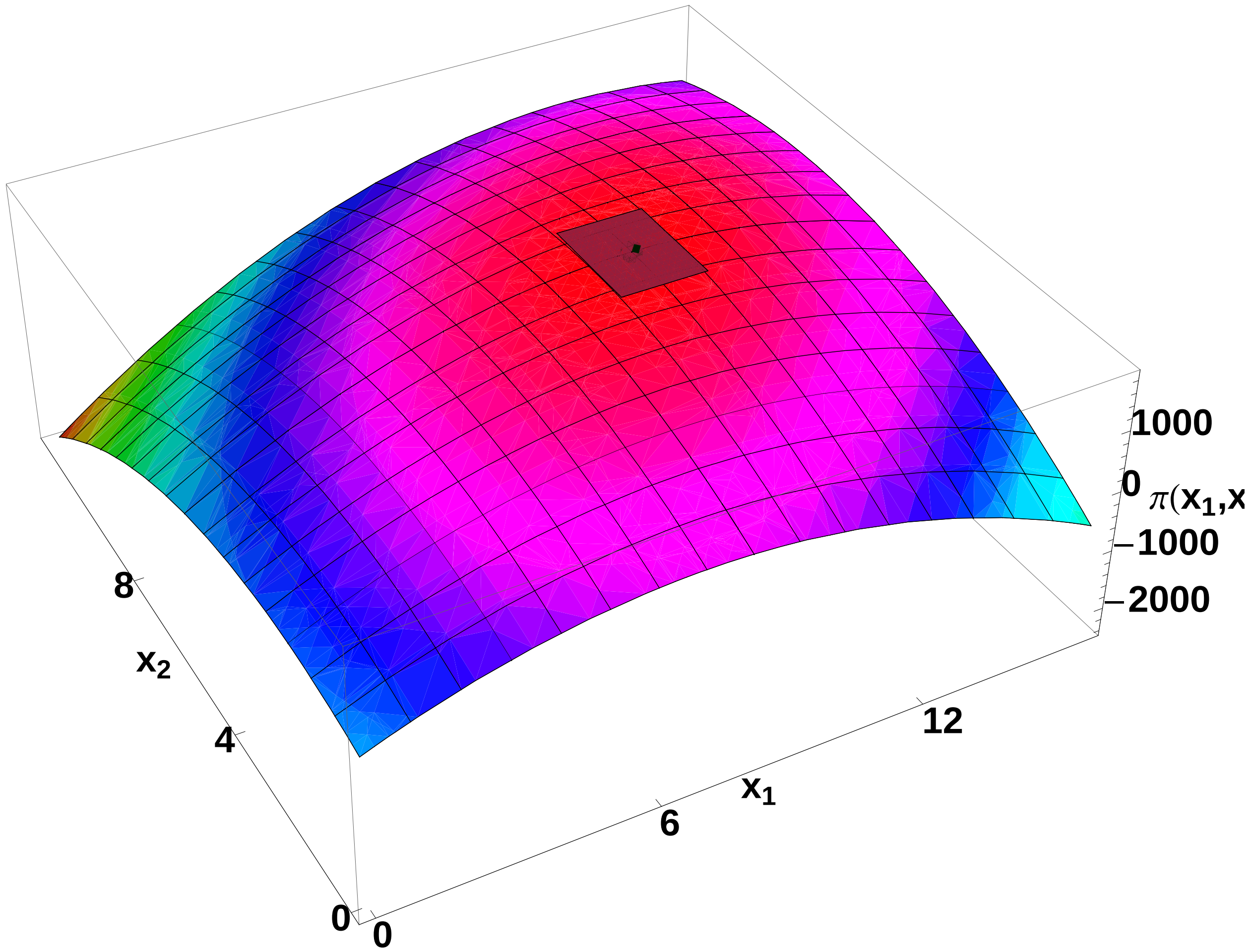
$$\begin{aligned} f(x_1, x_2) &= 30x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - 3x_2^2 \\ p &= 10 \\ w_1 &= 20, \qquad w_2 \; = \; 40 \end{aligned}$$

a. Write an equation that represents profit as a function of the two inputs x_1 and x_2. Simplify the expression.

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

FIGURE 1. Maximum Profit



c. Find potential profit maximizing levels of x_1 and x_2.

d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of x_1 and x_2 and then verify that the levels of x_1 and x_2 you found are either maximum, minimum or saddle points.

$$H = \begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ & \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} =$$

e. What is the optimal level of output?

f. How much does the firm spend on inputs?

Problem 2. Consider a consumer with a utility function given by

$$v = u(x_1, x_2) = 30x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - 3x_2^2$$

The consumer faces prices and an income constraint given by

$$p_1 = 20, \qquad p_2 = 40, \qquad m_0 = 440$$

Find potential levels of x_1 and x_2 to maximum utility for this consumer given the income constraint and the stated prices. Verify that these consumption levels maximize utility.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathfrak{L}(x_1, x_2, \lambda) =$$

$\dfrac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\dfrac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_2}$

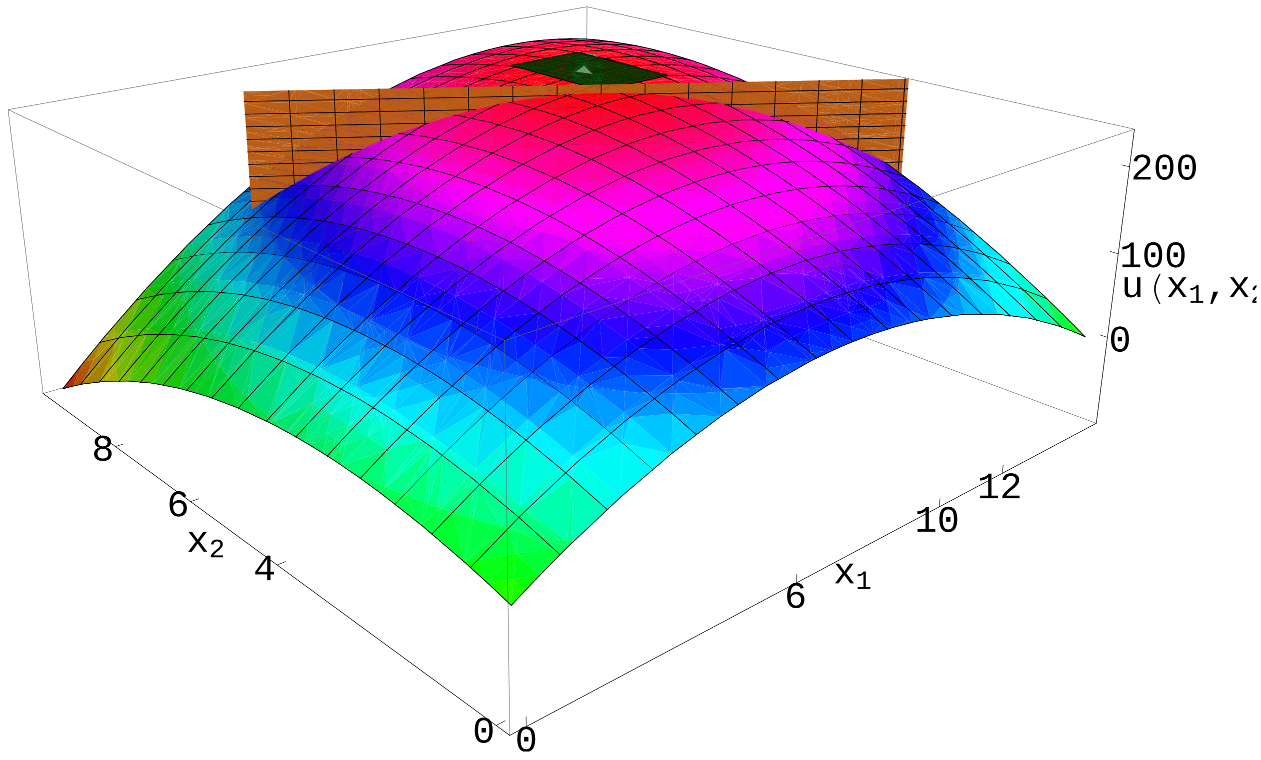
b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

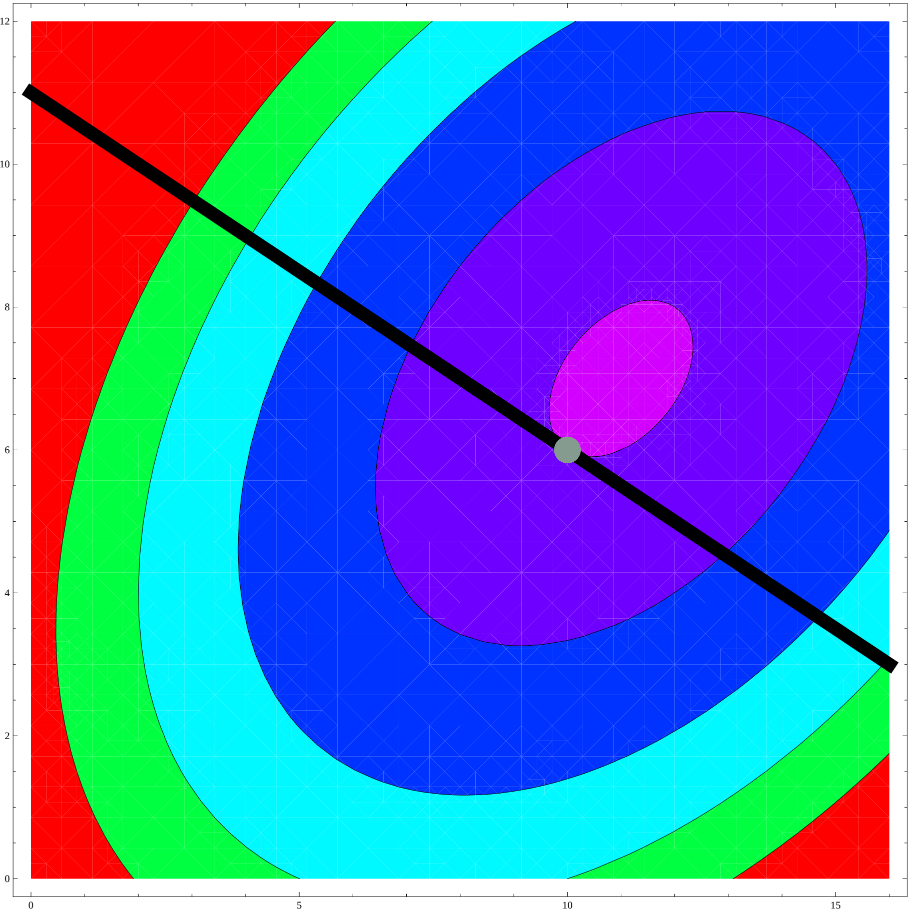
In figure 2 you can see the maximum utility point and how it is unattainable given the budget constraint.

FIGURE 2. Utility and the Budget Constraint



In figure 3 you can see the the tangency between one indifference curve and the budget line.

FIGURE 3. Tangency Between Indifference Curve and Budget Line



d. Use the information from 2a and 2b to find critical values for x_1, x_2 and λ. They are as follows: $x_1 = 10, x_2 = 6, \lambda = \frac{1}{10}$.

e. Use the answers from part 2d and the expressions from parts 2a and 2c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum, a minimum, or a saddle point. The determinant of the bordered Hessian matrix is 12000.

$$H_B = \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

f. If income went up by \$1.00, by how much would utility rise?

Problem 3. Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2.

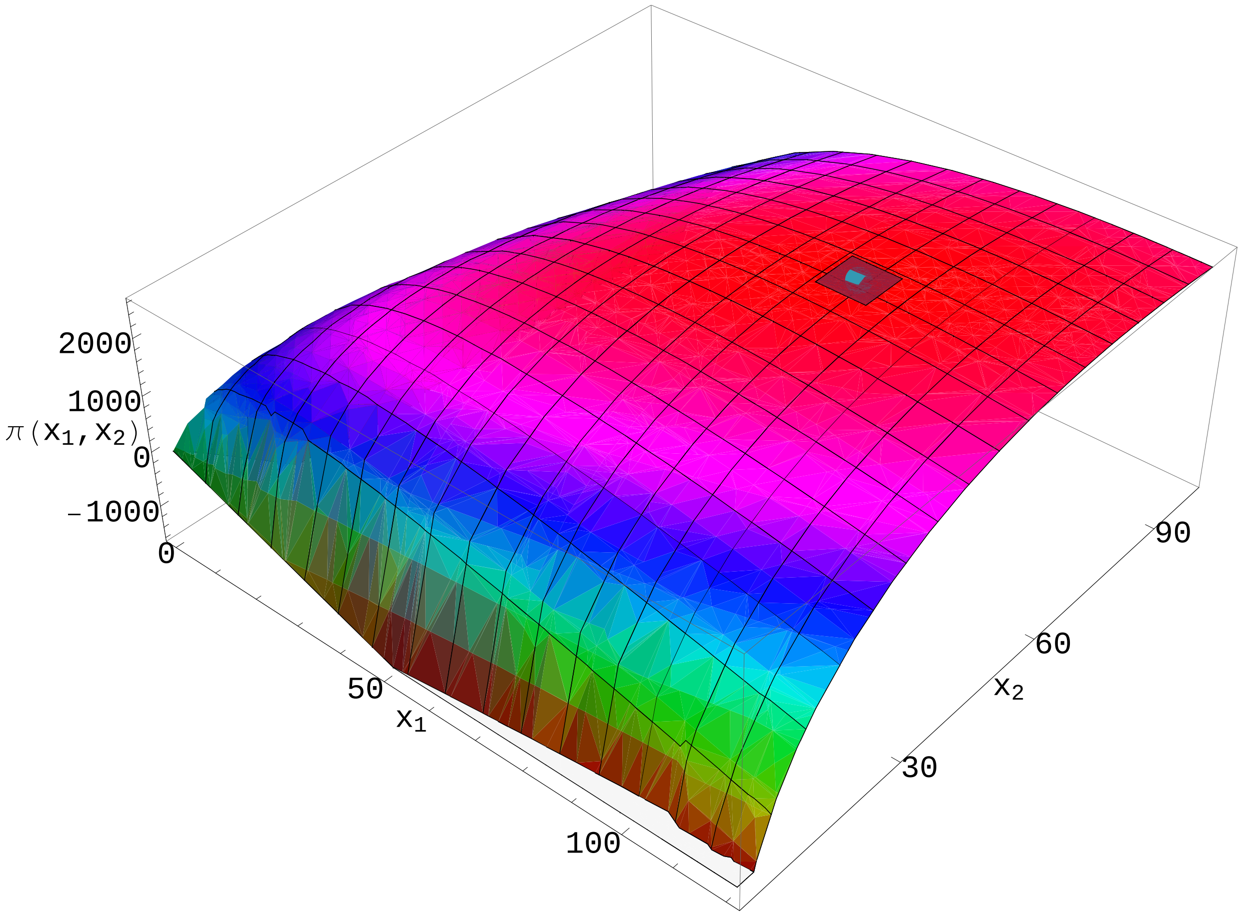
$$
\begin{aligned}
f(x_1, x_2) &= x_1^{1/4} x_2^{1/2} \\
p &= 432 \\
w_1 &= 32, \qquad w_2 \ = \ 81
\end{aligned}
$$

a. Write an equation that represents profit as a function of the two inputs x_1 and x_2. Simplify the expression.

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

FIGURE 4. Maximum Profit



c. Show that the potential profit maximizing levels of x_1 and x_2 are $x_1 = 81, x_2 = 64$.

d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of x_1 and x_2 and then verify that the levels of x_1 and x_2 you found are either maximum, minimum or saddle points.

$$H = \begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{81}{128} \\ \\ \end{vmatrix} =$$

The determinant of the Hessian matrix is $\frac{1}{8}$.

e. What is the optimal level of output?

f. How much does the firm spend on inputs?

g. Show that the marginal value product of x_1 at its optimal value is equal to w_1?

h. Explain in words why the value of the marginal product for each input for this firm is equal to the price of that input at the profit maximizing level of input use for that input.

Problem 4. Consider a firm with a production function given by

$$f(x_1, x_2) = x_1^{1/4} x_2^{1/2}$$

The firm faces prices and a cost constraint given by

$$\begin{aligned} w_1 &= 32 \\ w_2 &= 81 \\ c_0 &= 7776 \end{aligned}$$

Find potential levels of x_1, x_2 and λ to maximize output for this firm given the cost constraint and the stated prices. Verify that these input levels maximize output.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$\mathcal{L}(x_1, x_2, \lambda) \ =$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\dfrac{\partial g(x_1, x_2)}{\partial x_1} =$	$\dfrac{\partial g(x_1, x_2)}{\partial x_2} =$

d. Use the information from 4a and 4b to find critical values for x_1, x_2 and λ. Hint: $x_1 = 81, x_2 = 64, \lambda = \frac{1}{432}$

FIGURE 5. Output Maximization Subject to a Cost Constraint

e. Use the answers from part 4d and the expressions from parts 4a and 4c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum. The determinant of the bordered Hessian is 9.

$$H_B = \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{1728} & \frac{\partial g(x_1, x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

f. How much output can this firm produce given it spends only \$7776?

g. What is the marginal product of x_1 at its optimal value?

h. What is the marginal product of x_2 at its optimal value?

i. Show that the ratio of the marginal products is equal to the input price ratio?

j. Interpret the condition in part i.

Problem 5. Consider a producer with a production function given by

$$f(x_1, x_2) \;=\; x_1^{1/4} x_2^{1/2}$$

The firm faces prices and an output target given by

$$w_1 \;=\; 32, \qquad w_2 \;=\; 81, \qquad y_0 \;=\; 24$$

Find potential levels of x_1 and x_2 to minimize the cost for this producer to reach the target level of output given the stated prices. Verify that these input levels minimize cost.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathfrak{L}(x_1, x_2, \lambda) \;=$$

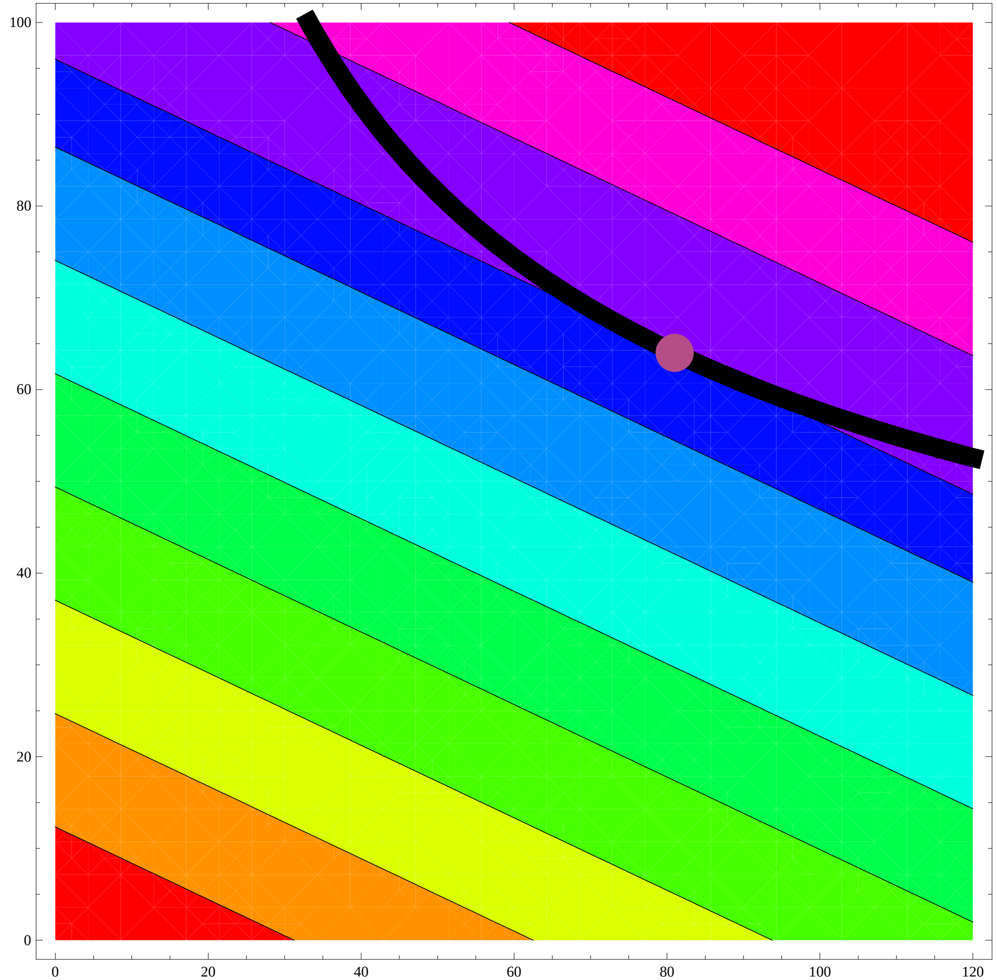
$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

Figure 6. Cost Minimization Subject to an Output Constraint



d. Use the information from 5a and 5b to find critical values for x_1, x_2 and λ.

e. Substitute the values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{1}{48}$.

$$H_B = \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{8}{27} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & \dfrac{\partial g(x_1, x_2)}{\partial x_1} = \\ \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \dfrac{\partial g(x_1, x_2)}{\partial x_2} = \\ \\ \dfrac{\partial g(x_1, x_2)}{\partial x_1} = & \dfrac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

f. How much does this firm spend on inputs?

g. How much output does it produce?

h. What is the marginal product of x_1 at its optimal value?

i. What is the marginal product of x_2 at its optimal value?

j. Show that the ratio of the marginal products is equal to the input price ratio?

k. Interpret the condition in part j.

l. Why did you not need to do any of the work in problem 5 given you had already worked problems 3 and 4?