ECONOMICS 207
SPRING 2008
PROBLEM SET 5
KEY

Problem 1. Solve the following systems of equations for x_1 and x_2 using the method of substitution.

a.

$$\{x_1 = 125,\ x_2 = 32\}$$

$$100x_1^{-2/3}x_2^{3/5} - 32 = 0$$

$$180x_1^{1/3}x_2^{-2/5} - 225 = 0$$

From the second equation,

$$\begin{aligned} 180x_1^{1/3}x_2^{-2/5} - 225 &= 0 \\ \Rightarrow \quad x_1^{1/3}x_2^{-2/5} &= 225/180 = 5/4 \\ \Rightarrow \quad x_1^{1/3} &= 5/4x_2^{2/5} \\ \Rightarrow \quad x_1^{-1/3} &= 4/5x_2^{-2/5} \\ \Rightarrow \quad x_1^{-2/3} &= (4/5)^2x_2^{-4/5} \end{aligned}$$

Substitute $x_1^{-2/3} = (4/5)^2x_2^{-4/5}$ into the first equation.

$$\begin{aligned} 100x_1^{-2/3}x_2^{3/5} - 32 &= 0 \\ \Rightarrow \quad 100\left((4/5)^2x_2^{-4/5}\right)x_2^{3/5} &= 32 \\ \Rightarrow \quad 64x_2^{-1/5} &= 32 \\ \Rightarrow \quad x_2^{-1/5} &= 2^{-1} \\ \Rightarrow \quad x_2 &= \left(2^{-1}\right)^{-5} = 2^5 = 32 \end{aligned}$$

Substitute $x_2 = 32$ into $x_1^{1/3} = 5/4x_2^{2/5}$.

$$\begin{aligned} x_1^{1/3} &= 5/4x_2^{2/5} \\ \Rightarrow \quad x_1^{1/3} &= 5/4 \times 32^{2/5} = 5 \\ \Rightarrow \quad x_1 &= 5^3 = 125 \end{aligned}$$

So the solution is

$$x_1 = 125,\ x_2 = 32$$

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b.

$$\{x_1 = 729,\ x_2 = 81\}$$

$$\begin{aligned} 162x_1^{-5/6}x_2^{1/4} - 2 &= 0 \\ 243x_1^{1/6}x_2^{-3/4} - 27 &= 0 \end{aligned}$$

From the first equation,

$$\begin{aligned} 162x_1^{-5/6}x_2^{1/4} - 2 &= 0 \\ \Rightarrow \quad x_1^{-5/6}x_2^{1/4} &= 2/162 \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{1}{81}x_1^{5/6} \\ \Rightarrow \quad x_2^{-3/4} &= \left(\tfrac{1}{81}x_1^{5/6}\right)^{-3} = 3^{12}x_1^{-5/2} \end{aligned}$$

Substitute $x_2^{-3/4} = 3^{12}x_1^{-5/2}$ into the second equation.

$$\begin{aligned} 243x_1^{1/6}x_2^{-3/4} - 27 &= 0 \\ \Rightarrow \quad 243x_1^{1/6}\left(3^{12}x_1^{-5/2}\right) &= 27 \\ \Rightarrow \quad 3^{17}x_1^{-7/3} &= 3^3 \\ \Rightarrow \quad x_1^{-7/3} &= 3^{-14} \\ \Rightarrow \quad x_1^{1/3} &= 3^2 \\ \Rightarrow \quad x_1 &= 3^6 = 729 \end{aligned}$$

Substitute $x_1 = 729$ into $x_2^{1/4} = \frac{1}{81}x_1^{5/6}$.

$$\begin{aligned} x_2^{1/4} &= \tfrac{1}{81}x_1^{5/6} \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{1}{81} \times 729^{5/6} = 3 \\ \Rightarrow \quad x_2 &= 3^4 = 81 \end{aligned}$$

So the solution is

$$x_1 = 729, \quad x_2 = 81$$

Problem 2. Solve the following systems of equations for x_1 and x_2 first using the method of substitution and then using the method of elimination.

a.

$$\{x_1 = -1,\ x_2 = 2\}$$

$$\begin{aligned} 2x_1 + 2x_2 &= 2 \\ 3x_1 + 4x_2 &= 5 \end{aligned}$$

Method of substitution:

Rearrange the first equation.

$$\begin{aligned} 2x_1 + 2x_2 &= 2 \\ \Rightarrow \quad x_1 &= 1 - x_2 \end{aligned}$$

Substitute $x_1 = 1 - x_2$ into the second equation.

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ \Rightarrow \quad 3(1 - x_2) + 4x_2 &= 5 \\ \Rightarrow \quad x_2 &= 2 \end{aligned}$$

Then $x_1 = 1 - x_2 = -1$.
So the solution is $x_1 = -1,\ x_2 = 2$.

Method of elimination:

Add the first equation multiplied by $-3/2$ to the second equation.

$$\begin{aligned} 3x_1 + 4x_2 + (2x_1 + 2x_2) \times (-3/2) &= 5 + 2 \times (-3/2) \\ \Rightarrow \quad x_2 &= 2 \end{aligned}$$

Add the first equation multiplied by -2 to the second equation.

$$\begin{aligned} 3x_1 + 4x_2 + (2x_1 + 2x_2) \times (-2) &= 5 + 2 \times (-2) \\ \Rightarrow \quad -x_1 &= 1 \\ \Rightarrow \quad x_1 &= -1 \end{aligned}$$

So the solution is $x_1 = -1,\ x_2 = 2$.

b.

$$\{x_1 = 1,\ x_2 = 3\}$$

$$7x_1 + 5x_2 = 22$$
$$3x_1 + 2x_2 = 9$$

Method of substitution:

Rearrange the first equation.

$$\begin{aligned} 7x_1 + 5x_2 &= 22 \\ \Rightarrow \quad 7x_1 &= 22 - 5x_2 \\ \Rightarrow \quad x_1 &= (22 - 5x_2)/7 \end{aligned}$$

Substitute $x_1 = (22 - 5x_2)/7$ into the second equation.

$$\begin{aligned} 3x_1 + 2x_2 &= 9 \\ \Rightarrow \quad 3\left((22 - 5x_2)/7\right) + 2x_2 &= 9 \\ \Rightarrow \quad 66/7 - x_2/7 &= 9 \\ \Rightarrow \quad 66/7 - x_2/7 &= 9 \\ \Rightarrow \quad -x_2/7 &= 9 - 66/7 = -3/7 \\ \Rightarrow \quad x_2 &= 3 \end{aligned}$$

Then $x_1 = (22 - 5 \times 3)/7 = 1$.
So the solution is $x_1 = 1,\ x_2 = 3$.

Method of elimination:

Add the first equation multiplied by $-2/5$ to the second equation.

$$\begin{aligned} 3x_1 + 2x_2 + (7x_1 + 5x_2) \times (-2/5) &= 9 + 22 \times (-2/5) \\ \Rightarrow \quad x_1/5 &= 1/5 \\ \Rightarrow \quad x_1 &= 1 \end{aligned}$$

Add the first equation multiplied by $-3/7$ to the second equation.

$$\begin{aligned} 3x_1 + 2x_2 + (7x_1 + 5x_2) \times (-3/7) &= 9 + 22 \times (-3/7) \\ \Rightarrow \quad -x_2/7 &= -3/7 \\ \Rightarrow \quad x_2 &= 3 \end{aligned}$$

So the solution is $x_1 = 3,\ x_2 = 3$.

Problem 3. Solve the following system of equations for x_1, x_2, and x_3 first using the method of substitution and then using the method of elimination.

$$\{x_1 = 3, x_2 = 1, x_3 = -2\}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 5x_1 + 6x_2 + 4x_3 &= 13 \\ 3x_1 - 2x_2 + 7x_3 &= -7 \end{aligned}$$

Method of substitution:

Rearrange the first equation.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ \Rightarrow \quad x_1 &= 2 - x_2 - x_3 \end{aligned}$$

Substitute $x_1 = 2 - x_2 - x_3$ into the second equation.

$$\begin{aligned} 5x_1 + 6x_2 + 4x_3 &= 13 \\ \Rightarrow \quad 5(2 - x_2 - x_3) + 6x_2 + 4x_3 &= 13 \\ \Rightarrow \quad x_2 - x_3 &= 3 \\ \Rightarrow \quad x_2 &= 3 + x_3 \qquad (1) \end{aligned}$$

Substitute $x_1 = 2 - x_2 - x_3$ into the third equation.

$$\begin{aligned} 3x_1 - 2x_2 + 7x_3 &= -7 \\ \Rightarrow \quad 3(2 - x_2 - x_3) - 2x_2 + 7x_3 &= -7 \\ \Rightarrow \quad -5x_2 + 4x_3 &= -13 \qquad (2) \end{aligned}$$

Substitute equation (1), i.e., $x_2 = 3 + x_3$, into equation (2).

$$\begin{aligned} -5x_2 + 4x_3 &= -13 \\ \Rightarrow \quad -5(3 + x_3) + 4x_3 &= -13 \\ \Rightarrow \quad -x_3 &= -13 + 15 \\ \Rightarrow \quad x_3 &= -2 \end{aligned}$$

Then $x_2 = 3 + x_3 = 3 - 2 = 1$; $x_1 = 2 - x_2 - x_3 = 2 - 1 - (-2) = 3$.

So the solution is $x_1 = 3,\ x_2 = 1,\ x_3 = -2$.

Method of elimination:

Add the first equation multiplied by -5 to the second equation.

$$5x_1 + 6x_2 + 4x_3 + (x_1 + x_2 + x_3) \times (-5) = 13 + 2 \times (-5)$$
$$\Rightarrow \qquad x_2 - x_3 = 3 \tag{3}$$

Add the first equation multiplied by -3 to the third equation.

$$3x_1 - 2x_2 + 7x_3 + (x_1 + x_2 + x_3) \times (-3) = -7 + 2 \times (-3)$$
$$\Rightarrow \qquad -5x_2 + 4x_3 = -13 \tag{4}$$

Add equation (3) multiplied by 5 to equation (4).

$$-5x_2 + 4x_3 + (x_2 - x_3) \times 5 = -13 + 3 \times 5$$
$$\Rightarrow \qquad -x_3 = 2$$
$$\Rightarrow \qquad x_3 = -2$$

Add equation (3) multiplied by 4 to equation (4).

$$-5x_2 + 4x_3 + (x_2 - x_3) \times 4 = -13 + 3 \times 4$$
$$\Rightarrow \qquad -x_2 = -1$$
$$\Rightarrow \qquad x_2 = 1$$

Add $x_2 = 1$ multiplied by -1 to the first equation; and add $x_3 = -2$ multiplied by -1 to the first equation.

$$x_1 + x_2 + x_3 - x_2 - x_3 = 2 - 1 - 2 \times (-1)$$
$$\Rightarrow \qquad x_1 = 3$$

So the solution is $x_1 = 3,\ x_2 = 1,\ x_3 = -2$.

Problem 4. Find the derivatives of each of the following functions with respect to x.

a. $y = 4x^2 - 3x^3$

$$\frac{dy}{dx} = 8x - 9x^2$$

b. $f(x) = 4e^x + \frac{1}{2}x^2$

$$\frac{dy}{dx} = 4e^x + x$$

c. $f(x) = 5x^3 - 2\log[x]$

$$f'(x) = 15x^2 - \frac{2}{x}$$

d. $f(x) = 2x^5 - 5x^4 + 4^x$

$$f'(x) = 10x^4 - 20x^3 + 4^x \log(4)$$

e. $f(x) = 15x^{1/2} + 9x^{2/3} - 18x^{1/6}$

$$f'(x) = \frac{15}{2}x^{-1/2} + 6x^{-1/3} - 3x^{-5/6}$$

f. $f(x) = 5x^3 - 2xe^x$

$$\begin{aligned} f'(x) &= 15x^2 - (2e^x + 2xe^x) \\ &= 15x^2 - 2e^x - 2xe^x \end{aligned}$$

g. $f(x) = 3x^5 \log[x]$

$$f'(x) = 15x^2 \log[x] + 3x^5 \frac{1}{x}$$
$$= 15x^2 \log[x] + 3x^4$$

h. $f(x) = (4x - 2)^2$ Find in two different ways.

Use chain rule

$$f'(x) = 2(4x - 2)$$
$$= 32x - 16$$

Find the derivative after expanding

$$f(x) = 16x^2 - 16x + 4$$

$$f'(x) = 32x - 16$$

i. $f(x) = \frac{5x^3}{2x^2+4x}$

$$f(x) = \frac{5x^2}{2x+4}$$

$$f'(x) = \frac{10x(2x+4) - 5x^2 \times 2}{(2x+4)^2} = \frac{20x^2 + 40x - 10x^2}{4(x+2)^2}$$
$$= \frac{5x^2 + 20x}{2(x+2)^2}$$

j. $f(x) = \frac{6x^2-x-2}{2x^2-7x-4}$

$$f(x) = \frac{(2x+1)(3x-2)}{(2x+1)(x-4)} = \frac{3x-2}{x-4}$$

$$f'(x) = \frac{3(x-4) - (3x-2)}{(x-4)^2}$$
$$= \frac{-10}{(x-4)^2}$$

Problem 5. Find the derivatives of each of the following functions with respect to x.

a. $f(x) = (2x^2 + 3x)^3$

$$f'(x) = 3(2x^2+3x)^2(4x+3)$$

b. $f(x) = (6x - 5)(2x + 3)$ Show two ways.

Method 1:

$$f'(x) = 6(2x+3) + (6x-5)2$$
$$= 24x + 8$$

Method 2:

$$f(x) = 6x(2x+3) - 5(2x+3) = 12x^2 + 8x - 3$$

$$f'(x) = 24x + 8$$

c. $f(x) = 3xe^{2x^2+3x}$

$$f(x) = 3e^{2x^2+3x} + 3xe^{2x^2+3x}(4x+3)$$
$$= 3e^{2x^2+3x}(1 + 3x + 4x^2)$$

d. $f(x) = 3x^2 \log[2x^2 + 3x]$

$$f'(x) = 6x\log[2x^2+3x] + 3x^2 \frac{1}{2x^2+3x}(4x+3)$$
$$= 6x\log[2x^2+3x] + \frac{12x^2+9x}{2x+3}$$

e. $f(x) \ = \ 2x^3 \log[2x^2 + 4x]$

$$\begin{aligned} f'(x) &= 6x^2 \log[2x^2 + 4x] + 2x^3 \frac{1}{2x^2 + 4x}(4x + 4) \\ &= 6x^2 \log[2x^2 + 4x] + \frac{x^3(4x + 4)}{x^2 + 2x} \end{aligned}$$

f. $f(x) \ = \ e^{(x^3 - 4x)^2}$

$$\begin{aligned} f'(x) &= e^{(x^3 - 4x)^2} \left(2(x^3 - 4x)(3x^2 - 4)\right) \\ &= 2(x^3 - 4x)(3x^2 - 4)e^{(x^3 - 4x)^2} \end{aligned}$$

g. $f(x) \ = \ \frac{2xe^{3x}}{\log(x)}$

$$\begin{aligned} f'(x) &= \frac{\left(2e^{3x} + 2xe^{3x} \times 3\right) \log(x) - 2xe^{3x} \frac{1}{x}}{[\log(x)]^2} \\ &= \frac{\left(2e^{3x} + 6xe^{3x}\right) \log(x) - 2e^{3x}}{[\log(x)]^2} \\ &= \frac{2e^{3x} + 6xe^{3x}}{\log(x)} - \frac{2e^{3x}}{[\log(x)]^2} \end{aligned}$$

h. $f(x) = \frac{3xe^{2x}}{4x^{1/2}+2}$

$$f'(x) = \frac{\left(3e^{2x} + 3xe^{2x} \times 2\right)(4x^{1/2}+2) - 3xe^{2x} \cdot 2x^{-1/2}}{(4x^{1/2}+2)^2}$$

$$= \frac{\left(3e^{2x} + 6xe^{2x}\right)(4x^{1/2}+2) - 6x^{1/2}e^{2x}}{(4x^{1/2}+2)^2}$$

$$= \frac{3e^{2x} + 6xe^{2x}}{4x^{1/2}+2} - \frac{6x^{1/2}e^{2x}}{(4x^{1/2}+2)^2}$$

i. $f(x) = 3xe^{2x^2}$

$$f'(x) = 3e^{2x^2} + 3xe^{2x^2} \cdot (4x)$$

$$= 3e^{2x^2} + 12x^2e^{2x^2}$$

j. $f(x) = \frac{3xe^{2x^2}}{x^2+2\log[x]}$

$$f'(x) = \frac{\left(3e^{2x^2} + 12x^2e^{2x^2}\right)(x^2 + 2\log[x]) - 3xe^{2x^2}\left(2x + \frac{2}{x}\right)}{\left(x^2 + 2\log[x]\right)^2}$$

$$= \frac{3e^{2x^2} + 12x^2e^{2x^2}}{x^2 + 2\log[x]} - \frac{6x^2e^{2x^2} + 6e^{2x^2}}{\left(x^2 + 2\log[x]\right)^2}$$

Problem 6. For each of the following, take the derivative with respect to x_1, set the derivative equal to zero and solve the resulting equation for x_1.

a. $\{x_1 = 243\}$

$$f(x_1) = 3645x_1^{1/5} - 9x_1$$

$$f'(x_1) = \frac{3645}{5}x_1^{-4/5} - 9$$

$$= 729x_1^{-4/5} - 9$$

Set $f'(x_1) = 0$ and then solve the equation.

$$\begin{aligned}
& f'(x_1) = 729x_1^{-4/5} - 9 &=& \ 0 \\
\Rightarrow & \quad x_1^{-4/5} &=& \ 9/729 = 3^{-4} \\
\Rightarrow & \quad x_1^{1/5} &=& \ 3 \\
\Rightarrow & \quad x_1 &=& \ 3^5 = 243
\end{aligned}$$

b. $\{x_1 = 216\}$

$$f(x_1) = 540x_1^{1/3} - 5x_1$$

$$f'(x_1) = 180x_1^{-2/3} - 5$$

Set $f'(x_1) = 0$ and then solve the equation.

$$\begin{aligned}
& f'(x_1) = 180x_1^{-2/3} - 5 &=& \ 0 \\
\Rightarrow & \quad 180x_1^{-2/3} &=& \ 5 \\
\Rightarrow & \quad x_1^{-2/3} &=& \ 1/36 = 6^{-2} \\
\Rightarrow & \quad x_1^{1/3} &=& \ 6 \\
\Rightarrow & \quad x_1 &=& \ 6^3 = 216
\end{aligned}$$

c. $\{x_1 = 256\}$
$f(x_1) = 768x_1^{1/4} - 3x_1$

$$f'(x_1) = \frac{768}{4}x_1^{-3/4} - 3$$
$$= 192x_1^{-3/4} - 3$$

Set $f'(x_1) = 0$ and then solve the equation.

$$\begin{aligned} & f'(x_1) = 192x_1^{-3/4} - 3 &=& \ 0 \\ \Rightarrow & \quad 192x_1^{-3/4} &=& \ 3 \\ \Rightarrow & \quad x_1^{-3/4} &=& \ 3/192 = 4^{-3} \\ \Rightarrow & \quad x_1^{1/4} &=& \ 4 \\ \Rightarrow & \quad x_1 &=& \ 4^4 = 256 \end{aligned}$$

Problem 7. Do the following problems from the book.

a. Section 6.2
   1) 1 (The comparison is with equation 6 in the text, not problem 6)
   2) 3
   3) 5
b. Section 6.3
   1) 1
c. Section 6.4
   1) 1
   2) 3
   3) 7
d. Section 6.5
   1) 1a
   2) 1b
   3) 1c
   4) 1d
e. Section 6.6
   1) 3a
   2) 3b
   3) 3c
   4) 3d
   5) 3e
   6) 3f
   7) 3g
   8) 3h