ECONOMICS 207
SPRING 2008
PROBLEM SET 6
KEY

Problem 1. Find the derivatives of each of the following functions with respect to x.

a. $f(x) \ = \ 4x^2e^{2x^3+4x}$

$$\begin{aligned} f'(x) &= 8xe^{2x^3+4x} + 4x^2e^{2x^3+4x}(6x^2+4) \\ &= 8xe^{2x^3+4x} + 8x^2(3x^2+2)e^{2x^3+4x} \end{aligned}$$

b. $f(x) \ = \ e^{(2x^3-4x^2)^4}$

$$\begin{aligned} f'(x) &= e^{(2x^3-4x^2)^4} \cdot \Big(4(2x^3-4x^2)^3(6x^2-8x)\Big) \\ &= 64(x^3-2x^2)^3(3x^2-4x)e^{(2x^3-4x^2)^4} \\ &= 64x^7(x-2)^3(3x-4)e^{(2x^3-4x^2)^4} \end{aligned}$$

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c. $f(x) = \frac{(2x^2+4x)(x^2+2x)^3}{4x^2e^{2x^3+2x}}$

$$f(x) = \frac{(x^2+2x)(x^2+2x)^3}{2x^2e^{2x^3+2x}} = \frac{(x^2+2x)^4}{2x^2e^{2x^3+2x}} = \frac{x^4(x+2)^4}{2x^2e^{2x^3+2x}} = \frac{x^2(x+2)^4}{2e^{2x^3+2x}}$$

$$= \frac{1}{2}x^2(x+2)^4e^{-2x^3-2x}$$

$$f'(x) = \frac{1}{2}\left(2x(x+2)^4 + x^2 \cdot (4(x+2)^3)\right)e^{-2x^3-2x} + \frac{1}{2}x^2(x+2)^4e^{-2x^3-2x}(-6x^2-2)$$

$$= e^{-2x^3-2x}(x+2)^3\left(x(x+2) + 2x^2 - x^2(x+2)(3x^2+1)\right)$$

$$= e^{-2x^3-2x}(x+2)^3\left(3x^2+2x-(3x^5+6x^4+x^3+2x^2)\right)$$

$$= -e^{-2x^3-2x}(x+2)^3\left(3x^5+6x^4+x^3-x^2-2x\right)$$

$$= -e^{-2x^3-2x}x(x+2)^3\left(3x^4+6x^3+x^2-x-2\right)$$

d. $f(x) = -1120x + 10(175x + 10x^2 - x^3)$

$$f'(x) = -1120 + 10(175 + 20x - 3x^2)$$

$$= 630 + 200x - 30x^2$$

Problem 2. Find the second derivative of each of the following functions with respect to x

a. $y = 22x^3 - 10x^2 - 10x^{1/2}$

$$\frac{dy}{dx} = 66x^2 - 20x - 5x^{-1/2}$$

$$\frac{d^2y}{dx^2} = 132x - 20 + \frac{5}{2}x^{-3/2}$$

b. $f(x) = (2x^3 + 4x^2 - 10x)^3$

$$\begin{aligned} f'(x) &= 3(2x^3 + 4x^2 - 10x)^2(6x^2 + 8x - 10) \\ &= 24(x^3 + 2x^2 - 5x)^2(3x^2 + 4x - 5) \end{aligned}$$

$$\begin{aligned} f''(x) &= 48(x^3 + 2x^2 - 5x)(3x^2 + 4x - 5)^2 + 24(x^3 + 2x^2 - 5x)^2(6x + 4) \\ &= 48(x^3 + 2x^2 - 5x)(3x^2 + 4x - 5)^2 + 48(x^3 + 2x^2 - 5x)^2(3x + 2) \end{aligned}$$

c. $f(x) = 4x^2 e^{2x^3+3x}$

$$\begin{aligned}
f'(x) &= 8xe^{2x^3+3x} + 4x^2 e^{2x^3+3x}(6x^2+3) \\
&= 8xe^{2x^3+3x} + 12x^2 e^{2x^3+3x}(2x^2+1) \\
&= e^{2x^3+3x}\left(8x + 12x^2(2x^2+1)\right) \\
&= e^{2x^3+3x}\left(8x + 12x^2 + 24x^4\right)
\end{aligned}$$

$$\begin{aligned}
f''(x) &= e^{2x^3+3x}(6x^2+3)\left(8x + 12x^2 + 24x^4\right) + e^{2x^3+3x}\left(8 + 24x + 96x^3\right) \\
&= 12e^{2x^3+3x}(2x^2+1)\left(2x + 3x^2 + 6x^4\right) + 8e^{2x^3+3x}\left(1 + 3x + 12x^3\right) \\
&= 4e^{2x^3+3x}\left(2 + 12x + 9x^2 + 36x^3 + 36x^4 + 36x^6\right)
\end{aligned}$$

d. $f(x) = 10(175x + 10x^2 - x^3) - 1120x$

$$f'(x) = 10\left(175 + 20x - 3x^2\right) - 1120$$

$$\begin{aligned}
f''(x) &= 10\,(20 - 6x) \\
&= 200 - 60x
\end{aligned}$$

Problem 3. In the following problems you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input. For each problem

1) Write down an equation that represents profit for the firm.
2) Maximize this function by taking its derivative with respect to the variable input x and setting it equal to zero.
3) Solve for the profit maximizing level of x.
4) Find the optimal level of output.
5) What is revenue at the optimal level of output?
6) What is cost at the optimal level of the input x?

a.

$$\begin{aligned} output\ price &= p = 10 \\ input\ price &= w = 1120 \end{aligned}$$

$$y = output = f(x) = 175x + 10x^2 - x^3$$

For this example, profit is given by

$$\begin{aligned} Profit &= 10(175x + 10x^2 - x^3) - 1120x \\ &= 1750x + 100x^2 - 10x^3 - 1120x \\ &= 630x + 100x^2 - 10x^3 \end{aligned}$$

For this example, the derivative of profit is given by

$$\begin{aligned} Profit &= 630x + 100x^2 - 10x^3 \\ \frac{d\,Profit}{dx} &= 630 + 200x - 30x^2 \\ \frac{d^2\,Profit}{dx^2} &= 200 - 60x \end{aligned}$$

Now set the derivative of profit equal to zero and solve for x as follows.

$$\begin{aligned} \frac{d\,Profit}{dx} = 630 + 200x - 30x^2 &= 0 \\ \Rightarrow 3x^2 - 20x - 63 &= 0 \\ \Rightarrow (3x+7)(x-9) &= 0 \\ \Rightarrow x = -\frac{7}{3} \text{ or } x &= 9 \end{aligned}$$

At x = -$\frac{7}{3}$, $\frac{d^2\,Profit}{dx^2} = 200 - 60(-7/3) = 200 + 140 = 340 > 0$.
At x = 9, $\frac{d^2\,Profit}{dx^2} = 200 - 60(9) = 800 - 540 = -340 < 0$. The optimal x =9.

The optimal output is given by

$$\begin{aligned} y &= 175x + 10x^2 - x^3 \\ &= (175)(9) + 10(9^2) - 9^3 \\ &= 1575 + (10)(81) - 729 \\ &= 1575 + 810 - 729 = 6000 - 3375 = 1656 \end{aligned}$$

Revenue is equal to the price of output multiplied by the output price.

$$revenue = py = (10)(1656) = 16560$$

Cost of production is equal to the level of the input used multiplied by the input price.

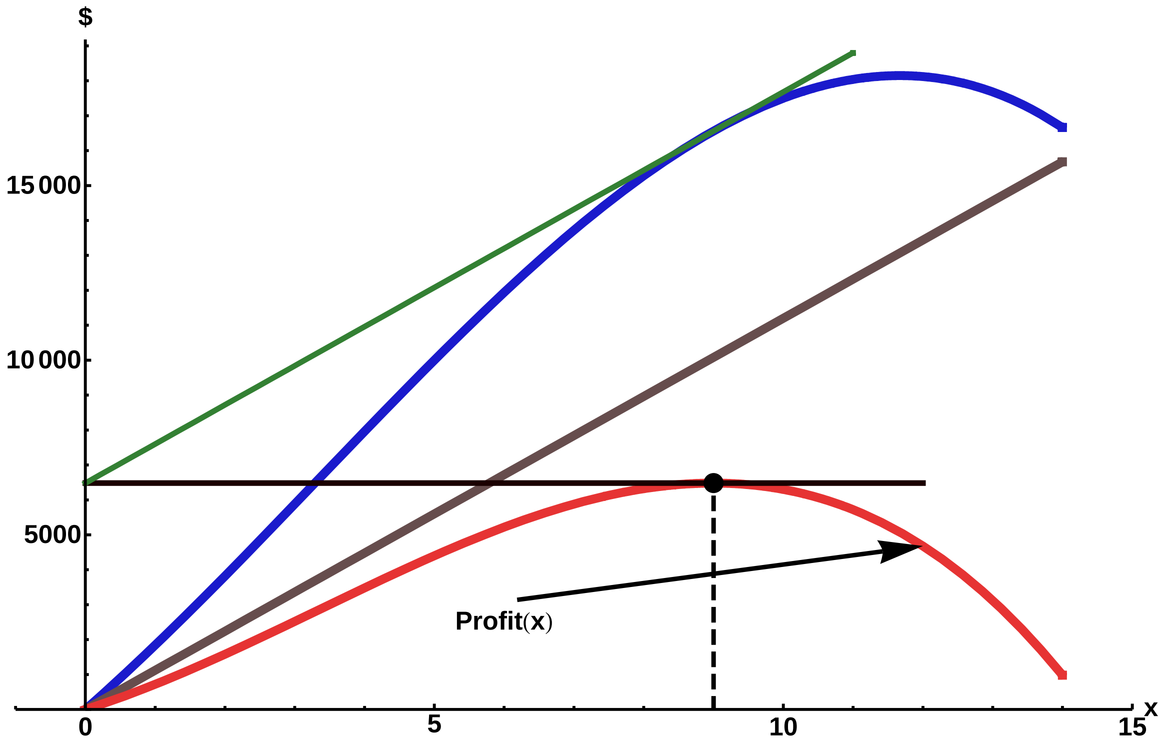
$$cost = wx = (1120)(9) = 10080$$

Profit is equal to revenue minus cost.

$$profit = revenue - cost = 16560 - 10080 = 6480.$$

A graph of revenue, cost and profit is given in figure 1

FIGURE 1. Revenue, Cost and Profit



b.

$$\begin{aligned} output\ price &= p = 3 \\ input\ price &= w = 3588 \\ y = output &= f(x) = 196x + 95x^2 - 2x^3 \end{aligned}$$

For this firm, profit is given by

$$\begin{aligned} Profit &= 3(196x + 95x^2 - 2x^3) - 3588x \\ &= -3000x + 190x^2 - 6x^3 \end{aligned}$$

Then the derivative of profit is given by

$$Profit = -3000x + 285x^2 - 6x^3$$

$$\frac{d\,Profit}{d\,x} = -3000 + 570x - 18x^2$$

$$\frac{d^2\,Profit}{d\,x^2} = 570 - 36x$$

Now set the derivative of profit equal to zero and solve for x as follows.

$$\begin{aligned} \frac{d\,Profit}{d\,x} = -3000 + 570x - 18x^2 &= 0 \\ \Rightarrow \qquad -6(-25 + x)(-20 + 3x) &= 0 \\ \Rightarrow \qquad x = 25 \ \text{ or } \ x &= 20/3 \end{aligned}$$

At $x = 20/3$, $\frac{d^2\,Profit}{d\,x^2} = 570 - 36 \times \frac{20}{3} = 330 > 0$.
At $x = 25$, $\frac{d^2\,Profit}{d\,x^2} = 570 - 36 \times 25 = -330 < 0$. The optimal $x = 25$.
The optimal output is given by

$$\begin{aligned} y &= 196x + 95x^2 - 2x^3 \\ &= 196 \times 25 + 95 \times 25^2 - 2 \times 25^3 \\ &= 33025 \end{aligned}$$

Revenue is equal to the price of output multiplied by the output.

$$Revenue = py = 3 \times 33025 = 99075$$

Cost of production is equal to level of the input used multiplied by the input price.

$$Cost = xw = 25 \times 3588 = 89700$$

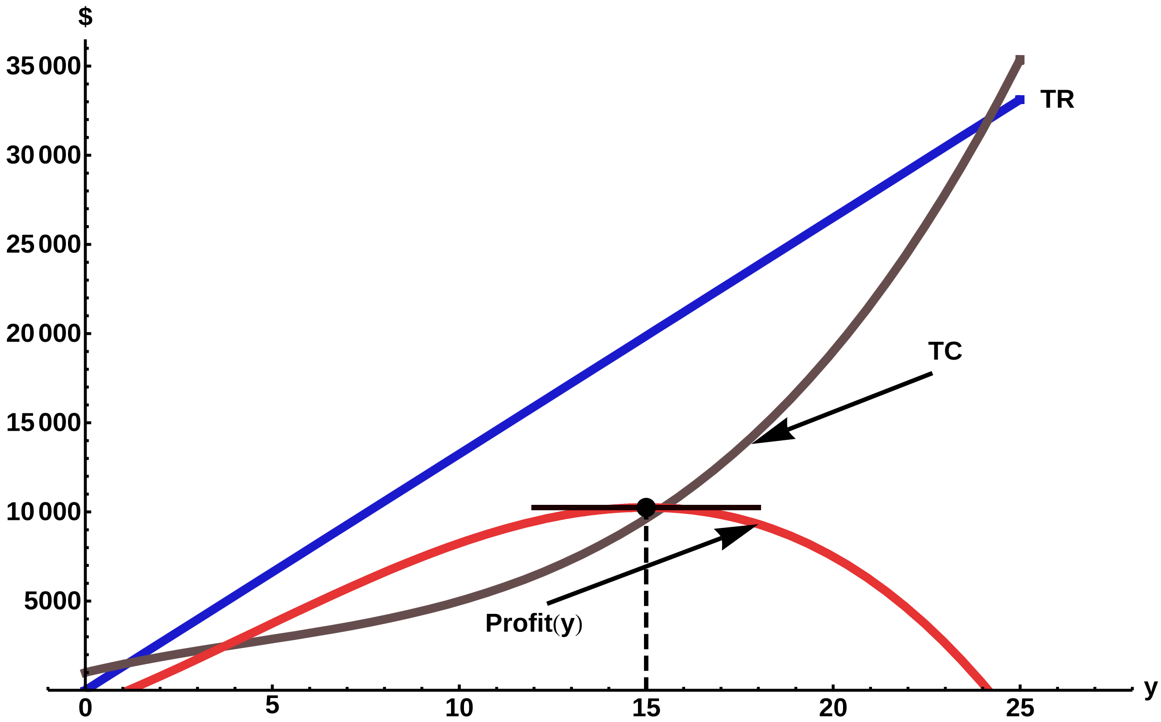
Profit is equal to revenue minus cost.

$$Profit = revenue - cost = 99075 - 89700 = 9375$$

Problem 4. In this problem (Problem 4), you will be given the price (p) for a competitive firm and the total cost function (TC[y]) for the firm. TC[y] specifies cost as a function of output which is denoted by y. Total revenue is given by TR[y] = py. Marginal cost (MC) is the derivative of the cost function with respect to output. At the optimal level of output, marginal cost will be equal to price.

A graph of total revenue (TR), total cost (TC) and profit for such a firm is given in figure 2. Maximum profit is at the point where total revenue is at the highest level above total cost.

FIGURE 2. Revenue, Cost and Profit



For each problem write down the equation for profit. Then find the profit maximizing level of output.

a.

$$\begin{aligned} price &= p &= 1325 \\ Total\ Cost &= TC(y) &= 1000 + 500y - 40y^2 + 3y^3 \end{aligned}$$

The profit is given by

$$\begin{aligned} Profit &= py - TC(y) \\ &= 1325y - \left(1000 + 500y - 40y^2 + 3y^3\right) \\ &= -3y^3 + 40y^2 + 825y - 1000 \end{aligned}$$

Then the derivatives of profit are given by

$$\frac{d\,Profit}{d\,y} = -9y^2 + 80y + 825 \tag{1}$$

$$\frac{d^2\,Profit}{d\,y^2} = -18y + 80 \tag{2}$$

Set the first derivative, equation (1), to zero.

$$\begin{aligned} & -9y^2 + 80y + 825 &=& \; 0 \\ \Rightarrow \quad & -(y-15)(9y+55) &=& \; 0 \\ \Rightarrow \quad & y = 15 &\text{or}& \; y = -55/9 \end{aligned}$$

At $y = -55/9$, $\dfrac{d^2\,Profit}{d\,y^2} = -18 \times (-55/9) + 80 = 190$.

At $y = 15$, $\dfrac{d^2\,Profit}{d\,y^2} = -18 \times 15 + 80 = -190 < 0$. The optimal $y = 15$.

b.

$$\begin{aligned} p &= 1000 \\ TC &= 500 + 1000y - 60y^2 + 2y^3 \end{aligned}$$

The profit is given by

$$\begin{aligned} Profit &= py - TC(y) \\ &= 1000y - \left(500 + 1000y - 60y^2 + 2y^3\right) \\ &= -500 + 60y^2 - 2y^3 \end{aligned}$$

Then the derivatives of profit are given by

$$\frac{d\,Profit}{d\,y} = -6y^2 + 120y \tag{3}$$

$$\frac{d^2\,Profit}{d\,y^2} = -12y + 120 \tag{4}$$

Set the first derivative, equation (3), to zero.

$$\begin{aligned} -6y^2 + 120y &= 0 \\ \Rightarrow \quad -6y(y-20) &= 0 \\ \Rightarrow \quad y = 20 \quad &\text{or} \quad y = 0 \end{aligned}$$

At $y = 0$, $\dfrac{d^2\,Profit}{d\,y^2} = -12 \times 0 + 120 = 120 > 0$.

At $y = 20$, $\dfrac{d^2\,Profit}{d\,y^2} = -12 \times 20 + 120 = -120 < 0$. The optimal $y = 20$.

Problem 5. Solve the following system of equations.

$$288x_1^{-2/5}x_2^{1/4} - 64 = 0$$
$$120x_1^{3/5}x_2^{-3/4} - 405 = 0$$

The x values that solve the system are $x_1 = 243$ and $x_2 = 16$.

Rearrange the first equation.

$$\begin{aligned} 288x_1^{-2/5}x_2^{1/4} - 64 &= 0 \\ \Rightarrow \quad x_1^{-2/5}x_2^{1/4} &= 64/288 = 2/9 \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{2}{9}x_1^{2/5} \\ \Rightarrow \quad x_2^{-1/4} &= \tfrac{9}{2}x_1^{-2/5} \\ \Rightarrow \quad x_2^{-3/4} &= \left(\tfrac{9}{2}\right)^3 x_1^{-6/5} \end{aligned}$$

Substitute $x_2^{-3/4} = \left(\frac{9}{2}\right)^3 x_1^{-6/5}$ into the second equation.

$$\begin{aligned} 120x_1^{3/5}x_2^{-3/4} - 405 &= 0 \\ \Rightarrow \quad 120x_1^{3/5}\left(\left(\tfrac{9}{2}\right)^3 x_1^{-6/5}\right) - 405 &= 0 \\ \Rightarrow \quad x_1^{-3/5} &= 3^{-3} \\ \Rightarrow \quad x_1^{1/5} &= 3 \\ \Rightarrow \quad x_1 &= 3^5 = 243 \end{aligned}$$

Substitute $x_1 = 243$ into $x_2^{1/4} = \frac{2}{9}x_1^{2/5}$.

$$\begin{aligned} x_2^{1/4} &= \tfrac{2}{9}x_1^{2/5} \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{2}{9}243^{2/5} = 2 \\ \Rightarrow \quad x_2 &= 2^4 = 16 \end{aligned}$$

So the solution is $x_1 = 243,\ x_2 = 16$.

Problem 6. For each of the following, find the points where f has critical values. Check to see whether these are maximum or minimum point.

a. $\{x = 80, x = ?\}$

$$f(x) = 120x^2 - x^3$$

The derivatives of $f(x)$ are given by

$$f'(x) = 240x - 3x^2 \tag{5}$$

$$f''(x) = 240 - 6x \tag{6}$$

Set the first derivative, equation (5), to zero.

$$\begin{aligned} f'(x) = 240x - 3x^2 &= 0 \\ \Rightarrow \qquad 3x(80 - x) &= 0 \\ \Rightarrow \qquad x = 0 \qquad \text{or} \qquad x = 80 \end{aligned}$$

Check the second derivatives, equation (6), for every critical points respectively.

$$f''(0) = 240 - 6 \times 0 = 240 > 0$$

$$f''(80) = 240 - 6 \times 80 = -240 < 0$$

So $x = 0$ is a relative minimum point; $x = 80$ is a relative maximum point.

b. $\{x = 5, x = ?\}$

$$f(x) = -2x^3 + 240x^2 - 2250x$$

The derivatives of $f(x)$ are given by

$$f'(x) = -2250 + 480x - 6x^2 \tag{7}$$

$$f''(x) = 480 - 12x \tag{8}$$

Set the first derivative, equation (7), to zero.

$$\begin{aligned} f'(x) = -2250 + 480x - 6x^2 &= 0 \\ \Rightarrow \qquad -6(x - 5)(x - 75) &= 0 \\ \Rightarrow \qquad x = 5 \qquad \text{or} \qquad x = 75 \end{aligned}$$

Check the second derivatives, equation (8), for every critical points respectively.

$$f''(5) = 480 - 12 \times 5 = 420 > 0$$

$$f''(75) = 480 - 12 \times 75 = -420 < 0$$

So $x = 5$ is a relative minimum point; $x = 75$ is a relative maximum point.

c. $f(x) = -2x^3 + 38x^2 + 400x$

The derivatives of $f(x)$ are given by

$$f'(x) = 400 + 76x - 6x^2 \tag{9}$$

$$f''(x) = 76 - 12x \tag{10}$$

Set the first derivative, equation (9), to zero.

$$\begin{aligned} f'(x) = 400 + 76x - 6x^2 &= 0 \\ \Rightarrow \qquad -2(x+4)(3x-50) &= 0 \\ \Rightarrow \qquad x = -4 \qquad \text{or} \qquad & x = 50/3 \end{aligned}$$

Check the second derivatives, equation (10), for every critical points respectively.

$$f''(-4) = 76 - 12 \times (-4) = 124 > 0$$

$$f''(50/3) = 76 - 12 \times (50/3) = -124 < 0$$

So $x = -4$ is a relative minimum point; $x = 50/3$ is a relative maximum point.

d. $f(x) = -4x^3 + 76x^2 - 220x$

The derivatives of $f(x)$ are given by

$$f'(x) = -220 + 152x - 12x^2 \tag{11}$$

$$f''(x) = 152 - 24x \tag{12}$$

Set the first derivative, equation (11), to zero.

$$\begin{aligned} f'(x) = -220 + 152x - 12x^2 &= 0 \\ \Rightarrow \qquad -4(x-11)(3x-5) &= 0 \\ \Rightarrow \qquad x = 5/3 \qquad \text{or} \qquad & x = 11 \end{aligned}$$

Check the second derivatives, equation (12), for every critical points respectively.

$$f''(5/3) = 152 - 24 \times (5/3) = 112 > 0$$

$$f''(11) = 152 - 24 \times 11 = -112 < 0$$

So $x = 5/3$ is a relative minimum point; $x = 11$ is a relative maximum point.

e. $f(x) = 15x^{3/5} - x$

The derivatives of $f(x)$ are given by

$$f'(x) = 9x^{-2/5} - 1 \tag{13}$$

$$f''(x) = \frac{-18}{5}x^{-7/5} \tag{14}$$

Set the first derivative, equation (13), to zero.

$$\begin{aligned} f'(x) = 9x^{-2/5} - 1 &= 0 \\ \Rightarrow \qquad x^{-2/5} &= 3^{-2} \\ \Rightarrow \qquad x^{1/5} &= 3 \\ \Rightarrow \qquad x &= 243 \end{aligned}$$

Check the second derivative, equation (14).

$$f''(243) = \left(\frac{-18}{5}\right) \times 243^{-7/5} < 0$$

So $x = 243$ is a relative maximum point.

Problem 7. Find the indefinite integral of each of the following functions. Write in the form F(x) + c.

a. $f(x) \;=\; 24x^2 - 10x + 5$

$$\int f(x)dx = 8x^3 - 5x^2 + 5x + c$$

b. $f(x) \;=\; 500 - 80x + 9x^2$

$$\int f(x)dx = 500x - 40x^2 + 3x^3 + c$$

c. $y = \frac{12}{x}$

$$\int y dx = \int \frac{12}{x} dx = 12 \log |x| + c$$

d. $y = -\frac{8x}{3x^{3/4}}$

$$\int y dx = \int -\frac{8x}{3x^{3/4}} dx = \int -\frac{8}{3} x^{1/4} dx = -\frac{8}{3} \left(\frac{1}{1+1/4} \right) x^{1+1/4}$$
$$= -\frac{32}{15} x^{5/4}$$

Problem 8. Find the definite integral of each of the following functions.

a. $\int_2^4 \left(3x^2 + 4x\right)\, dx$

$$\begin{aligned}\int_2^4 \left(3x^2 + 4x\right)\, dx &= \left(x^3 + 2x^2\right)\Big|_2^4 \\ &= (64 + 32) - (8 + 8) = 80\end{aligned}$$

b. $\int_0^{15} \left(9x^2 - 80x + 500\right)\, dx$

$$\begin{aligned}\int_0^{15} \left(9x^2 - 80x + 500\right)\, dx &= \left(3x^3 - 40x^2 + 500x\right)\Big|_0^{15} \\ &= 3 \times 15^3 - 40 \times 15^2 + 500 \times 15 \\ &= 8625\end{aligned}$$

Problem 9. Do the following problems from the book.

a. Section 9.1
   1) 1a
   2) 1b
   3) 1c
   4) 3a
   5) 3b
   6) 5a
   7) 5b
   8) 5c

b. Section 9.2
   1) 5a
   2) 5b
   3) 5c
   4) 5d
   5) 5e
   6) 7

c. Section 9.3
   1) 1a
   2) 1b
   3) 1c
   4) 1d
   5) 1e