ECONOMICS 207
PROBLEM SET 8
KEY

Problem 1. Consider the following matrices.

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -7 & -2 \\ 3 & 7 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 7 & 19 & 17 \\ -3 & -8 & -7 \\ 0 & -1 & -1 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Compute the following

a. A + C

$$A + C = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 7 & 4 \end{bmatrix}$$

b. A + 3C

$$A + 3C = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 13 & 6 \end{bmatrix}$$

c. A + B

The dimension of matrix A is 2×2. However, the dimension of matrix B is 2×3, which is different from that of matrix A. As a result, there is not definition of $A + B$.

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d. AC

$$AC = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3\times 1 + 2\times 3 & 3\times 0 + 2\times 1 \\ 4\times 1 + 3\times 3 & 4\times 0 + 3\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 2 \\ 13 & 3 \end{bmatrix}$$

e. a′

$$a' = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}' = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$$

f. a′b

$$a'b = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times 1 - 2\times 1 + 2\times 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

g. AB

$$AB = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3\times 3 + 2\times(-4) & 3\times(-2) + 2\times 3 & 3\times 5 + 2\times 1 \\ 4\times 3 + 3\times(-4) & 4\times(-2) + 3\times 3 & 4\times 5 + 3\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 23 \end{bmatrix}$$

h. Ba

$$Ba = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\times 1 + (-2)\times(-2) + 5\times 2 \\ (-4)\times 1 + 3\times(-2) + 1\times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 17 \\ -8 \end{bmatrix}$$

i. Bb

$$Bb = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + (-2) \times 1 + 5 \times 1 \\ (-4) \times 1 + 3 \times 1 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

j. c′a

$$c\prime = \begin{bmatrix} 2 \\ 3 \end{bmatrix}' = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

The dimension of $c\prime$ is 1×2, where the dimension of a is 3×1. So these two matrices are not compatible for multiplication.

k. c′B

$$c\prime B = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 3 + 3 \times (-4) & 2 \times (-2) + 3 \times 3 & 2 \times 5 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} -6 & 5 & 13 \end{bmatrix}$$

l. FG

$$FG = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -7 & -2 \\ 3 & 7 & 1 \end{bmatrix} \begin{bmatrix} 7 & 19 & 17 \\ -3 & -8 & -7 \\ 0 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 7 + 2 \times (-3) & 1 \times 19 + 2 \times (-8) + 3 \times (-1) & 1 \times 17 + 2 \times (-7) + 3 \times (-1) \\ -3 \times 7 - 7 \times (-3) & -3 \times 19 - 7 \times (-8) - 2 \times (-1) & -3 \times 17 - 7 \times (-7) - 2 \times (-1) \\ 3 \times 7 + 7 \times (-3) & 3 \times 19 + 7 \times (-8) + 1 \times (-1) & 3 \times 17 + 7 \times (-7) + 1 \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

m. Fa

$$Fa = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -7 & -2 \\ 3 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times (-2) + 3 \times 2 \\ (-3) \times 1 + (-7) \times (-2) + (-2) \times 2 \\ 3 \times 1 + 7 \times (-2) + 1 \times 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -9 \end{bmatrix}$$

n. b′G

$$b'G = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 19 & 17 \\ -3 & -8 & -7 \\ 0 & -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 10 & 9 \end{bmatrix}$$

o. A + 2D

$$A + 2D = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} + 2 \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 8 & 9 \end{bmatrix}$$

p. B′D

$$B'D = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix}' \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \times 5 - 4 \times 2 & 3 \times 2 - 4 \times 3 \\ -2 \times 5 + 3 \times 2 & -2 \times 2 + 3 \times 3 \\ 5 \times 5 + 1 \times 2 & 5 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -4 & 5 \\ 27 & 13 \end{bmatrix}$$

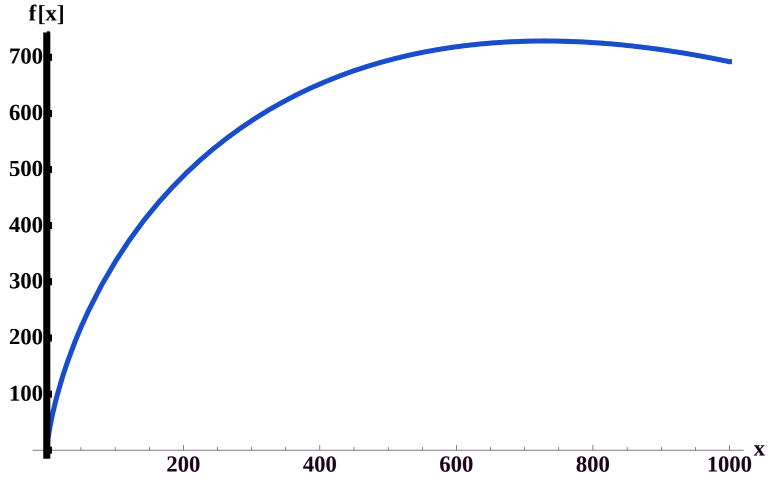
q. BG

$$BG = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 7 & 19 & 17 \\ -3 & -8 & -7 \\ 0 & -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \times 7 - 2 \times (-3) & 3 \times 19 - 2 \times (-8) + 5 \times (-1) & 3 \times 17 - 2 \times (-7) + 5 \times (-1) \\ -4 \times 7 + 3 \times (-3) & -4 \times 19 + 3 \times (-8) + 1 \times (-1) & -4 \times 17 + 3 \times (-7) + 1 \times (-1) \end{bmatrix}$$
$$= \begin{bmatrix} 27 & 68 & 60 \\ -37 & -101 & -90 \end{bmatrix}$$

Problem 2. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

a. $f(x) = 18x^{5/6} - 5x$

FIGURE 1. $f(x) = 18x^{5/6} - 5x$



The derivatives of $f(x) = 18x^{5/6} - 5x$ are given by

$$f'(x) = 15x^{-1/6} - 5 \tag{1}$$

$$f''(x) = \frac{-15}{6}x^{-7/6} \tag{2}$$

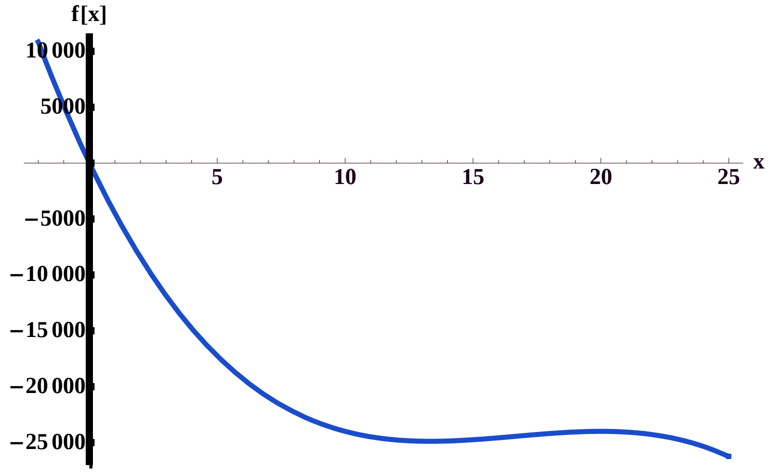
Set the first derivative, equation (1), to be zero.

$$\begin{aligned} 15x^{-1/6} - 5 &= 0 \\ \Rightarrow \quad x^{-1/6} &= 3^{-1} \\ \Rightarrow \quad x &= 3^6 = 729 \end{aligned}$$

Since the second derivative, $f''(x) = \frac{-15}{6}x^{-7/6} < 0$, $x = 729$ is a local maximum point and there is no infecltion point.

b. $f(x) = -6x^3 + 300x^2 - 4800x$

FIGURE 2. $f(x) = -6x^3 + 300x^2 - 4800x$



The derivatives of $f(x) = -6x^3 + 300x^2 - 4800x$ are given by

$$f'(x) = -18x^2 + 600x - 4800 \tag{3}$$

$$f''(x) = -36x + 600 \tag{4}$$

$$f^{(3)}(x) = -36 \tag{5}$$

Set the first derivative, equation (3), to be zero.

$$\begin{aligned} f'(x) = -18x^2 + 600x - 4800 &= 0 \\ \Rightarrow \qquad 3x^2 - 100x + 800 &= 0 \\ \Rightarrow \qquad (x-20)(3x-40) &= 0 \\ \Rightarrow \qquad x = 20 \quad \text{or} \quad x &= 40/3 \end{aligned}$$

Check the second derivative, equation (4), when $x = 20$ and $x = 40/3$ respectively.

$$f''(20) = -36 \times 20 + 600 = -120 < 0$$

$$f''(40/3) = -36 \times (40/3) + 600 = 120 > 0$$

So $x = 20$ is a local maximum point; $x = 40/3$ is a local minimum point.

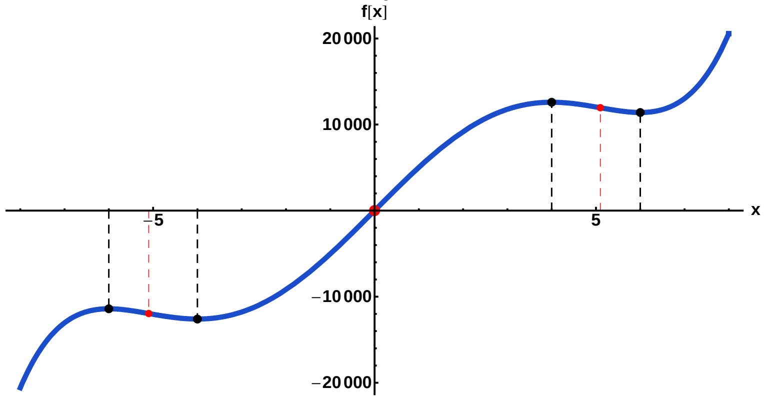
Set the second derivative, equation (4), to be zero.

$$\begin{aligned} f''(x) = -36x + 600 &= 0 \\ \Rightarrow \qquad x &= 50/3 \end{aligned}$$

Since $f^{(3)}(x) = -36 \neq 0$, $x = 50/3$ is an inflection point.

c. $f(x) = \frac{9}{5}x^5 - 156x^3 + 5184x$
Hint: The inflection points are $x = \{0, -\sqrt{26}, \sqrt{26}\}$

FIGURE 3. $f(x) = \frac{9}{5}x^5 - 156x^3 + 5184x$



The derivatives of $f(x) = \frac{9}{5}x^5 - 156x^3 + 5184x$ are given by

$$f'(x) = 9x^4 - 468x^2 + 5184 \tag{6}$$

$$f''(x) = 36x^3 - 936x = 36x(x^2 - 26) \tag{7}$$

$$f^{(3)}(x) = 108x^2 - 936 = 36(3x^2 - 26) \tag{8}$$

Set the first derivative, equation (6), to be zero.

$$\begin{aligned} f'(x) = 9x^4 - 468x^2 + 5184 &= 0 \\ \Rightarrow \quad 9(x^2 - 16)(x^2 - 36) &= 0 \\ \Rightarrow \quad x = \pm 4 \quad &\text{or} \quad x = \pm 6 \end{aligned}$$

Check the second derivative, equation (7), for $x = \pm 4$ and $x = \pm 6$.

$$f''(4) = 36 \times 4 \times (4^2 - 26) = 144 \times (-10) < 0$$

$$f''(-4) = 36 \times (-4) \times ((-4)^2 - 26) = -144 \times (-10) > 0$$

$$f''(6) = 36 \times 6 \times (6^2 - 26) = 216 \times 10 > 0$$

$$f''(-6) = 36 \times (-6) \times ((-6)^2 - 26) = -216 \times 10 < 0$$

So $x = 4$ and $x = -6$ are local maximum points; $x = -4$ and $x = 6$ are local minimum points.

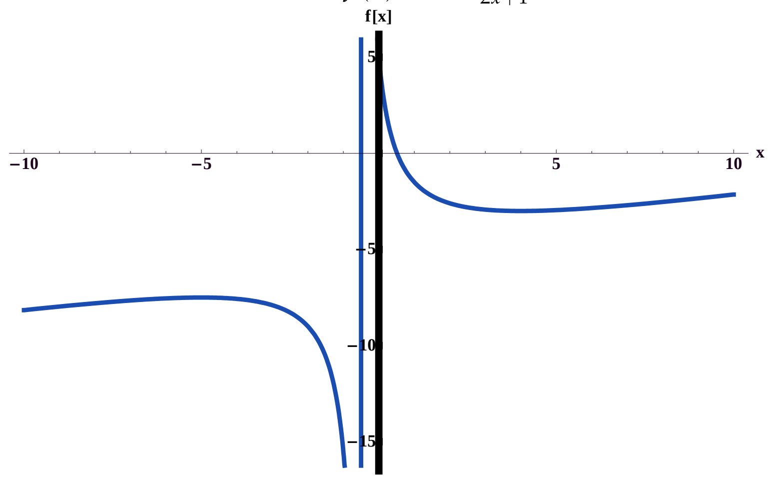
Set the second derivative, equation (7), to be zero.

$$\begin{aligned} f''(x) = 36x(x^2 - 26) &= 0 \\ \Rightarrow \quad x = 0 \quad \text{or} \quad &x = \pm\sqrt{26} \end{aligned}$$

Since $f^{(3)}(x) = 36(3x^2 - 26)$ equals to zero only when $x^2 = 26/3$, i.e., $x = \pm\sqrt{\frac{26}{3}}$, $x = 0$ and $x = \pm\sqrt{26}$ are inflection points.

d. $f(x) = \frac{\frac{1}{2}x^2-10x+5}{2x+1}$

FIGURE 4. $f(x) = \frac{\frac{1}{2}x^2-10x+5}{2x+1}$



Simplify the function.

$$f(x) = \frac{\frac{1}{2}x^2 - 10x + 5}{2x+1} = \frac{\frac{1}{2}x^2 + \frac{1}{4}x - \frac{41}{4}x - \frac{41}{8} + \frac{81}{8}}{2x+1}$$
$$= \frac{x}{4} - \frac{41}{8} + \frac{81/8}{2x+1} = \frac{x}{4} - \frac{41}{8} + \frac{81}{16}(x+1/2)^{-1}$$

Then the derivatives of $f(x) = \frac{\frac{1}{2}x^2-10x+5}{2x+1}$ are given by

$$f'(x) = \frac{1}{4} - \frac{81}{16}(x+1/2)^{-2} \tag{9}$$
$$f''(x) = \frac{81}{8}(x+1/2)^{-3} \neq 0 \tag{10}$$

Set the first derivative, equation (9), to be zero.

$$\begin{aligned}
f'(x) = \frac{1}{4} - \frac{81}{16}(x+1/2)^{-2} &= 0 \\
\Rightarrow \quad (x+1/2)^{-2} &= \frac{4}{81} = (9/2)^{-2} \\
\Rightarrow \quad (x+1/2)^{2} &= (9/2)^2 \\
\Rightarrow \quad x = 4 \quad &\text{or} \quad x = -5
\end{aligned}$$

Check the second derivative, equation (10), for $x = 4$ and $x = -5$.

$$f''(4) = \frac{81}{8}(4 + 1/2)^{-3} = \frac{1}{9} > 0$$

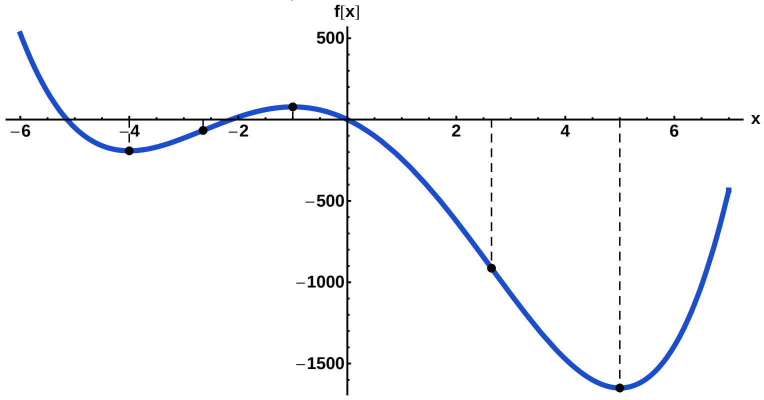
$$f''(-5) = \frac{81}{8}(-5 + 1/2)^{-3} = -\frac{1}{9} < 0$$

So $x = 4$ is a local minimum point; $x = -5$ is a local minimum point.

Since the second derivative, equation (10), is not zero for any x, there is no inflection point.

e. $f(x) = 2x^4 - 84x^2 - 160x$
Hint: The inflection points are $x = \pm\sqrt{7}$

FIGURE 5. $f(x) = 2x^4 - 84x^2 - 160x$



The derivatives of $f(x) = 2x^4 - 84x^2 - 160x$ are given by

$$f'(x) = 8x^3 - 168x - 160 \tag{11}$$

$$f''(x) = 24x^2 - 168 \tag{12}$$

$$f^{(3)}(x) = 48x \tag{13}$$

By setting the first derivative, equation (11), to be zero, we obtain

$$f'(x) = 8x^3 - 168x - 160 = 0 \tag{14}$$

Substitute $x = -1$ into equation (14), $f'(-1) = -8 + 168 - 160 = 0$. Thus $f'(x)$ can be factorized by $(x+1)$. That is $f'(x) = (x+1)(ax^2 + bx + c)$.

$$\begin{array}{r|rrr}
 & 8x^2 & -8x & -160 \\ \hline
x+1 & 8x^3 & -168x & -160 \\
 & -8x^3 - 8x^2 & & \\ \hline
 & -8x^2 & -168x & \\
 & 8x^2 & +8x & \\ \hline
 & & -160x & -160 \\
 & & 160x & +160 \\ \hline
 & & & 0
\end{array}$$

Thus, we can solve the equation (14).

$$\begin{aligned}
8x^3 - 168x - 160 &= 0 \\
\Rightarrow \quad (x+1)(8x^2 - 8x - 160) &= 0 \\
\Rightarrow \quad 8(x+1)(x-5)(x+4) &= 0 \\
\Rightarrow \quad x = -1 \quad \text{or} \quad x = -4 \quad &\text{or} \quad x = 5
\end{aligned}$$

Check the second derivatives for $x = -1$, $x = -4$, and $x = 5$ respectively.

$$f''(-1) = 24(-1)^2 - 168 = -144 < 0$$

$$f''(-4) = 24(-4)^2 - 168 = 216 > 0$$

$$f''(5) = 24(5)^2 - 168 = 432 > 0$$

So $x = -1$ is a local maximum point; $x = -4$ and $x = 5$ are local minimum points.

Set the second derivative, equation (12), to be zero.

$$\begin{aligned} f''(x) = 24x^2 - 168 &= 0 \\ \Rightarrow \qquad x^2 - 7 &= 0 \\ \Rightarrow \qquad x &= \pm\sqrt{7} \end{aligned}$$

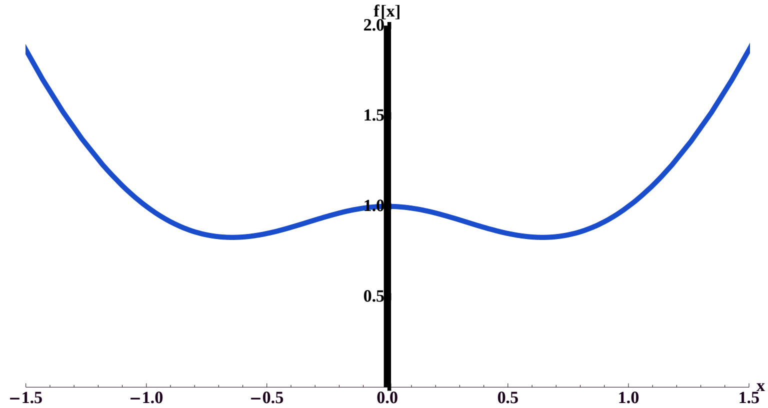
Since $f^{(3)}(x) = 48x$ equals to zero only when $x = 0$, $x = \pm\sqrt{7}$ are two inflection points.

f. $f(x) = \frac{x^4+1}{x^2+1} \quad (x \neq 1, x \neq -1)$

Hint Substitute $z = x^2$ in the first order conditions and then solve for the roots. The roots are are follows.

$x = \{0, \quad -\sqrt{-1+\sqrt{2}}, \quad \sqrt{-1+\sqrt{2}}, \quad -i\sqrt{-1+\sqrt{2}}, \quad i\sqrt{-1+\sqrt{2}}\}$

FIGURE 6. $f(x) = \frac{x^4+1}{x^2+1}$



Simplify $f(x) = \frac{x^4+1}{x^2+1}$.

$$f(x) = \frac{x^4+1}{x^2+1} = \frac{x^4+x^2-x^2-1+2}{x^2+1} = x^2-1+\frac{2}{x^2+1}$$

Then the derivatives of $f(x) = \frac{x^4+1}{x^2+1}$ are given by

$$f'(x) = 2x - 2(x^2+1)^{-2}\cdot(2x) = 2x[1-2(x^2+1)^{-2}] \tag{15}$$

$$\begin{aligned} f''(x) &= 2[1-2(x^2+1)^{-2}] + 2x\left[4(x^2+1)^{-3}\cdot(2x)\right] \\ &= 2[1-2(x^2+1)^{-2}] + 16x^2(x^2+1)^{-3} \end{aligned} \tag{16}$$

Set the first derivative, equation (15), to be zero.

$$\begin{aligned} f'(x) = 2x[1-2(x^2+1)^{-2}] &= 0 \\ \Rightarrow \quad 1-2(x^2+1)^{-2} = 0 \quad &\text{or} \quad x = 0 \\ \Rightarrow \quad x^2+1 = \pm\sqrt{2} \quad &\text{or} \quad x = 0 \\ \Rightarrow \quad x^2 = \sqrt{2}-1 \quad &\text{or} \quad x = 0 \\ \Rightarrow \quad x = \pm\sqrt{\sqrt{2}-1} \quad &\text{or} \quad x = 0 \end{aligned}$$

Check the second derivative, equation (16), for $x = 0$ and $x = \pm\sqrt{\sqrt{2}-1}$.

$$f''(0) = 2\left[1-2(0^2+1)^{-2}\right] + 16\times 0^2 \times (0^2+1)^{-3} = 2(1-2)+0 = -2 < 0$$

$$f''(\pm\sqrt{\sqrt{2}-1}) = 2\left[1-2\left(\sqrt{2}\right)^{-2}\right] + 16\times\left(\sqrt{2}-1\right)\times\sqrt{2}^{-3} = 16\times\left(\sqrt{2}-1\right)\times\sqrt{2}^{-3} > 0$$

So $x = 0$ is a local maximum point; $x = \pm\sqrt{\sqrt{2} - 1}$ are local minimum points.

Set the second derivative, equation (16), to be zero.

$$\begin{aligned}
f''(x) = 2[1 - 2(x^2+1)^{-2}] + 16x^2(x^2+1)^{-3} &= 0 \\
\Rightarrow \qquad 2 - \frac{4}{(x^2+1)^2} + \frac{16x^2}{(x^2+1)^3} &= 0 \\
\Rightarrow \qquad 2 - \frac{4(x^2+1)}{(x^2+1)^3} + \frac{16x^2}{(x^2+1)^3} &= 0 \\
\Rightarrow \qquad 2 + \frac{12x^2 - 4}{(x^2+1)^3} &= 0 \\
\Rightarrow \qquad 1 + \frac{6x^2 - 2}{(x^2+1)^3} &= 0 \\
\Rightarrow \qquad (x^2+1)^3 &= 2 - 6x^2 \\
\Rightarrow \qquad x^6 + 3x^4 + 3x^2 + 1 &= 2 - 6x^2 \\
\Rightarrow \qquad x^6 + 3x^4 + 9x^2 - 1 &= 0
\end{aligned}$$

We know that the solutions to equation $az^3 + bz^2 + cz + d = 0$ are given by

$$z_1 = A + B - r$$

$$z_{2,3} = -\frac{1}{2}(A+B) \pm i\frac{\sqrt{3}}{2}(A - B) - r$$

where $A = \sqrt[3]{q/2 + \sqrt{D}}$, $B = \sqrt[3]{q/2 - \sqrt{D}}$, $D = (p/3)^3 + (q/2)^2$, where $p = \frac{c}{a} - \frac{b^2}{3a^2}$, $q = \frac{-2b^3}{27a^3} + \frac{bc}{3a^2} - \frac{d}{a}$, and $r = \frac{b}{3a}$.

By substituting $z = x^2$ into the $x^6 + 3x^4 + 9x^2 - 1 = 0$, we obtain

$$z^3 + 3z^2 + 9z - 1 = 0 \tag{17}$$

Then regarding equation (17),

$$p = \frac{c}{a} - \frac{b^2}{3a^2} = 9 - \frac{9}{3} = 6$$

$$q = \frac{-2b^3}{27a^3} + \frac{bc}{3a^2} - \frac{d}{a} = \frac{-54}{27} + \frac{27}{3} - (-1) = 8$$

$$r = \frac{b}{3a} = 1$$

$$D = \frac{q^2}{4} + \frac{p^3}{27} = 16 + \frac{3^3}{3^3} = 16 + 8 = 24 > 0$$

$$A = \sqrt[3]{4 + \sqrt{24}} = \sqrt[3]{4 + 2\sqrt{6}}$$

$$B = \sqrt[3]{4 - \sqrt{24}} = \sqrt[3]{4 - 2\sqrt{6}}$$

And the only real solution is then given by

$$z_1 = A + B - r = \sqrt[3]{4+2\sqrt{6}} + \sqrt[3]{4-2\sqrt{6}} - 1$$

So $x^2 = \sqrt[3]{4+2\sqrt{6}} + \sqrt[3]{4-2\sqrt{6}} - 1$ is the only real solution to $f^{(2)}(x) = 0$. In addition, we know that when $x^2 = 0$, $f^{(2)}(0) < 0$, and when $x^2 = \sqrt{2} - 1$, $f^{(2)}(\pm\sqrt{\sqrt{2}-1}) > 0$. So by mean value theormem, $\sqrt[3]{4+2\sqrt{6}} + \sqrt[3]{4-2\sqrt{6}} - 1$ is in between 0 and $\sqrt{2} - 1$ and thus it is bigger than 0. As a result, $x = \pm\sqrt{\sqrt[3]{4+2\sqrt{6}} + \sqrt[3]{4-2\sqrt{6}} - 1}$ are two potential inflection points.

Problem 3. Find the definite integral of each of the following functions.

a. $\int_1^4 \left(2x^3 + 6x^2 - 15x\right)\, dx$

$$\begin{aligned}\int_1^4 \left(2x^3 + 6x^2 - 15x\right)\, dx &= \left(\frac{x^4}{2} + 2x^3 - \frac{15}{2}x^2\right)\Big|_1^4 \\ &= \left(\frac{4^4}{2} + 2\times 4^3 - \frac{15}{2}\times 4^2\right) - \left(\frac{1^4}{2} + 2\times 1^3 - \frac{15}{2}\times 1^2\right) \\ &= 136 + 5 \\ &= 141\end{aligned}$$

b. $\int_0^{12} (9x^2 - 60x + 200)\, dx$

$$\begin{aligned}\int_0^{12} (9x^2 - 60x + 200)\, dx &= \left(3x^3 - 30x^2 + 200x\right)\Big|_0^{12} \\ &= 3\times 12^3 - 30\times 12^2 + 200\times 12 \\ &= 6264\end{aligned}$$

Problem 4. Solve the following systems of equations.

$$\begin{aligned} 243x_1^{-5/6}x_2^{2/3} - 16 &= 0 \\ 972x_1^{1/6}x_2^{-1/3} - 729 &= 0 \end{aligned}$$

Rearrange the second equation.

$$\begin{aligned} 972x_1^{1/6}x_2^{-1/3} - 729 &= 0 \\ \Rightarrow \quad x_1^{1/6}x_2^{-1/3} &= 729/972 = 3/4 \\ \Rightarrow \quad x_2^{-1/3} &= \tfrac{3}{4}x_1^{-1/6} \\ \Rightarrow \quad x_2^{1/3} &= \tfrac{4}{3}x_1^{1/6} \\ \Rightarrow \quad x_2^{2/3} &= \tfrac{16}{9}x_1^{1/3} \end{aligned}$$

Substitute $x_2^{2/3} = \frac{16}{9}x_1^{1/3}$ into the first equation.

$$\begin{aligned} 243x_1^{-5/6}x_2^{2/3} - 16 &= 0 \\ \Rightarrow \quad 243x_1^{-5/6}\left(\tfrac{16}{9}x_1^{1/3}\right) - 16 &= 0 \\ \Rightarrow \quad 243x_1^{-5/6}\left(\tfrac{16}{9}x_1^{1/3}\right) &= 16 \\ \Rightarrow \quad x_1^{-1/2} &= 1/27 \\ \Rightarrow \quad x_1 &= 27^2 = 729 \end{aligned}$$

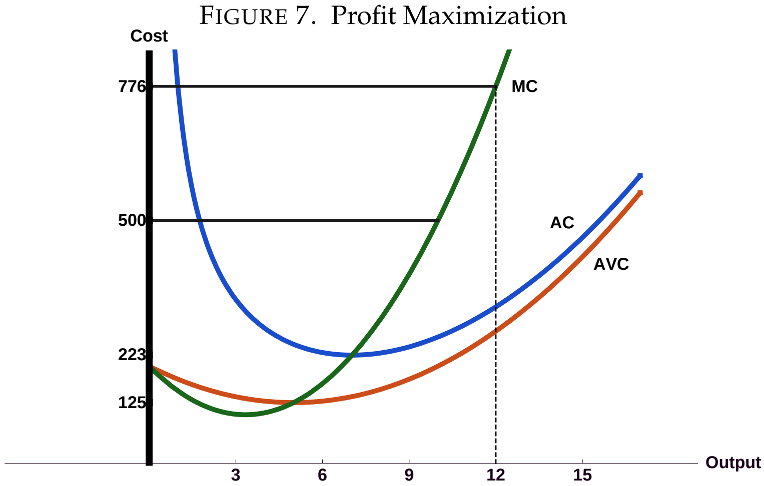
Substitute $x_1 = 729$ into $x_2^{1/3} = \frac{4}{3}x_1^{1/6}$.

$$\begin{aligned} x_2^{1/3} &= \tfrac{4}{3}x_1^{1/6} \\ \Rightarrow \quad x_2^{1/3} &= \tfrac{4}{3} \times 729^{1/6} = 4 \\ \Rightarrow \quad x_2 &= 4^3 = 64 \end{aligned}$$

So the solution is $x_1 = 729,\ x_2 = 64$.

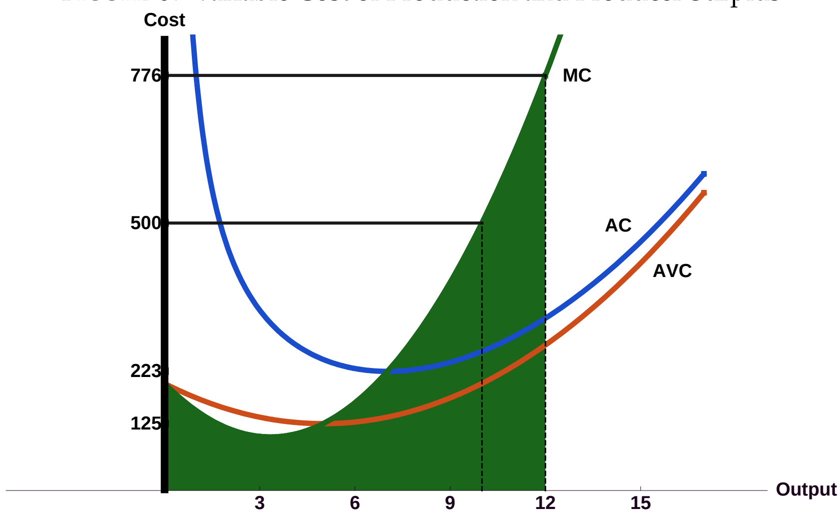
Problem 5. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 7.

FIGURE 7. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 8 represents the variable cost of production for the cost function $c(y) = 600 + 200y - 30\,y^2 + 3y^3$.

FIGURE 8. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 776 in figure 8. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$\begin{aligned} price &= p &= \$776 \\ cost &= c(y) &= 600 + 200y - 30y^2 + 3y^3 \end{aligned}$$

The profit is given by

$$\begin{aligned} Profit &= revenue - cost = py - c(y) = 776y - \left(600 + 200y - 30y^2 + 3y^3\right) \\ &= -\left(600 - 576y - 30y^2 + 3y^3\right) \end{aligned}$$

Then the first and second derivatives of profit are given by

$$\begin{aligned} \frac{d\,Profit}{d\,y} &= \frac{d\,\left(-\left(600 - 576y - 30y^2 + 3y^3\right)\right)}{d\,y} = -\left(-576 - 60y + 9y^2\right) \\ \frac{d^2\,Profit}{d\,y^2} &= -\left(-60 + 18y\right) = 60 - 18y \end{aligned}$$

Set the first derivative of profit to zero and solve for the critical values.

$$\begin{aligned} -\left(-576 - 60y + 9y^2\right) &= 0 \\ \Rightarrow \quad -(y - 12)(9y + 48) &= 0 \\ \Rightarrow \quad y = 12 \quad &\text{or} \quad y = -48/9 \end{aligned}$$

At $y = -48/9$, $\frac{d^2\,Profit}{d\,y^2} = 60 - 18 \times (-48/9) = 60 + 96 > 0$.
At $y = 12$, $\frac{d^2\,Profit}{d\,y^2} = 60 - 18 \times 12 = 60 - 216 < 0$. So the optimal level $y = 12$.

b. Find the profit maximizing level of output when the price is \$500. Demonstrate that the level you choose maximizes profit.

Here the profit is given by

$$\begin{aligned} Profit &= revenue - cost = py - c(y) = 500y - \left(600 + 200y - 30y^2 + 3y^3\right) \\ &= -\left(600 - 300y - 30y^2 + 3y^3\right) \end{aligned}$$

Then the first and second derivatives of profit are given by

$$\frac{d\,Profit}{d\,y} = \frac{d\,\left(-\left(600 - 300y - 30y^2 + 3y^3\right)\right)}{d\,y} = -\left(-300 - 60y + 9y^2\right)$$

$$\frac{d^2\,Profit}{d\,y^2} = -\left(-60 + 18y\right) = 60 - 18y$$

Set the first derivative of profit to zero and solve for the critical values.

$$\begin{aligned} -\left(-300 - 60y + 9y^2\right) &= 0 \\ \Rightarrow \quad -(y-10)(9y+30) &= 0 \\ \Rightarrow \quad y = 10 \quad \text{or} \quad y &= -30/9 = -10/3 \end{aligned}$$

At $y = -10/3$, $\frac{d^2\,Profit}{d\,y^2} = 60 - 18 \times (-10/3) = 120 > 0$.

At $y = 10$, $\frac{d^2\,Profit}{d\,y^2} = 60 - 18 \times 10 = -120 < 0$. So the optimal level $y = 10$.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$Profit = py - c(y)$$
$$Profit = 776y - \left[600 + 200y - 30\,y^2 + 3y^3\right]$$

Setting the first derivative of proft to zero and solve for the optiam level can be expressed using following equation.

$$\frac{d\,Profit}{d\,y} = \frac{d\ (py - c(y))}{d\,y} = p - c'(y) = 0 \tag{18}$$

And equation (18) is equivalent to

$$p = c'(y) \tag{19}$$

The left hand side of equation (19) is the price; the right hand side of equation (19) is marginal cost.

Therefore, setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y.

d. What is variable cost for this firm when it maximizes profit with a price of \$776?

The variable cost for this firm is given by

$$\begin{aligned} VC(y) &= \int_0^y MC(y)d\,y \\ &= \int_0^y \left(200 - 60y + 9y^2\right) d\,y \\ &= 200y - 30y^2 + 3y^3 \end{aligned}$$

The optimal level for maximizing profit with a price of \$776 is that $y = 12$. And at $y = 12$ the variable cost is given by

$$200y - 30y^2 + 3y^3 = 200 \times 12 - 30 \times 12^2 + 3 \times 12^3 = 3264$$

e. What is producer surplus for this profit maximizing firm when the price is \$776?

For price \$776, the optimal level $y = 12$. The producer surplus is given by

$$\begin{aligned} Produer\ surplus &= Reveune - Variable\ Cost = py - VC(y) \\ &= 776 \times 12 - 3264 \\ &= 6048 \end{aligned}$$

f. What is variable cost for this firm when when it maximizes profit with a price of \$500?

The optimal level for maximizing profit with a price of \$500 is that $y = 10$. And at $y = 10$ the variable cost is given by

$$200y - 30y^2 + 3y^3 = 200 \times 10 - 30 \times 10^2 + 3 \times 10^3 = 2000$$

g. What is producer surplus for this profit maximizing firm when the price is \$500?

For price \$500, the optimal level $y = 10$. The producer surplus is given by

$$\begin{aligned} Produer\ surplus &= Reveune - Variable\ Cost = py - VC(y) \\ &= 500 \times 10 - 2000 \\ &= 3000 \end{aligned}$$

h. How much is the firm worse off when price falls from \$776 to \$500?

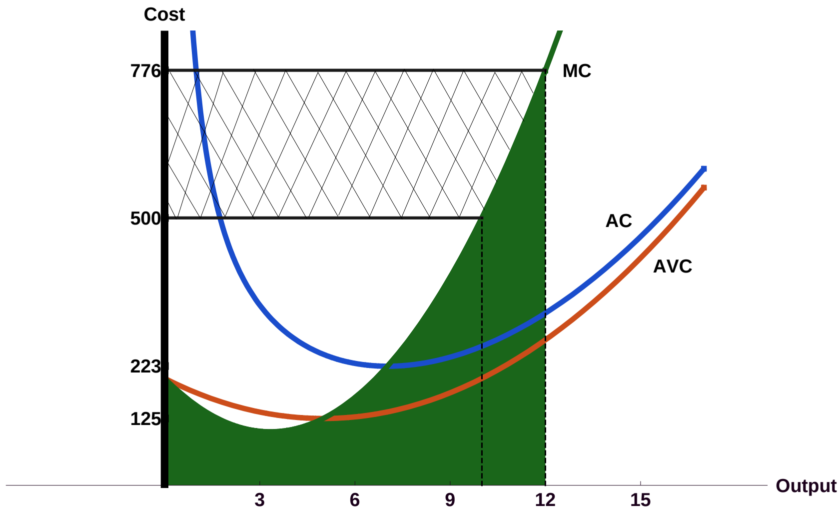
From part (e) and part (g), the difference of producer surplus is given by

$$6048 - 3000 = 3048$$

So the furm is worse off \$3048 when price falls from \$776 to \$500.

i. Cross-hatch the change in producer surplus in Figure 8.

FIGURE 9. Change in producer surplus in Figure 8



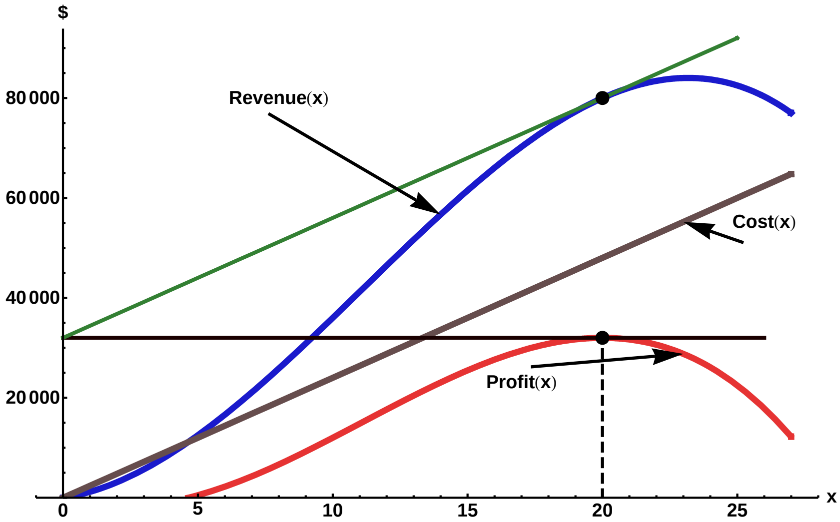
Problem 6. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 4$$

$$input\ price = w = 2400$$

$$y = output = f(x) = 200x + 100x^2 - 3x^3$$

FIGURE 10. Revenue, Cost and Profit



a. Write down an equation that represents profit for the firm.

The profit is given by

$$\begin{aligned} Profit &= Revenue - cost = py - wx \\ &= 4\left(200x + 100x^2 - 3x^3\right) - 2400x \\ &= 4\left(-400x + 100x^2 - 3x^3\right) \end{aligned}$$

b. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

Set the derivative of profit with respect to the variable input x to zero.

$$\begin{aligned} \frac{d\,Porfit}{d\,x} = 4\left(-400+200x-9x^2\right) &= 0 \\ \Rightarrow \qquad 400-200x+9x^2 &= 0 \\ \Rightarrow \qquad (x-20)(9x-20) &= 0 \\ \Rightarrow \qquad x = 20 \qquad \text{or} \qquad & x = 20/9 \end{aligned}$$

c. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

First, the second drivative is given by

$$\frac{d^2\,Porfit}{d\,x^2} = 4\,(200-18x)$$

Check the second derivative for $x = 20$ and $x = 20/9$.

$$\left.\frac{d^2\,Porfit}{d\,x^2}\right|_{x=20} = 4(200-18\times 20) = 4(200-360) < 0$$

$$\left.\frac{d^2\,Porfit}{d\,x^2}\right|_{x=20/9} = 4(200-18\times(20/9)) = 4(200-40) > 0$$

So it is $x = 20$ that maximizes profit.

d. What is the optimal level of output for this firm?

When $x = 20$, the output is given by

$$\begin{aligned} 200x + 100x^2 - 3x^3 &= 200 \times 20 + 100 \times 20^2 - 3 \times 20^3 \\ &= 20000 \end{aligned}$$

e. Explain in words why the slope of the total revenue function for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the graph 10 and the following information in explaining this phenomenon.

$$\begin{aligned} Revenue &= pf(x) \\ Cost &= wx \\ Profit &= Revenue - Cost = pf(x) - wx \end{aligned}$$

By setting the derivative of profit to zero to solve for the maximizng level of input use, we obtain

$$\frac{d\,Profit}{d\,x} = \frac{d\,Revenue}{d\,x} - \frac{d\,Cost}{d\,x} = \frac{d\,Revenue}{d\,x} - w = 0,$$

which is equivalent to

$$\frac{d\,Revenue}{d\,x} = w \tag{20}$$

The left hand side of equation (20) is the slope of the total revenue function for this firm and the right hand side of equation (20) is the price of the single variable input. Therefore, the slope of the total revenue function for this firm is equal to the price of the single variable input at the profit maximizing level of input use.

Problem 7. Solve the following system of equations for x_1, x_2, and x_3 using the method of elimination.

$$\{x_1 = 4,\ x_2 = -1,\ x_3 = 2\}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ -3x_1 - 7x_2 - 2x_3 &= -9 \\ 3x_1 + 7x_2 + x_3 &= 7 \end{aligned}$$

Add the first equation multiplied by 3 to the second equation;

$$\begin{aligned} & -3x_1 - 7x_2 - 2x_3 + (x_1 + 2x_2 + 3x_3) \times 3 = -9 + 8 \times 3 \\ \Rightarrow \quad & -x_2 + 7x_3 = 15 \end{aligned}$$

Add the second equation to the third equation.

$$\begin{aligned} & 3x_1 + 7x_2 + x_3 + (-3x_1 - 7x_2 - 2x_3) = 7 - 9 \\ \Rightarrow \quad & -x_3 = -2 \\ \Rightarrow \quad & x_3 = 2 \end{aligned}$$

Add $x_3 = 2$ multiplied by -7 to $-x_2 + 7x_3 = 15$.

$$\begin{aligned} & -x_2 + 7x_3 + x_3 \times (-7) = 15 + 2 \times (-7) \\ \Rightarrow \quad & -x_2 = 1 \\ \Rightarrow \quad & x_2 = -1 \end{aligned}$$

Add $x_2 = -1$ multiplied by -2 and $x_3 = 2$ multiplied by -3 to the first equation.

$$\begin{aligned} & x_1 + 2x_2 + 3x_3 + x_2 \times (-2) + x_3 \times (-3) = 8 - 1 \times (-2) + 2 \times (-3) \\ \Rightarrow \quad & x_1 = 4 \end{aligned}$$

So the solution is $x_1 = 4,\ x_2 = -1,\ x_3 = 2$.