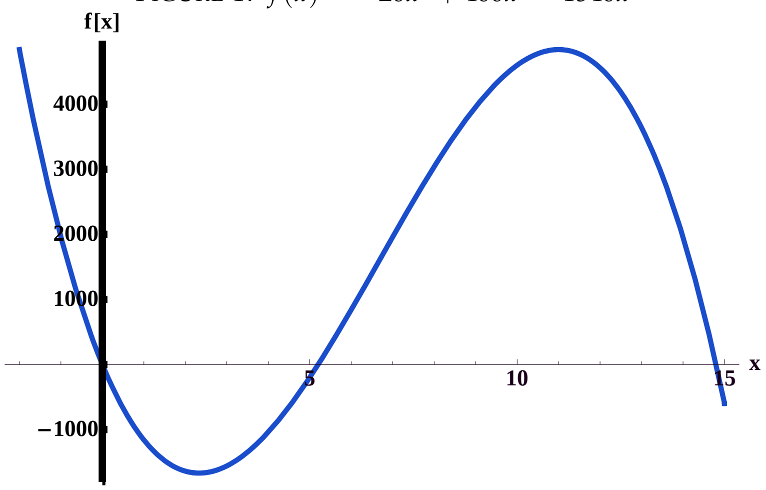
ECONOMICS 207
SPRING 2008
PROBLEM SET 9
KEY

Problem 1. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

a. $f(x) = -20x^3 + 400x^2 - 1540x$
Hint: The inflection point is $x = \frac{20}{3}$

FIGURE 1. $f(x) = -20x^3 + 400x^2 - 1540x$



The derivatives of $f(x) = -20x^3 + 400x^2 - 1540x$ are given by

$$f'(x) = -60x^2 + 800x - 1540 \quad (1)$$

$$f''(x) = -120x + 800 \quad (2)$$

$$f^{(3)}(x) = -120 \quad (3)$$

Date: August 25, 2008.

Set the first derivative, equation (1), to be zero.

$$\begin{aligned} f'(x) = -60x^2 + 800x - 1540 &= 0 \\ \Rightarrow \qquad 3x^2 - 40x + 77 &= 0 \\ \Rightarrow \qquad (x-11)(3x-7) &= 0 \\ \Rightarrow \qquad x = 11 \qquad &\text{or} \qquad x = 7/3 \end{aligned}$$

Check the second derivative, equation (2), for $x = 11$ and $x = 7/3$ respectively.

$$f''(11) = -120 \times 11 + 800 = -520 < 0$$

$$f''(7/3) = -120 \times (7/3) + 800 = 520 > 0$$

So $x = 11$ is a relative maximum point; $x = 7/3$ is a relative minimum point.

Set the second derivative, equation (2), to be zero.

$$f''(x) = -120x + 800 = 0$$

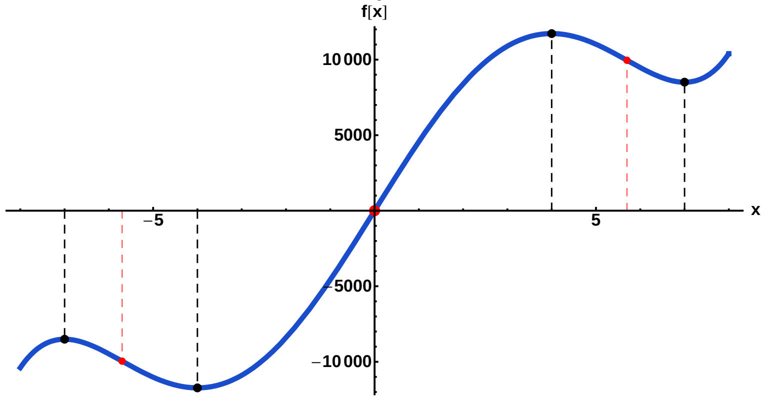
$$\Rightarrow \qquad x = 20/3$$

Since $f^{(3)}(x) = -120 \neq 0$, $x = 20/3$ is an inflection point.

b. $f(x) = \frac{6}{5}x^5 - 130x^3 + 4704x$

Hint: The inflection points are $x = 0, \pm\sqrt{\frac{65}{2}}$

FIGURE 2. $f(x) = \frac{6}{5}x^5 - 130x^3 + 4704x$



The derivatives of $f(x) = \frac{6}{5}x^5 - 130x^3 + 4704x$ are given by

$$f'(x) = 6x^4 - 390x^2 + 4704 \tag{4}$$

$$f''(x) = 24x^3 - 780x = 12x(2x^2 - 65) \tag{5}$$

$$f^{(3)}(x) = 72x^2 - 780 \tag{6}$$

Set the first derivative, equation (4), to be zero.

$$\begin{aligned} & f'(x) = 6x^4 - 390x^2 + 4704 &=& \ 0 \\ \Rightarrow & \quad x^4 - 65x^2 + 784 &=& \ 0 \\ \Rightarrow & \quad (x^2 - 16)(x^2 - 49) &=& \ 0 \\ \Rightarrow & \quad x = \pm 4 & \text{or} & \quad x = \pm 7 \end{aligned}$$

Check the second derivative, equation (5), for $x = \pm 4$ and $x = \pm 7$ respectively.

$$f''(4) = 12 \times 4 \times (2 \times 4^2 - 65) = 48 \times (32 - 65) < 0$$

$$f''(-4) = 12 \times (-4) \times \left(2 \times (-4)^2 - 65\right) = -48 \times (32 - 65) > 0$$

$$f''(7) = 12 \times 7 \times (2 \times 7^2 - 65) = 84 \times (98 - 65) > 0$$

$$f''(-7) = 12 \times (-7) \times \left(2 \times (-7)^2 - 65\right) = -84 \times (98 - 65) < 0$$

So $x = 4$ and $x = -7$ are local maximum points; $x = -4$ and $x = 7$ are local minimum points.

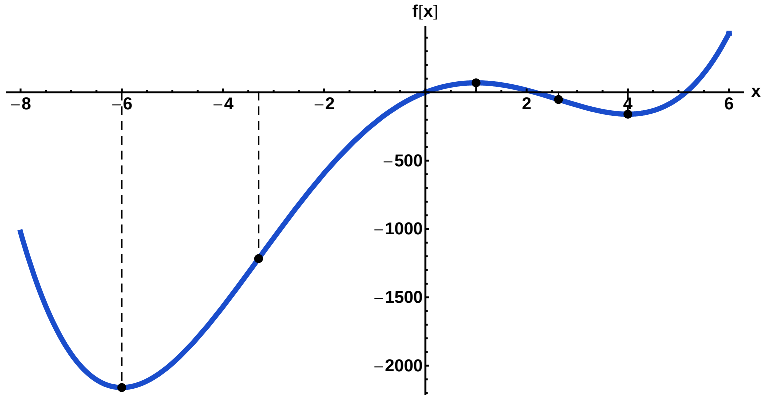
Set the second derivative, equation (5), to be zero.

$$\begin{aligned} f''(x) = 12x(2x^2 - 65) &= 0 \\ \Rightarrow \qquad x^2 = 65/2 \qquad &\text{or} \qquad x = 0 \\ \Rightarrow \qquad x = \pm\sqrt{65/2} \qquad &\text{or} \qquad x = 0 \end{aligned}$$

Since $f^{(3)}(x) = 72x^2 - 780$ is zero only when $x^2 = 65/6$, i.e., $x = \pm\sqrt{65/6}$, $x = 0$ and $x = \pm\sqrt{65/6}$ are inflection points.

c. $f(x) = \frac{3}{2}x^4 + 2x^3 - 78x^2 + 144x$
Hint: The inflection points are $x = \frac{-1 \pm \sqrt{79}}{3}$

FIGURE 3. $f(x) = \frac{3}{2}x^4 + 2x^3 - 78x^2 + 144x$



The derivatives of $f(x) = \frac{3}{2}x^4 + 2x^3 - 78x^2 + 144x$ are given by

$$f'(x) = 6x^3 + 6x^2 - 156x + 144 \tag{7}$$

$$f''(x) = 18x^2 + 12x - 156 \tag{8}$$

$$f^{(3)}(x) = 36x + 12 \tag{9}$$

Set the frits derivative, equation (7), to be zero.

$$f'(x) = 6x^3 + 6x^2 - 156x + 144 = 0 \tag{10}$$

By guess $x = 1$ is a solution of equation (10). So we try to factorize $f'(x)$ by $x - 1$.

$$\begin{array}{r|l}
 & 6x^2 \quad + 12x - 144 \\
\hline
x - 1 & 6x^3 \quad + 6x^2 - 156x + 144 \\
 & -6x^3 \quad + 6x^2 \\
\hline
 & 12x^2 - 156x \\
 & -12x^2 \quad + 12x \\
\hline
 & -144x + 144 \\
 & 144x - 144 \\
\hline
 & 0
\end{array}$$

Then we can solve the equation (10).

$$\begin{aligned}
f'(x) = 6x^3 + 6x^2 - 156x + 144 &= 0 \\
\Rightarrow \quad (x-1)(6x^2 + 12x - 144) &= 0 \\
\Rightarrow \quad 6(x-1)(x+6)(x-4) &= 0 \\
\Rightarrow \quad x = 1 \quad \text{or} \quad x = 4 \quad &\text{or} \quad x = -6
\end{aligned}$$

Check the second derivative, equation (8), for $x = 1$, $x = 4$, and $x = -6$ respectively.

$$f''(1) = 18 \times 1^2 + 12 \times 1 - 156 = -126 < 0$$

$$f''(4) = 18 \times 4^2 + 12 \times 4 - 156 = 180 > 0$$

$$f''(-6) = 18 \times (-6)^2 + 12 \times (-6) - 156 = 420 > 0$$

So $x = 1$ is a local maximum point; $x = 4$ and $x = -6$ are local minimum points.

Set the second derivative, equation (8), to be zero.

$$f''(x) = 18x^2 + 12x - 156 = 0$$

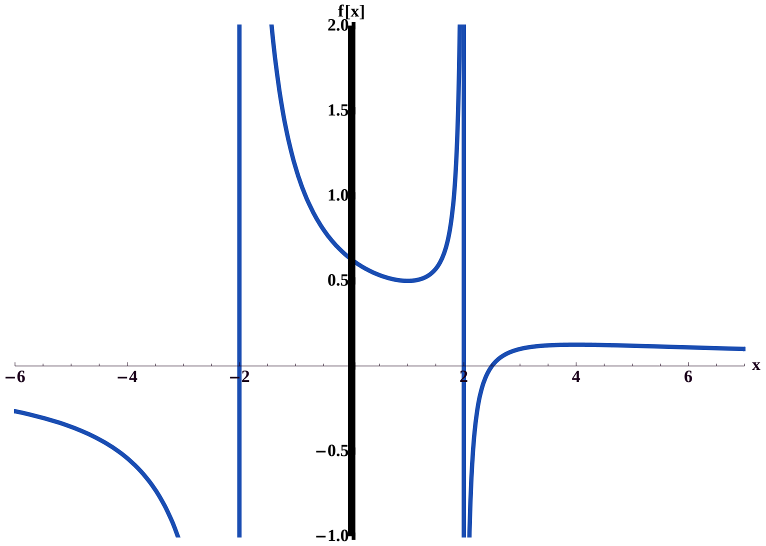
$$\Rightarrow \qquad x = \frac{-12 \pm \sqrt{12^2 - 4 \times 18 \times (-156)}}{36}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{79}}{3}$$

Since $f^{(3)}(x) = 36x + 12$ is zero only when $x = -3$, $x = \dfrac{-1 \pm \sqrt{79}}{3}$ are inflection points.

d. $f(x) = \frac{x-5/2}{x^2-4}$
Hint: The second derivative evaluated at the x = 1 is $\frac{1}{3}$. The second derivative evaluated at the x = 4 is $\frac{-1}{48}$. You need not find the inflection points.

FIGURE 4. $f(x) = \frac{x-5/2}{x^2-4}$



The derivatives for $f(x) = \frac{x-5/2}{x^2-4}$ are given by

$$f'(x) = \frac{x^2-4-(x-5/2)\cdot(2x)}{(x^2-4)^2} = \frac{-x^2+5x-4}{(x^2-4)^2} \tag{11}$$

$$= \frac{-(x-4)(x-1)}{(x^2-4)^2} \tag{12}$$

$$f''(x) = \frac{\left((-2x+5)(x^2-4)^2\right) - (-x^2+5x-4)\cdot\left(4x(x^2-4)\right)}{(x^2-4)^4}$$

$$= \frac{(-2x+5)(x^2-4) - 4x(-x^2+5x-4)}{(x^2-4)^3}$$

$$= \frac{2x^3-15x^2+24x-20}{(x^2-4)^3} \tag{13}$$

Set the first derivative, equation (12), to be zero.

$$f'(x) = \frac{-(x-4)(x-1)}{(x^2-4)^2} = 0$$

$$\Rightarrow \qquad x = 4 \quad \text{or} \quad x = 1$$

Check the second derivative, equation (13), for $x = 1$ and $x = 4$ respectively.

$$f''(1) = \frac{2 \times 1^3 - 15 \times 1^2 + 24 \times 1 - 20}{(1^2 - 4)^3} = \frac{-9}{-27} = \frac{1}{3} > 0$$

$$f''(4) = \frac{2 \times 4^3 - 15 \times 4^2 + 24 \times 4 - 20}{(4^2 - 4)^3} = \frac{-36}{12^3} = \frac{-1}{48} < 0$$

So $x = 1$ is a local minimum point; $x = 4$ is a local maximum point.

Problem 2. a. Use elementary row operations to solve the following system of equations. The answers are x_1 = -1, x_2 = 4, x_3 = 5.

$$Ax = b$$

$$\begin{pmatrix} 1 & -1 & 2 \\ -4 & 5 & -2 \\ 3 & -4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ -24 \end{pmatrix}$$

First, create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = \begin{pmatrix} A & b \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 5 \\ -4 & 5 & -2 & 14 \\ 3 & -4 & -1 & -24 \end{pmatrix}$$

Then use elementary row operations on $\tilde{A}$ to create an identity matrix on the left side of $\tilde{A}$. To begin with, based on $\tilde{A}$, add the first row multiplied by 4 to the second row.

$$\begin{pmatrix} 1 & -1 & 2 & 5 \\ -4+1\times 4 & 5-1\times 4 & -2+2\times 4 & 14+5\times 4 \\ 3 & -4 & -1 & -24 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 6 & 34 \\ 3 & -4 & -1 & -24 \end{pmatrix} \tag{14}$$

Based on the matrix on the right side of equation (14), add the first row multiplied by -3 to the third row.

$$\begin{pmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 6 & 34 \\ 3+1\times(-3) & -4-1\times(-3) & -1+2\times(-3) & -24+5\times(-3) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 6 & 34 \\ 0 & -1 & -7 & -39 \end{pmatrix} \tag{15}$$

Based on the matrix on the right side of equation (15), add the second row to the first row.

$$\begin{pmatrix} 1 & -1+1 & 2+6 & 5+34 \\ 0 & 1 & 6 & 34 \\ 0 & -1 & -7 & -39 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 8 & 39 \\ 0 & 1 & 6 & 34 \\ 0 & -1 & -7 & -39 \end{pmatrix} \tag{16}$$

Based on the matrix on the right side of equation (16), add the second row to the third row.

$$\begin{pmatrix} 1 & 0 & 8 & 39 \\ 0 & 1 & 6 & 34 \\ 0 & -1+1 & -7+6 & -39+34 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 8 & 39 \\ 0 & 1 & 6 & 34 \\ 0 & 0 & -1 & -5 \end{pmatrix} \tag{17}$$

Based on the matrix on the right side of equation (17), add the third row multiplied by 8 to the first row.

$$\begin{pmatrix} 1 & 0 & 8-8 & 39-5\times 8 \\ 0 & 1 & 6 & 34 \\ 0 & 0 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 6 & 34 \\ 0 & 0 & -1 & -5 \end{pmatrix} \tag{18}$$

Based on the matrix on the right side of equation (18), add the third row multiplied by 6 to the second row.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 6-6 & 34-5\times 6 \\ 0 & 0 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -5 \end{pmatrix} \tag{19}$$

Based on the matrix on the right side of equation (19), multiply the third row by -1.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1\times(-1) & -5\times(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{pmatrix} \tag{20}$$

So the solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$.

b. Use elementary row operations to solve the following system of equations. The answers are x_1 = -2, x_2 = 1, x_3 = 4.

$$Fx = c$$

$$\begin{pmatrix} 2 & -1 & 2 \\ -3 & 1 & -2 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -11 \end{pmatrix}$$

First, create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = \begin{pmatrix} F & c \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 3 \\ -3 & 1 & -2 & -1 \\ 4 & 1 & -1 & -11 \end{pmatrix}$$

Then use elementary row operations on $\tilde{A}$ to create an identity matrix on the left side of $\tilde{A}$. To begin with, based on $\tilde{A}$, add the first row multiplied by $3/2$ to the second row.

$$\begin{pmatrix} 2 & -1 & 2 & 3 \\ -3+2\times(3/2) & 1-1\times(3/2) & -2+2\times(3/2) & -1+3\times(3/2) \\ 4 & 1 & -1 & -11 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & \frac{-1}{2} & 1 & \frac{7}{2} \\ 4 & 1 & -1 & -11 \end{pmatrix} \tag{21}$$

Based on the matrix on the right side of equation (21), multiply the second row by -2.

$$\begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & \frac{-1}{2}\times(-2) & 1\times(-2) & \frac{7}{2}\times(-2) \\ 4 & 1 & -1 & -11 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & 1 & -2 & -7 \\ 4 & 1 & -1 & -11 \end{pmatrix} \tag{22}$$

Based on the matrix on the right side of equation (22), add the first row multiplied by -2 to the third row.

$$\begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & 1 & -2 & -7 \\ 4+2\times(-2) & 1-1\times(-2) & -1+2\times(-2) & -11+3\times(-2) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & 1 & -2 & -7 \\ 0 & 3 & -5 & -17 \end{pmatrix} \tag{23}$$

Based on the matrix on the right side of equation (23), add the second row multiplied by -3 to the third row.

$$\begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & 1 & -2 & -7 \\ 0 & 3+1\times(-3) & -5+(-2)\times(-3) & -17+(-7)\times(-3) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 3 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{24}$$

Based on the matrix on the right side of equation (24), add the second row to the first row.

$$\begin{pmatrix} 2 & -1+1 & 2+(-2) & 3+(-7) \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -4 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{25}$$

Based on the matrix on the right side of equation (25), add the third row multiplied by 2 to the second row.

$$\begin{pmatrix} 2 & 0 & 0 & -4 \\ 0 & 1 & -2+1\times 2 & -7+4\times 2 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{26}$$

Based on the matrix on the right side of equation (26), divide the first row by 2.

$$\begin{pmatrix} 2/2 & 0 & 0 & -4/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{27}$$

So the solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

c. What is the following product?

$$\begin{pmatrix} 13 & 9 & 8 \\ 10 & 7 & 6 \\ -1 & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 14 \\ -24 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 13 & 9 & 8 \\ 10 & 7 & 6 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 14 \\ -24 \end{pmatrix} &= \begin{pmatrix} 13 \times 5 + 9 \times 14 + 8 \times (-24) \\ 10 \times 5 + 7 \times 14 + 6 \times (-24) \\ -1 \times 5 + (-1) \times 14 + (-1) \times (-24) \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} \end{aligned}$$

d. What is the following product?

$$\begin{pmatrix} -1 & -1 & 0 \\ 11 & 10 & 2 \\ 7 & 6 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ -1 \\ -11 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} -1 & -1 & 0 \\ 11 & 10 & 2 \\ 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -11 \end{pmatrix} &= \begin{pmatrix} -1 \times 3 + (-1) \times (-1) + 0 \times (-11) \\ 11 \times 3 + 10 \times (-1) + 2 \times (-11) \\ 7 \times 3 + 6 \times (-1) + 1 \times (-11) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \end{aligned}$$

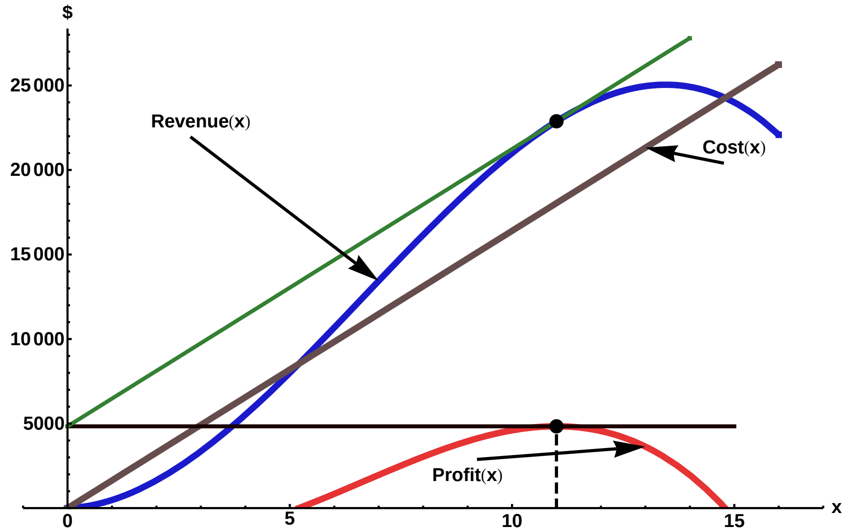
Problem 3. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 10$$

$$input\ price = w = 1640$$

$$y = output = f(x) = 10x + 40x^2 - 2x^3$$

FIGURE 5. Revenue, Cost and Profit



a. Write down an equation that represents profit for the firm.

Profit is given by

$$\begin{aligned} Profit &= Revenue - cost = py - wx \\ &= 10(10x + 40x^2 - 2x^3) - 1640x \\ &= 10(-154x + 40x^2 - 2x^3) \end{aligned}$$

b. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

The derivatives of profit function are given by

$$\frac{d\,Profit}{d\,x} = 10(-154 + 80x - 6x^2) \tag{28}$$

$$\frac{d^2\,Profit}{d\,x^2} = 10(80 - 12x) \tag{29}$$

Set the first derivative, equation (28), to zero.

$$\begin{aligned} & 10(-154 + 80x - 6x^2) &= 0 \\ \Rightarrow \quad & 77 - 40x + 3x^2 &= 0 \\ \Rightarrow \quad & (x - 11)(3x - 7) &= 0 \\ \Rightarrow \quad & x = 11 \quad \text{or} \quad x = 7/3 \end{aligned}$$

c. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

Check the second derivative, equation (29), for $x = 11$ and $x = 7/3$.

$$\left.\frac{d^2\,Profit}{d\,x^2}\right|_{x=11} = 10(80 - 12x)|_{x=11} = 10 \times (80 - 12 \times 11) = -520 < 0$$

$$\left.\frac{d^2\,Profit}{d\,x^2}\right|_{x=7/3} = 10(80 - 12x)|_{x=7/3} = 10 \times (80 - 12 \times (7/3)) = 520 > 0$$

So $x = 11$ is the optimal level which maximizes profit.

d. What is the optimal level of input for this for this firm?

From part (c) of this problem, the optimal level of input for this firm is that $x = 11$.

e. What is the optimal level of output for this for this firm?

When $x = 11$, the output for this firm is given by

$$f(11) = 10 \times 11 + 40 \times 11^2 - 2 \times 11^3 = 2288$$

f. Explain in words why the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the following information in explaining this phenomenon.

$$\begin{aligned} Output =& y = f(x) \\ MP =& \text{Marginal Product} = \frac{d\,f(x)}{dx} = f'(x) = \frac{\Delta y}{\Delta x} \\ Revenue =& pf(x) \\ Cost =& wx \\ Profit =& Revenue - Cost = pf(x) - wx \end{aligned}$$

By setting the derivative of profit with respect to the variable input, we obtain

$$\frac{d\,Profit}{d\,x} = \frac{d\ (pf(x) - wx)}{d\,x} = pf'(x) - w = 0,$$

which is equivalent to

$$pf'(x) = w \tag{30}$$

The left hand side of equation (30) is the price multiplied by the marginal product, i.e., the value of the marginal product while the right hand side of equation (30) is the price of the single variable input. Therefore, at the profit maximizing level of input use, the value of the marginal product is equal to the price of the single variable input.