Chapter 16. Risk and Uncertainty

Part A

Expected Value

X_i = outcome i, p_i = probability of X_i

$$EV = \sum_i p_i X_i.$$

For instance, suppose a person has an idle fund, $100, for one month, and is considering buying a stock for $100, which he needs to sell a month later. Here is a table with possible events and their probabilities.

Probability	Prize
40%	$120
50%	$100
10%	$80

Expected value of the stock is:

$$.4 \times 120 + .1 \times 80 + .5 \times 100 = 48 + 8 + 50 = 106.$$

If $100 is deposited in a savings account, monthly interest payment is .3%, and the total amount of money at the end of the money rises to $100.30. In this case, expected value of stocks is greater than that of savings.

Fair gamble A fair gamble is a lottery $L = [z, -z; 1/2, 1/2]$ **such that** $Ez = 0$.	Remark: In this lottery, a person has an even chance of winning or losing z dollars.
z is a random variable	**Ez = 0. In the long run, one expects to earn nothing from this lottery.**

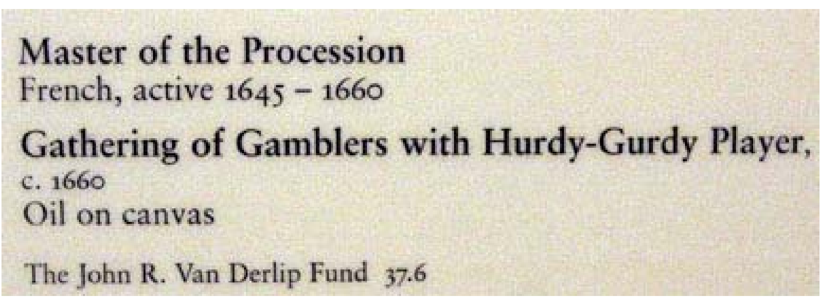
The St. Petersburg Paradox

In 1738, Daniel Bernouilli proposed the following lottery to illustrate that people do not maximize expected value. [In a different version, his cousin Nicholas Bernouilli proposed the lottery, and Daniel offered a solution].

Flip a fair coin until it lands heads up. The coin has a 50:50 chance of landing heads up on any given flip. *A player flips a coin until the first head appears*. (and the game ends.)  Medallion of Diocletian	Payoff (X_i) If the first head appears on the first toss, the player wins \$2 and the game ends. If the first head appears on the second toss, the player wins \$4 and the game ends. If the first head appears on the third toss, the player wins \$8 and the game ends, and so on. ... If the first head appears on the n^{th} toss, then the player wins $\$2^n$, and the game ends.

Outcome		prize (X_i)	probability (p_i)
1	H	2^1	1/2
2	TH	2^2	$1/2^2$
3	TTH	2^3	$1/2^3$
4	TTTH	2^4	$1/2^4$
...	…		
n	T…TH	2^n	$1/2^n$

What is the expected value of the St. Petersburg's lottery?	Bernouilli's answer Expected Value $2 \times (1/2) + 2^2 \times (1/2^2) + 2^3 \times (1/2^3) + ...$ $= 1 + 1 + 1 + ... = \infty$.
	Choi: no.
Why is it a paradox?	If a person is maximizing expected monetary value, he would be willing to pay an infinite amount of money to take part in the gamble. The paradox occurs because individuals will NOT pay an unlimited amount of money to play this game. A reasonable estimate for the amount that an individual is willing to pay is $20.



Daniel Bernouille's (1730) solution.	Choi's alternative
The utility function is monotone increasing and concave, $U'(w) > 0, U''(w) < 0$.	Even if the utility function is linear, the lottery is not worth much money, because the game cannot continue indefinitely because of the inability of the house to pay the prize beyond a certain amount.

	$\$2^{10} = 1024 \simeq \$10^3$, $\$2^{20} \simeq \$10^6$, $\$2^{30} \simeq \$10^9$, $\$2^{40} \simeq \$10^{12}$, ($1 trillion) $\$2^{44} \simeq \$16 \times 10^{15}$, ($16 trillion).
	Thus, the banker (who sells the lottery tickets) cannot pay $16 trillion, and the game ends when no head appears for the first 44 tosses. The maximum possible value of the lottery ticket is $44 for gamblers maximizing expected value (risk neutral gamblers)

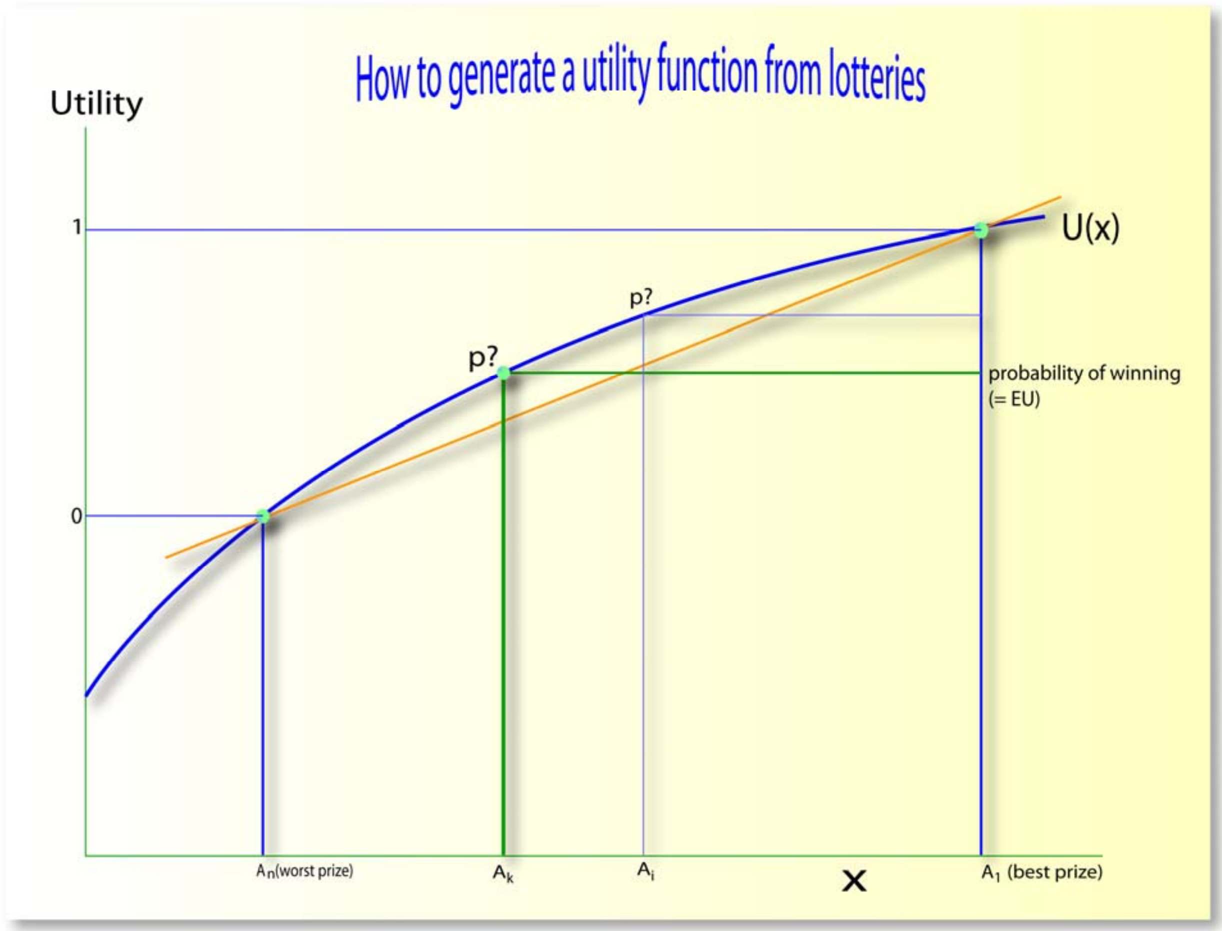
Ordinal Utility

Under certainty (perfect information), the assumptions of completeness, reflexivity, and transitivity imply the existence of an ordinal utility function that represents rankings of consumption bundles.

When there is uncertainty in the world, we need a stronger utility concept than ordinal utility.

How to Construct a Cardinal Utility Function (to deal with decision making under uncertainty)

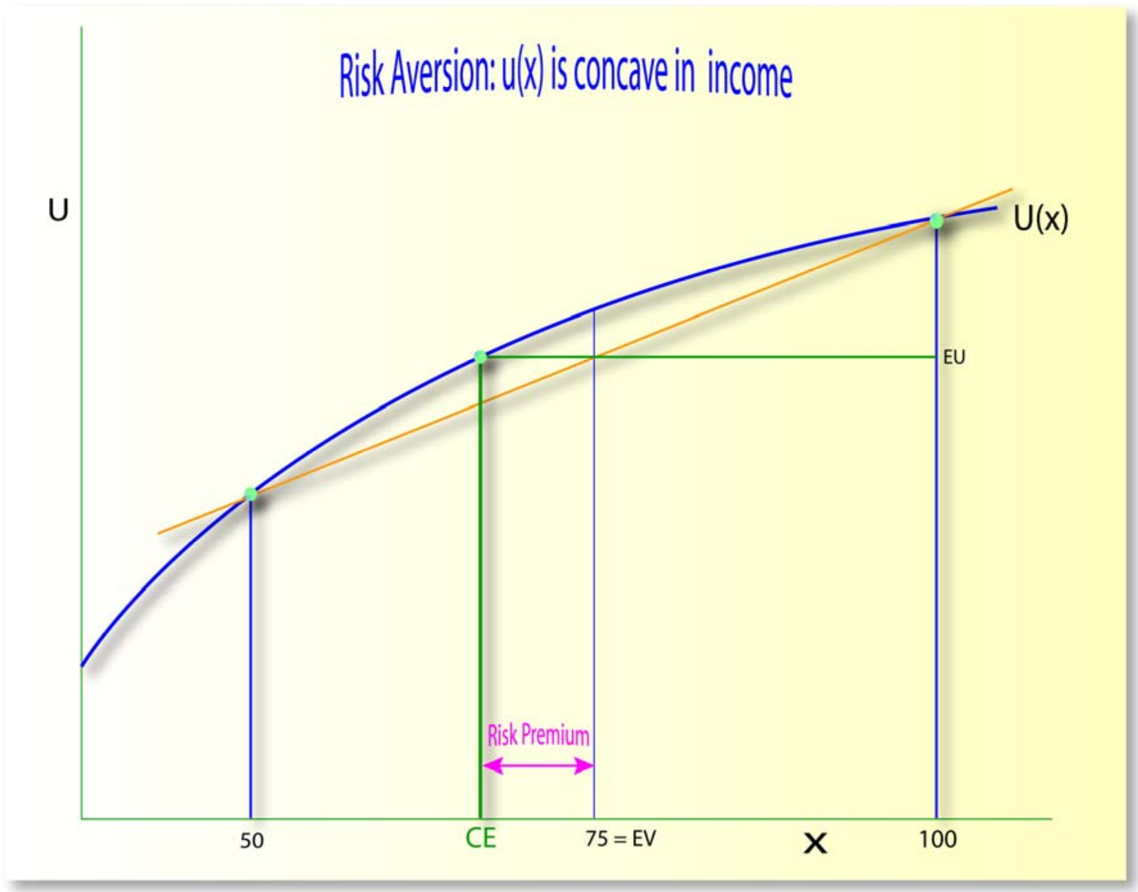
How does one construct cardinal utilities for these prizes?	First, assign a utility number of 0 to the worst prize A_n, and a utility number of 1 to the best prize A_1. $U(\text{worst prize}) = 0,$ $U(\text{best prize}) = 1.$
How about a gamble between the two prizes? $z = [A_1, A_n, p, 1-p].$	Then expected utility of this gamble is: $EU = pU(A_1) + (1-p)U(A_n)$ $= p \times 1 + (1-p) \times 0 = p.$
Wait a minute! The utility of the worst prize is ZERO and hence does not count. Then ask what probability *p* will the gambler be indifferent between any prize A_k **with certainty** and the gamble?	Yes! The utility of any prize A_k is exactly that probability, $U(A_k) = p.$



Repeat this process to get the cardinal utility of each prize. This yields the cardinal utility function *U* desired.

Risk Premium

Consider a gamble with an even chance of winning \$100 or \$50. Its **expected value** of z = [100, 50; 1/2,1/2] is: $$Ez = \frac{1}{2}\times 100 + \frac{1}{2}50 = 75.$$	**Expected utility** is $$EU = \frac{1}{2}\times U(100) + \frac{1}{2}\times U(50).$$
Expected utility is then the subjective value, whereas expected value is the objective value of the gamble?	Right. Moreover, if the utility function is concave, expected utility is less than the utility of expected value of the lottery.
Show me!	For example, if an individual has a square root utility function, $U(x) = \sqrt{x}$, which is by the way concave ($U''(z) < 0$), then $$EU(z) = \frac{1}{2}\times 10 + \frac{1}{2}\times\sqrt{50} < \sqrt{75} = U(Ez).$$ That is, expected utility of the gamble (about 8.5035) on the left hand side of the inequality in (1) is less than the utility of expected value of the gamble on the right hand side. This example shows that a risk averse individual prefers the certain income (EV) to the uncertain income or the gamble.
What is the implication?	A risk averse individual is willing to pay some amount of money to avoid the gamble. The maximum amount he is willing to pay to avoid the gamble is called the risk premium.



How is the risk premium calculated?	First, note that any gamble is equivalent to a certain income, which is called Certainty Equivalent.
	Certainty Equivalent (CE) of a gamble is the certain value which yields the same level of utility as the gamble. For the square root utility function, $\sqrt{x} = 8.5035$, certainty equivalent is $x = (8.5035)^2$, or $$CE = 72.32 < 75.$$

Does that mean the individual is willing to put up with a certain amount, which is less than the fair or expected value?	**Yes.** **Risk Premium** π of a gamble is the maximum amount the individual is willing to pay, and is implicitly defined by $EU(z) = U(Ez - \pi).$
Do we need to calculate expected utility of the lottery first?	**Yes.** Given the square root utility function, expected utility of the above lottery is $EU(z) = U(75 - \pi) =$ $\frac{10 + \sqrt{50}}{2} \approx \frac{17.07}{2} = 8.5035.$
The same level of utility could have been obtained with sure income of $(8.5035)^2 = 72.31$	**Right.** Then risk premium is $Ez - CE = 75 - 72.31 = 2.69.$ That is, the individual would rather take the certain income of $72.31, which is $2.69 less than the expected value of the gamble ($75).

Risk Preference

Utility is monotone increasing and convex ($U' > 0, U'' > 0$).