Chapter 16, Part C
Investment Portfolio

	Risk is often measured by variance. For the binary gamble $L = [z, -z; 1/2, 1/2]$, recall that expected value is $Ez = \frac{1}{2}z + \frac{1}{2}(-z) = 0.$ For this binary gamble, z represents the gamble size, and the variance of the gamble is z^2. That is, the variance of this lottery is simply the gamble size squared.
How is variance measured for general lotteries?	In general, gambles are more complex. Variance is generally defined as $Var(X) = E(X - EX)^2 = \sum_{i=1}^{n} p_i (X_i - EX)^2.$ The idea is to punish large deviations from the mean, and hence the errors or deviations from the mean are squared. Variance is then the expected value of the squared errors or deviations.

Example:

Probability	Prize
40%	$120
50%	$100
10%	$80

Mean of this lottery was $106. What is the variance of this lottery?

$$Var(X) = .4 \times (120-106)^2 + .5 \times (100-106)^2 + .1 \times (80-106)^2$$
$$= .4 \times 14^2 + .5 \times 6^2 + .1 \times 26^2 = .4 \times 56 + .5 \times 36 + .1 \times 676.$$

Portfolio Choice with One Risky Asset

Consider an individual investing in two assets: a risky asset (stocks) and a safe asset (Treasury bills). Let X and Y denote the returns from the risky and riskless assets, respectively. Let σ^2 denote the variance of X, and let α denote the fraction of the portfolio invested in the risky asset.

What is the problem with risky assets?	Expected return on the risky asset is higher than that of the riskless asset, i.e., $EX > Y$. However, there is a small chance that one may earn less than from the safe asset. Accordingly, the risky asset has a higher standard deviation (or variance) than the riskless asset.
Then one can choose a convex combination of the safe and risky asset.	Right. Let the initial asset w = 1. (If not, the total return will be wR. In the following analysis, initial wealth w does not play a role.) The total return is a random variable, $R = \alpha X + (1-\alpha)Y.$ (1) Expected return is $ER = \alpha \bar{X} + (1-\alpha)Y,$ (2) where a bar denotes expected value, and $\bar{X} = EX$ is expected return from the risky asset. The variance of the portfolio is $\text{var} = \alpha^2 \sigma^2.$ (3)

If one is risk averse, he will maximize expected utility of the portfolio.	For portfolio analyses, expected utility is often expressed as a weighted average of the expected return and variance, $EU = ER - k\alpha^2\sigma^2$ $= \alpha \bar{X} + (1-\alpha)Y - k\alpha^2\sigma^2. \qquad (4)$ Here, k represents the penalty per unit of variance.
If the investor is risk neutral, then k is 0. If he is risk averse, k > 0. Thus, k can be treated as a risk aversion measure.	Right. This is different from the Arrow-Pratt absolute risk aversion measure, $A = -\frac{U''(w)}{U'(w)}$ or relative risk aversion measure, $R = -w\frac{U''(w)}{U'(w)},$ where *w* is income or wealth. These two measures are widely used in the theoretical literature. For more details, see another note.
I notice that the absolute and relative risk aversion measures are NOT parameters, but functions, which depend on the level of wealth.	Right. For this reason, they are impractical for everyday use. Business people buying and selling risky assets need rough and ready measures of risk aversion.

	Also, the mean-variance approach is often used for practical reasons that mean and variance of risky assets are easily obtainable.
Expected utility then depends on three parameters: mean (EX), variance (σ^2), and the risk aversion measure, k	Right. Just differentiate expected utility in (4) with respect to α, we can find its optimal value.
The first order condition is: $\frac{dEU}{d\alpha} = (\bar{X} - Y) - 2k\alpha\sigma^2 = 0.$	You can solve for α.
The optimal fraction of your investment in the risky asset is: $\alpha^* = \frac{\bar{X} - Y}{2k\sigma^2}.$	It is interesting to note that if the two assets have identical expected rates of return, i.e., $\bar{X} = Y$, then α* = 0.
That makes sense. If two assets have the same expected rates of return, there is no point in taking risk.	Right. A risk averse investor holds a risky asset only if it yields a higher expected return.
If EX > Y, then one will hold some risk asset and some riskless asset.	Right.
If the risk of the risky asset increases, Wouldn't the investment in the risky asset decline?	Right. Differentiating (6) with respect to σ^2, we get

	$\frac{\partial \alpha^*}{\partial \sigma^2} = -\frac{2k(\bar{X} - Y)}{\left(2k\sigma^2\right)^2} < 0.$ (7) That is, as the variance of the risky asset increases, the investor's holding of the risky asset declines.
	However, (6) shows that no matter how large the variance is, the investor always holds some portion of its portfolio in the risky asset, for the simple reason that it has a higher expected rate of return.
If Jill is more risk averse than Jack, wouldn't she invest less in the risky asset?	You can verify that by differentiating (6) with respect to k gives $\frac{\partial \alpha^*}{\partial k} = -\frac{2\sigma^2(\bar{X} - Y)}{\left(2k\sigma^2\right)^2} < 0.$ (8) That is, a more risk averse person invests less in the risky asset.

Two Risky Assets

I understand now that one needs to invest in the risky assets if expected returns are higher than in safe assets. But there are many risky assets. Which ones does one choose?	There is a proverb that says "Don't put all your eggs in one basket." The current financial crisis is partly caused by big banks and insurance companies that have ignored this adage.
If one risky asset has a higher expected return than the other, should one invest all his money into the asset with higher expected return?	No. If one's wealth is infinite or one is risk neutral, in the long run it might be the best strategy. However, long run survival requires that the investor survive each period.

How does one choose among many risky assets?	For simplicity, consider the case with two risky assets. Assume that the return on the two risky assets are independently distributed or uncorrelated, i.e., their covariance is zero.
Suppose that one asset is riskier than another, i.e., the variance of one asset is greater than that of the other. In this case, should we abandon the riskier asset altogether?	Not if it has a higher expected return.
How do we choose between two risky assets?	Expected utility of a portfolio with two risky assets can then be written as: $EU = ER - k\alpha^2\sigma^2$ $= \alpha\bar{X} + (1-\alpha)\bar{Y}$ (9) $-k\left(\alpha^2\sigma_x^2 + \beta^2\sigma_y^2\right),$ where σ_i^2 is the variance of the risky asset i, and α and β are the fractions of the portfolio held in the risky assets, X and Y, respectively. Thus, $\alpha + \beta = 1.$
I know how to do this. The Lagrangian function associated with this problem is: $\mathcal{L} = \alpha\bar{X} + \beta\bar{Y}$ $-k\left(\alpha^2\sigma_x^2 + \beta^2\sigma_y^2\right)$ (10) $+\lambda[1-\alpha-\beta].$	Partially differentiating (10) with respect to α and β gives the FOCs: $\bar{X} - 2k\alpha\sigma_x^2 - \lambda = 0,$ (12) $\bar{Y} - 2k\beta\sigma_y^2 - \lambda = 0,$ (13)

	$1-\alpha-\beta=0.$
I regret that I asked the question. But from (12) and (13) , here is the solution: $\alpha^* = \dfrac{2k\sigma_y^2 + \bar{X} - \bar{Y}}{2k(\sigma_x^2 + \sigma_y^2)}.$	Note that if $\sigma_y^2 = 0$, that is, if Y is a riskless asset, then (14) reduces to (6). If the two assets have the same return ($\bar{X} = \bar{Y}$), then optimal portfolio reduces to: $\alpha^* = \dfrac{\sigma_y^2}{(\sigma_x^2 + \sigma_y^2)}$, (15) which is independent of risk aversion. It simply depends on the variances of the two risky assets. For all risk averters, investment in the riskier asset is smaller.
Question: If Y has a higher variance than X, one's investment in X asset is greater than that of Y. Shouldn't he get rid of the riskier asset altogether?	No. The solution does not say that α = 1. (This proves the wisdom of the proverb "Don't put all eggs in one basket," even if one asset is riskier than the other) Just because one asset has less variance and hence less risky, it does not follow that one should get rid of the riskier asset at all.
If the two risky assets have the same variances and same expected rates of return, then $\alpha = \dfrac{1}{2}.$	If two assets have identical rates of return and risks, then half-and-half is an optimal solution. Such a portfolio will yield the same expected return as that when all eggs are in one basket but will minimize the variance of the return of the total portfolio.
What if the variances of the two risky assets were also	Then the variance of the portfolio is:

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$$\sigma^2 = \alpha^2\sigma_x^2 + (1-\alpha)^2\sigma_x^2$$
$$= \alpha^2\sigma_x^2 + (1-2\alpha+\alpha^2)\sigma_x^2 \quad (1.17)$$
$$= \sigma_x^2(1+2\alpha^2-2\alpha).$$

If $\alpha = 0$, this reduces to σ_x^2. If $\alpha = 1$, it also reduces to σ_x^2. But somewhere between both ends, variance of the portfolio is smaller.
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I can see that.

For instance, if $\alpha = 1/3$, it reduces to

$$\sigma^2 = \sigma_x^2(1+2/9-6/9)$$
$$= \sigma_x^2(5/9) < \sigma_x^2.$$
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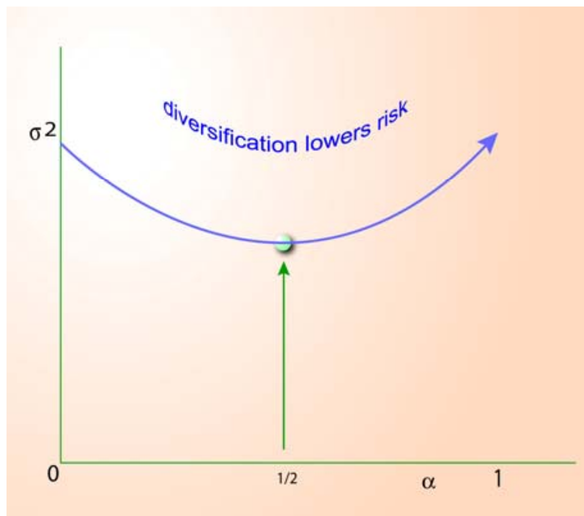
We can see how the variance changes as α increases as in the diagram below.



The optimal solution is

$$\alpha^* = \frac{1}{2}.$$
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About the Emerging Markets

If the financial assets (e.g., stocks) of two countries have the same expected rates of return and variances, shouldn't a shrewd investor should hold an even share of both assets?
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Yes and No.

American investors generally hold more than 50% of their portfolio in domestic assets, and only recently began to buy stocks in emerging markets.

Foreign securities may have higher rates of return, but they are also considered riskier.
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