

Ch.3 Growth and Accumulation

I. Introduction

- A. Growth accounting and source of economic growth
- B. The neoclassical growth model: the Simple Solow growth model
 - No population growth
 - No technical progress
- C. Golden rule level of capital stock: maximizing consumption
- D. The Solow growth model with the population growth and the technical progress

II. Production function and constant return to scale

- A. Production function: $Y = AF(K, N)$

Y: output or GDP

K: capital input

N: labor input

A: a measure of technology efficiency (or productivity)

- B. Constant returns to scale (CRTS)

- A function has CRTS if an equal percentage increases in *all* its variables results in that same percentage increase in the function value
- $\lambda Y = AF(\lambda K, \lambda N)$

- C. Euler's theorem

- For a function that has constant returns to scale (or is homogeneous of degree one), the sum of the marginal products multiplied by their levels will give the

level of the function. Specially for the case of a constant returns to scale production function it implies

$$Y = AF(K, N) = A(MPK * K + MPN * N) = A * MPK * K + A * MPN * N$$

D. Changes in factors of production

- Total change in Y consists of three parts

- a. Changes in Y as K changes

$$\Delta Y = \frac{A\Delta F(K, N)}{\Delta K} \Delta K = A * MPK * \Delta K$$

- b. Changes in Y as N changes

$$\Delta Y = \frac{A\Delta F(K, N)}{\Delta N} \Delta N = A * MPN * \Delta N$$

- c. Changes in Y as A changes

$$\Delta Y = \Delta A * F(K, N)$$

- d. Total change in Y (sum of a, b, and c)

$$\Delta Y = \Delta A * F(K, N) + A * MPK * \Delta K + A * MPN * \Delta N$$

- e. Growth rate in Y can be derived by dividing both sides with Y

$$\begin{aligned} \frac{\Delta Y}{Y} &= \frac{\Delta A}{A} + \frac{A * MPK * \Delta K}{Y} + \frac{A * MPN * \Delta N}{Y} \\ &= \frac{\Delta A}{A} + \frac{A * K * MPK}{Y} * \frac{\Delta K}{K} + \frac{A * N * MPN}{Y} * \frac{\Delta N}{N} \end{aligned}$$

- f. Growth accounting equation

Let $\frac{A * K * MPK}{Y}$ be θ . Then, from the Euler's theorem (see

appendix), $\frac{A * N * MPN}{Y} = 1 - \theta$. Therefore the above growth rate can

be expressed as the following.

$$\frac{\Delta Y}{Y} = \theta \frac{\Delta K}{K} + (1 - \theta) \frac{\Delta N}{N} + \frac{\Delta A}{A}$$

Output growth = (labor share X labor growth) + (capital share X capital growth) + technical progress (or *growth of total factor productivity*)

III. Source of economic growth

- The rate of economic growth depends on the growth rates of factors of production, their marginal productivities, and technical changes

A. Growth accounting

$$\frac{\Delta Y}{Y} = \theta \frac{\Delta K}{K} + (1 - \theta) \frac{\Delta N}{N} + \frac{\Delta A}{A}$$

- i. Growth accounting is an attempt to quantify different elements that contribute to economic growth over time
- ii. Example
Suppose that $\theta = 0.3$ and $1 - \theta = 0.7$. Now let capital grow by 5% and labor by 10%. Assume that technology is constant. Then, the percentage increase in Y is:

$$\frac{\Delta Y}{Y} = 0.3 * 0.05 + 0.7 * 0.1 = 0.085 = 8.5\%$$

B. Accounting for growth per capita output

- In many cases, we care about the income of an average person (GDP per capita) and the growth rate of GDP per capita.
With simple algebra, we can derive the accounting for growth per capita output from the above the growth accounting

Define $y \equiv \frac{Y}{N}$ and $k \equiv \frac{K}{N}$ (GDP per capita and capital-labor ratio,

respectively). Notice that $Y = yN$ and $K = kN$.

$$\Delta Y = N\Delta y + y\Delta N$$

$$\rightarrow \frac{\Delta Y}{Y} = \frac{N\Delta y}{Y} + \frac{y\Delta N}{Y} = \frac{\Delta y}{y} + \frac{\Delta N}{N}$$

Similarly,

$$\Delta K = N\Delta k + k\Delta N$$

$$\rightarrow \frac{\Delta K}{K} = \frac{N}{K}\Delta k + \frac{k}{K}\Delta N = \frac{\Delta k}{k} + \frac{\Delta N}{N}$$

Substitute these two formulas into the growth accounting.

$$\frac{\Delta Y}{Y} = \theta \frac{\Delta K}{K} + (1-\theta) \frac{\Delta N}{N} + \frac{\Delta A}{A}$$

$$\leftrightarrow \frac{\Delta y}{y} + \frac{\Delta N}{N} = \theta \left(\frac{\Delta k}{k} + \frac{\Delta N}{N} \right) + (1-\theta) \frac{\Delta N}{N} + \frac{\Delta A}{A}$$

$$\leftrightarrow \frac{\Delta y}{y} = \theta \frac{\Delta k}{k} + \frac{\Delta A}{A}$$

This is the accounting for growth per capita output

C. Empirical estimates of growth

- Study by Robert Solow

Data for the U.S. from 1909-1949

- Average annual growth of total GDP: 2.9%

Source of growth

Capital growth: 0.32%

Labor growth: 1.09%

Technical progress: 1.49%

- Average annual growth of GDP per capita: 1.81%

Source of growth

Growth of Capital-labor ration: 0.32%

Technical progress: 1.49%

D. Other factors affecting the growth

– Natural resource

– Human capital

Society's stock of skills is increases by investment in human capital through schooling, on-the job training, and other means. These stocks of skills affect the growth of GDP

IV. The neoclassical growth model: the Simple Solow model

A. Constant return to scale (CRTS) production function

– CRTS production function implies $\lambda Y = F(\lambda K, \lambda N)$

Let $\lambda = 1/N$. Then:

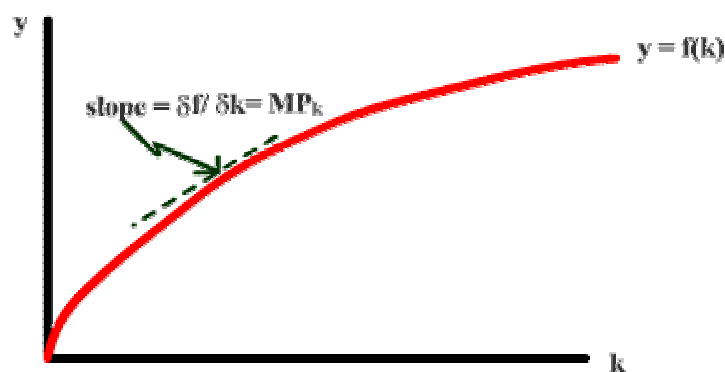
$$\frac{Y}{N} = F\left(\frac{K}{N}, \frac{N}{N}\right)$$

$$\Leftrightarrow y = F(k, 1) = f(k)$$

where $y = Y/N$, $k = K/N$, and $F(k, 1) = f(k)$. y is GDP per capita (or output per worker), and k is the capital-labor ratio (or capital per capita)

Note: $f(k)$ is called the intensive form of the production function

– A graphical depiction of the production relation is:



- The production function shows diminishing marginal returns to capital
- Slope of the production function is the marginal product of capital
- Production function shows the production of goods

B. Consumption and investment identities

- Now, we look at the demand for goods. In this simple model, the demand for goods consists of consumption plus investment.

$$Y = C + I$$

Then, the demand of an average worker is

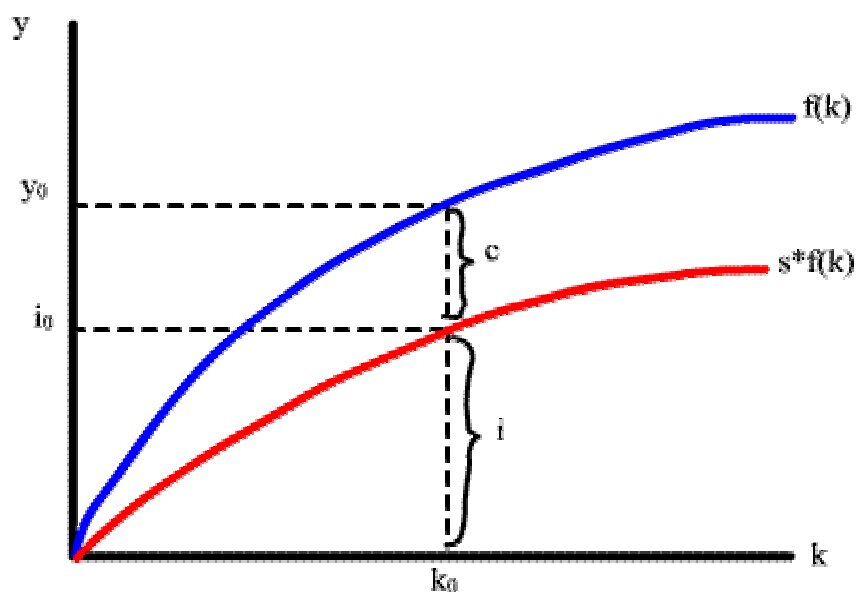
$$y = c + i$$

where $c = C/N$ and $i = I/N$.

- Assume that the average worker invests a constant fraction of income. That is, $i = sy$. Then, the average worker consumes a constant fraction of income also:

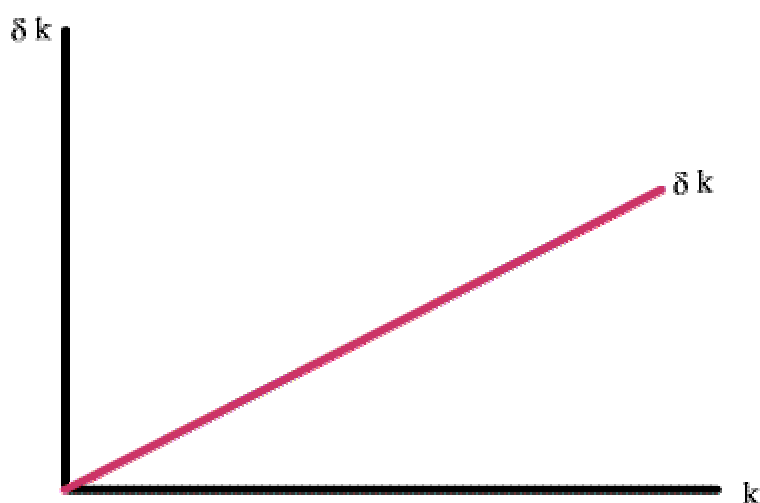
$$c = (1-s)y$$

- s is called the *saving rate*



C. Investment and the capital stock

- Stock of capital in this economy is used to produce goods.
- Investment of workers (or consumers) will add to the stock of capital and will increase the output level. However, some capital will wear out each year with constant rate δ . δ is called the depreciation rate.



- Therefore, the change in the capital stock consists of the newly added capital due to the investment and the foregone amount of the capital stock due to the depreciation:

$$\Delta k = i - \delta k$$

Notice that $i = sy = sf(k)$. Therefore, the change in the capital stock can be rewritten as:

$$\Delta k = sf(k) - \delta k$$

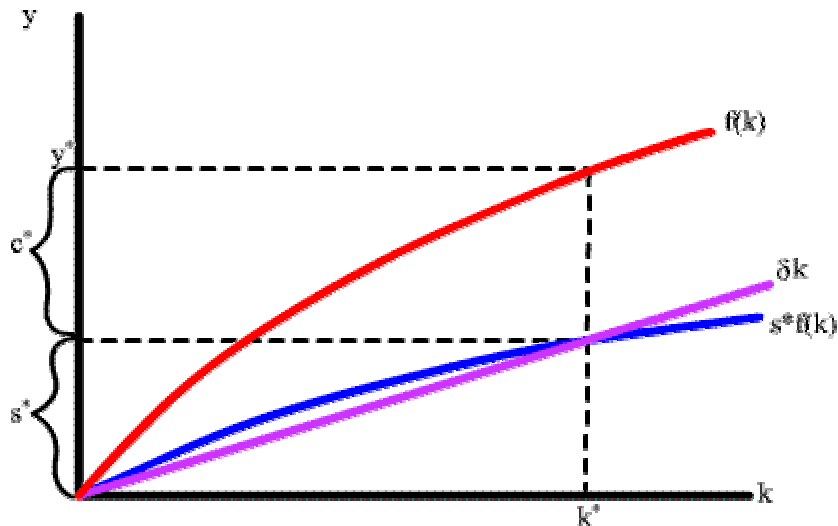
D. The steady state level of capital

- The steady state level of capital stock is the stock of capital at which investment and depreciation just offset each other: $\Delta k = 0$.

- In the steady state level of capital stock,

$$\Delta k = 0 = sf(k) - \delta k$$

$$\rightarrow sf(k) = \delta k$$



- If $k < k^*$, then $i > \delta k$, so $k \uparrow$ toward k^* .
When $i > \delta k$, saving and investment is larger than the depreciation, and the capital stock in this economy grows toward k^* (steady state capital level)
- If $k > k^*$, then $i < \delta k$, so $k \downarrow$ toward k^* .
When $i < \delta k$, saving and investment is less than the depreciation, and the capital stock falls toward k^*

Note: $sf(k) = i$

- Once the economy gets to k^* , the capital stock doesn't change
- In this simple model, there is *no growth* in the output in the steady state because the capital stock doesn't change.

– Example

$y = \sqrt{k}$, $s = 0.25$, $\delta = 0.1$. What is the steady state capital level k^* ?

In the steady state, $\Delta k = 0 = sf(k^*) - \delta k^* \leftrightarrow sf(k^*) = \delta k^*$.

For this example,

$$sf(k^*) = \delta k^* \rightarrow 0.25\sqrt{k^*} = 0.1k^*$$

$$\rightarrow k^* / \sqrt{k^*} = 0.25 / 0.1$$

$$\rightarrow k^* = 6.25$$

Note 1

In this simple model, in the steady state, the capital per worker (y) doesn't change.

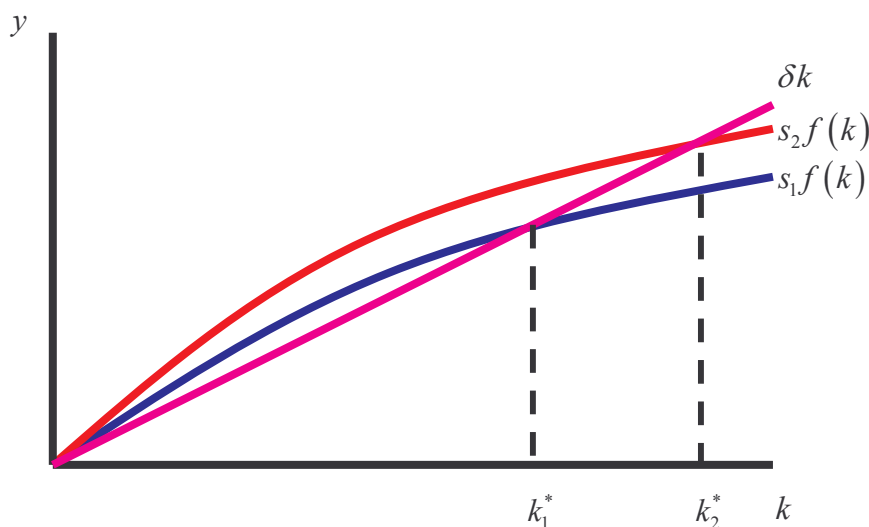
Therefore, the output per worker (k) also doesn't change. Because there is no population growth, *total output* ($Y = yN$) and *capital stock* ($K = kN$) also don't grow, which is unrealistic.

Note 2

The steady state is important in the growth theory because it is a good approximation for long run analysis.

E. Changes in the rate of saving

- As the saving rate changes, so will the steady state capital stock. In the above numerical example, if the saving rate were 0.3, the steady state capital stock would be 0.9.
- Graphical analysis
 - Saving rate increases from s_1 to s_2



- The steady state capital stock increases.

- Higher saving rates lead to higher level of capital and output in the steady state. But they may also lead to very low levels of consumption.

F. The Golden rule (see Ch.4)

- As we saw in the above example, government can move the economy to a new steady state. What is the best steady state? The best steady state that the government can choose is the steady state at which consumption is maximized (the golden rule steady state).
- The Golden Rule level of capital accumulation is the steady state with the highest level of consumption
- Remind $y = c + i \rightarrow c = y - i$. This can be rewritten as $c = f(k) - sf(k)$.

In the steady state, $sf(k^*) = \delta k^*$. Therefore, the consumption in the steady state becomes

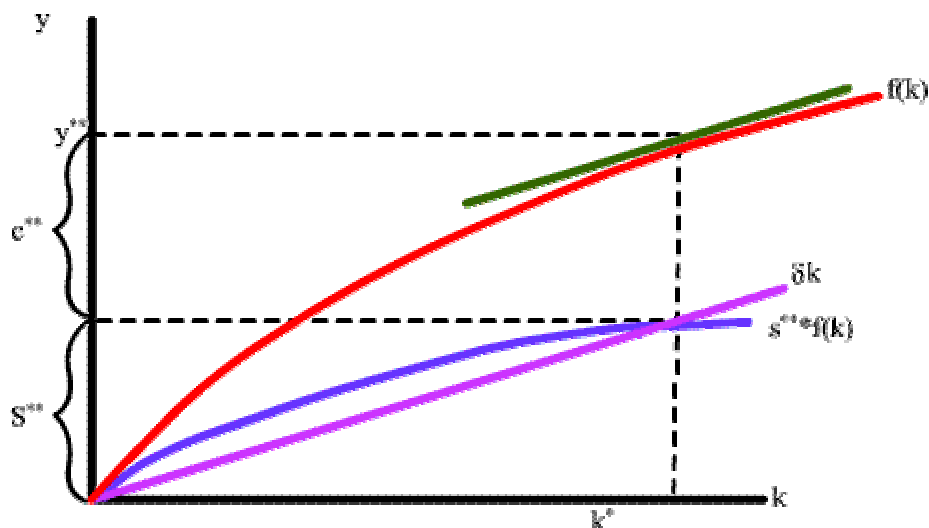
$$c^* = f(k^*) - \delta k^*$$

Since we want to maximize the consumption, we take the derivative of the above

formula and set it equal to zero

$$\frac{dc^*}{dk^*} = \frac{df(k^*)}{dk^*} - \delta = 0$$

$$\rightarrow MPK = \delta$$



- The golden rule level of capital stock is k_{gold} that satisfies $MPK = \delta$.

Note: the golden rule level of capital stock is a steady state capital stock.

- Example

$$y = \sqrt{k}, \quad \delta = 0.1, \quad \text{and current saving rate } (s) = 0.25$$

What is the golden rule level of capital stock?

How government can achieve the golden rule level?

In the golden rule level of capital stock,

$$MPK = \delta$$

$$\rightarrow \frac{d}{dk} \sqrt{k} = \frac{1}{2\sqrt{k}} = 0.1$$

$$\rightarrow k_{gold} = 25$$

When the saving rate is s , the steady state satisfies the following equation:

$$sf(k) = \delta k$$

$$\rightarrow s\sqrt{k} = 0.1k$$

$$\rightarrow k^{ss} = 100s^2$$

When the saving rate (s) satisfies $k^{ss} = 100s^2 = 25$, then the golden rule steady state can be achieved.

$$100s^2 = 25$$

$$\rightarrow s = 0.5$$

Therefore, the government can achieve the golden rule steady state by increasing the saving rate up to 0.5.

G. Introducing population growth

- In the previous simple growth model, the output and the capital stock don't grow in the steady state. Clearly it is unrealistic. Now, we will consider the population growth in the Solow model. In this version of the Solow model, the output and the capital grow to feed more population.
- Let n represent growth in the labor force. As this growth occurs, $k = K/N \downarrow$ and $y = Y/N \downarrow$ over time.
- First of all, notice that the production function and consumption/investment rule are the same as those in the previous simple model.

$$y = f(k)$$

$$i = sy$$

$$c = (1-s)y$$

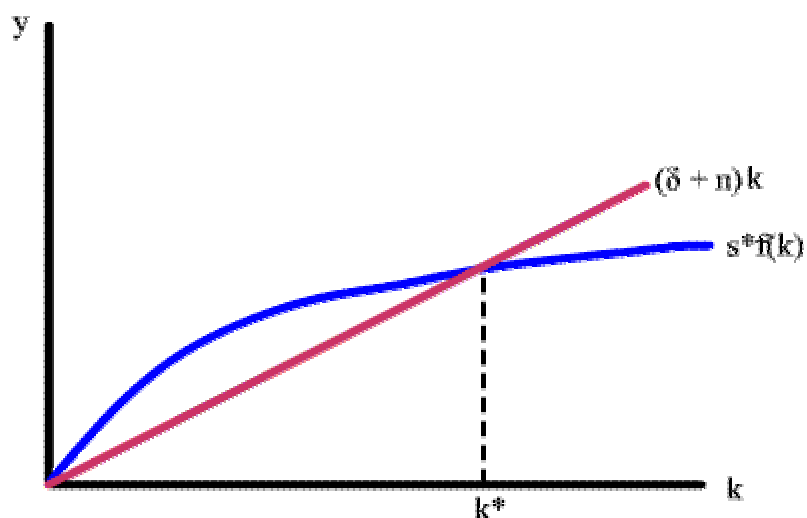
- However, the population growth affects capital accumulation. As N increases at the rate n , the change in k is now:

$$\Delta k = sf(k) - \delta k - nk$$

where nk represents the decrease in the capital stock per unit of labor from

having more labor. Now, the economy accumulates more capital to feed more population.

- From the definition of the steady state level of capital. In the steady state, $\Delta k = 0$. Thus, the steady state condition becomes $sf(k^*) = (\delta + n)k^*$.



- Notice that the straight line shows the outflow of capital stock, and the concave curve (saving schedule) shows the inflow of capital stock. Therefore, at k^* , the inflow and the outflow of capital stock are balanced, and $\Delta k = 0$.
- In the steady state, there's no change in k so there's no change in y , which means that *output per worker* and *capital per worker* are both constant. Since, however, the labor force is growing at the rate n (i.e., $N \uparrow$ at the rate n), Y (not y) is also increasing at the rate n . Similarly, K (not k) is increasing at the rate n .
 - $Y = yN$: Increases at the rate n
 - $K = kN$: Increases at the rate n
 - y : No change
 - k : No change
- Golden rule

In the steady state, the consumption level is $c = f(k^*) - sf(k^*)$. Since, in the steady state, $sf(k^*) = (\delta + n)k^*$, the consumption in the steady state can be rewritten as $c^* = f(k^*) - (\delta + n)k^*$.

The golden rule steady state is the steady state that maximizing consumption level.

$$\frac{dc^*}{dk^*} = \frac{df(k^*)}{dk^*} - (\delta + n) = 0$$

$$\rightarrow MPK = \delta + n$$

Now, the condition for the golden rule steady state becomes

$$MPK = n + \delta$$

H. Introducing Technological Progress

- The production function may change over time as the technology improve
- We will assume that technological progress occurs because of increased efficiency of labor.

$$Y = F(K, N * E)$$

where E represents the efficiency of labor. Now, $N * E$ is the effective labor input.

- We will assume that E grows at the rate g .

Assuming constant returns to scale, the production function can now be written as:

$$\hat{y} = \frac{Y}{N * E} = F\left(\frac{K}{N * E}, \frac{N * E}{N * E}\right) = f(\hat{k})$$

where $\hat{k} = K / (N * E)$.

Note: In this version of Solow model, we will consider the output per 'effective' labor and the capital per 'effective' labor.

- Consumption and investment rules

Like before, $Y = C + I$

Divide both sides by $N * E$. Then,

$$Y / (N * E) = C / (N * E) + I / (N * E)$$

$$\rightarrow y = \hat{c} + \hat{i}$$

Assume that the average worker invest a constant fraction of income. Then,

$$\hat{i} = s\hat{y}$$

$$\hat{c} = (1-s)y$$

- Now, the change in \hat{k} is now:

$$\Delta\hat{k} = sf(\hat{k}) - (n + g + \delta)\hat{k}$$

The above formula can be derived from the total differentiation of $\hat{k} = K / (N * E)$ (see appendix 2). Intuitively, the economy needs more capital to compensate the foregone amount of capital due to the depreciation ($\delta\hat{k}$) and to feed more population ($n\hat{k}$). In addition, the improved technology increases the output, and this more production needs more capital ($g\hat{k}$).

- The steady state condition with the technical progress is:

$$sf(\hat{k}) = (n + g + \delta)\hat{k}$$

- At the steady state, \hat{y} and \hat{k} are constant.

Since $\hat{y} = Y / (N * E)$ and N grows at the rate n while E grows at the rate g , then Y must grow at the rate $n + g$.

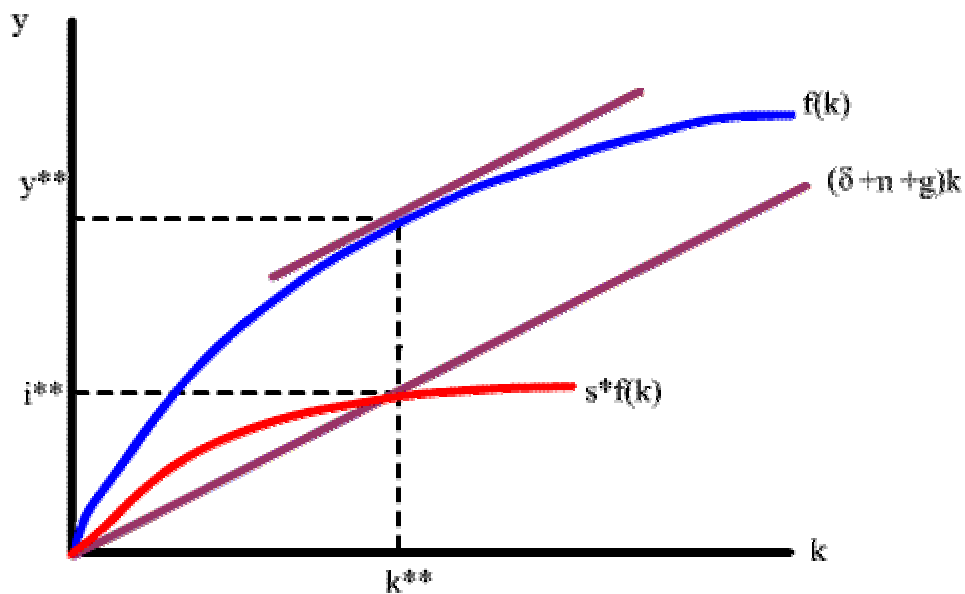
Similarly since $\hat{k} = K / (N * E)$, K also grow at the rate of $n + g$.

- The Golden Rule level of capital accumulation satisfies the following condition:

$$MPK = f'(\hat{k}) = (n + g + \delta)\hat{k}$$

$$\leftrightarrow MPK - \delta\hat{k} = (n + g)\hat{k}$$

The implication is that the marginal product of capital net of depreciation must equal the sum of population and technological progress.



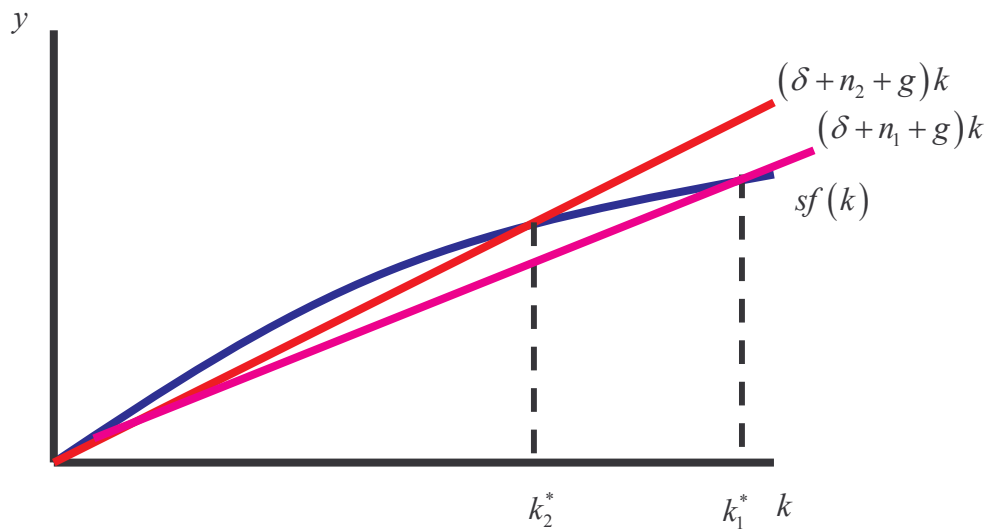
E. Experiments and policy implication

Consider the following situations

- Effect of the change in the birth rate (or population growth rate) on the growth rates

Increase in n (from n_1 to n_2)

- Investment requirement (straight line) becomes steeper
- The steady state level of capital (per worker) is reduced
- The steady state output (per worker) is reduced



Growth rate:

In the steady state, the growth rates of Y and K increases

Note: the growth rate of y and k are always zero in the steady state from the definition

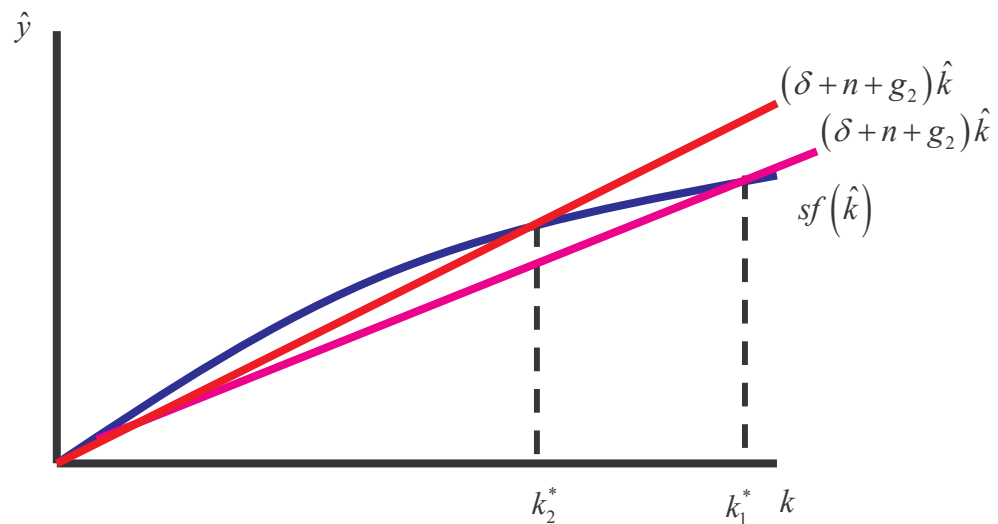
- Effect of the change in the technical growth rate (research, education, etc) on the growth rates

Increase in g (from g_1 to g_2)

→ Investment requirement (straight line) becomes steeper

→ The steady state level of capital (per worker) is reduced

→ The steady state output (per worker) is reduced



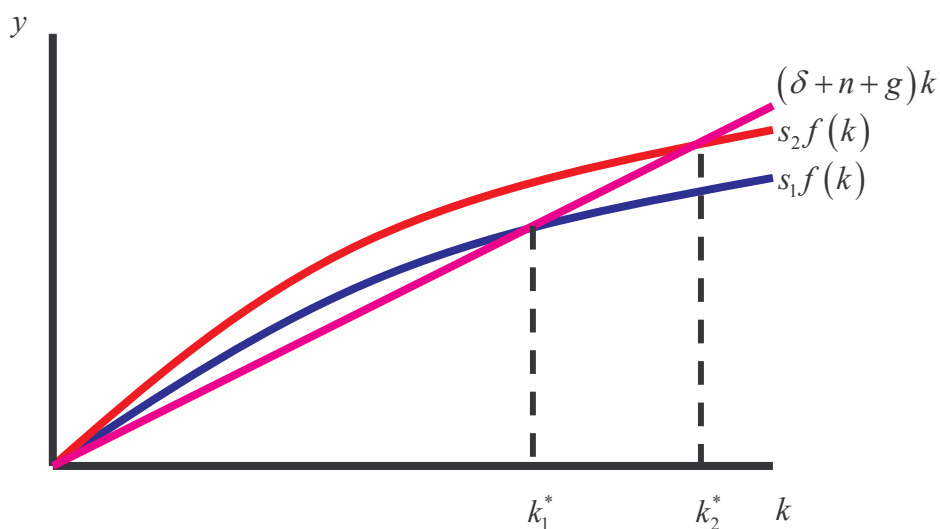
Growth rate:

In the steady state, the growth rates of Y and K increases

Note: the growth rate of y and k are always zero in the steady state from the definition

- Change in the saving rate

Increase in s (from s_1 to s_2)



The steady state levels of capital and outcome increase

But, there are no changes in the growth rates since any growth rate doesn't depend on the saving rate

V. Endogenous growth model (chapter 4.1)

A. Neoclassical growth model and empirical evidences

- The neoclassical growth model (Solow model) predicts that the saving rate doesn't affect the growth rate of economy.
- But empirical studies show that there is positive relation between the saving rate and the growth rate

- One possible approach to explain this relation is the endogenous growth model
- There are many endogenous growth models (Ramsey growth model, AK model, two-sector growth model, growth with learning-by-doing etc). Here, we will take a look at one simple model called the AK model

B. AK model

- Production function:
Now, assume that the production function shows a constant marginal product of capital. Especially, the production needs only capital input:

$$Y = AK$$

where A is the productivity (or technology).

Note 1: from the form of the production function, this model is called the AK model

Note 2: in the Solow model, we assumed the diminishing marginal product of capital

- Capital accumulation and growth
Assume that the saving rate is constant at s and there is neither population growth nor depreciation of capital. Then, all saving goes to the capital accumulation:

$$\Delta K = sY = sAK$$

$$\rightarrow \Delta K / K = sA$$

Since the output is proportional to capital,

$$\Delta Y = A\Delta K = AsY$$

$$\rightarrow \Delta Y / Y = sA$$

In this example, the higher the saving rate, the higher the growth rates of output and capital

VI. Appendix 1

Let $\frac{A^*K^*MPK}{Y}$ be θ . From Euler's theorem ($Y = A^*MPK^*K + A^*MPN^*N$),

$$\frac{A^*N^*MPN}{Y} = \frac{Y - A^*K^*MPK}{Y} = 1 - \frac{A^*K^*MPK}{Y} = 1 - \theta$$

VII. Appendix 2

From the definition of \hat{k} ,

$$\begin{aligned} \Delta\hat{k} &= \Delta\left(\frac{K}{N^*E}\right) = \frac{(N^*E)\Delta K - K(N^*\Delta E + E^*\Delta N)}{N^2 * E^2} \\ &= \frac{\Delta K}{N^*E} + \hat{k} \left[\frac{\Delta E}{E} + \frac{\Delta N}{N} \right] \\ &= \frac{\Delta K}{N^*E} - kg - kn \end{aligned}$$

$$\rightarrow \Delta K = (N^*E)\Delta\hat{k} + (N^*E)kg + (N^*E)kn$$

Change in the total capital stock K (not \hat{k} or k) consists of the total investment and the depreciation:

$$\Delta K = I - \delta K$$

Note: change in the capital per worker depends on the population growth, but the total capital stock doesn't

By combining the above two equations,

$$(N^*E)\Delta\hat{k} + (N^*E)kg + (N^*E)kn = I - \delta K$$

Divide the both sides with (N^*E) and rearrange the equation.

$$\Delta\hat{k} = i - (n + g + \delta)\hat{k}$$