Notes 8: Models of Spatial Competition

I. Product differentiation

A. Definition

Products are said to be differentiated if consumers consider them to be imperfect substitutes. For example, all consumers may agree about the relative quality of two types of the same good, but may differ in their willingness to pay for such differences. If consumers view brands in an industry as imperfect substitutes in this way, a firm may raise its price above that of its rivals without losing all its customers. This type of differentiation is usually called vertical product differentiation.

B. Spatial differentiation

Consider instead the case where consumers are more or less alike in terms of their basic willingness to pay for a product, but instead differ regarding the features of a product they regard as attractive. A common way to model such features is spatially. Consumers may prefer a product that is located close to their home or place of consumption. As a result, consumers who are located far from an outlet selling a product may only buy it if the price is low. Those who live close by may be willing to pay a higher price. This type of product differentiation is called horizontal differentiation.

A firm in a spatially differentiated market may choose to locate the distribution point in order to maximize profits. Or it may choose to have more than one distribution point in order to increase sales.

While the spatial model clearly fits products which are differentiated by location in physical space, it also fits a model where products are differentiated by their position in "product" space. The position in "product" space may be characterized by color, style, horsepower, taste, etc.

II. The Basic Spatial Model with a Single Monopolist

A. Idea of a simple one-dimensional spatial model

The idea of this model is to set up a market in one dimension where the customers are uniformly distributed in this dimension. By uniformly distributed we mean that there are N customers spaced evenly along this dimension. The easiest case is to consider a town with a Main Street of unit (say one mile) length. This market is supplied by a monopolist who owns two shops, one at each end of the street. The shops sell identical goods. One shop—located at the west end of town—has the address \(x = 0\). The other shop—located at the east end of town has the location, \(x = 1\). Thus the second shop is located one mile from the first shop.

Each point on the street (line) is associated with an \(x\) value measuring the location of that point relative to the west or left end of town. For example, the address \(x = 0.4\) is 4/10 of a mile from the west end of town.

A firm whose most preferred product or location is \(x^j\) is called firm \(x^j\). We can think of firm \(j\) as being located at point \(x^j\) on the street. Each firm will buy at most one unit of the product.

While firms differ about which variant of the good is best, they are the same in that each has the same reservation demand price, \(v\). This means that firm \(j\) is willing to pay the amount \(v\) for the product if it is delivered to point \(x^j\). However, if the product is supplied at another point, the firm incurs a cost. For example, if a farmer purchases feed at a location which is "far away" from her farm, she incurs a transportation cost. In particular, firm \(x^j\) incurs the cost \(tx^j\) if it purchases the good from the shop located at \(x = 0\), and the cost \(t(1-x^j)\) if it purchases the good from the shop located at \(x = 1\). The next figure describes the market setting.
**Main Street Spatial Model**

Except for their locations, firms are identical to each other. Each is willing to pay a total price—purchase price plus transportation cost—of $v$ dollars to obtain one and only one unit of the commodity. If we denote the price at store 1 as $p_1$ and the price at store 2 as $p_2$, then the customer enjoys a surplus of $v - tx^j - p_1$ if she buys from store 1 and a surplus of $v - t(1-x^j) - p_2$ if she buys at store 2. If she doesn’t buy the product her surplus is zero.

### B. Monopoly pricing in the spatial model

#### 1. Surplus for the customers

If the monopolist charged a very high price at one of the stores, all the customers of the good would go to the other store. Say the second shop charged the higher price so that $p_2 > p_1$. Suppose also that $p_2 - p_1 > t$. This means that even the firm who is located next to store 2 will prefer to shop from store 1. In this case every customer will have a larger potential surplus by going to the first store. Consider the figure below where we plot surplus as a function of position along the street.
The transportation cost determines the slope of the line. As we move away from a store, the surplus declines. The consumer who is located at store 2 \((x = 1)\) receives a surplus equal to \(v - p_1 - t\). In the diagram this is larger than \(v - p_2\) so even this customer shops at store 1. The fact that the surplus line from store 2 eventually hits the horizontal axis implies that some consumers would prefer to go without the good rather than buying from store 2 if they can’t buy from store 1. It is unlikely that a monopolist will set \(p_2\) so high. If all the customers shop at store 1, why have store 2 anyway?

There are two possible price configurations that make sense in this model: one in which some customers are not served (do not buy) and one in which all customers are served.

2. Equilibrium in which some potential customers do not buy

One possible equilibrium is that prices will be so high that some potential customers choose not to buy the product. This is represented in the next figure where the \(p_1 = p_2\) and both prices are fairly high.

Surplus to Customers In Spatial Model

\[ p_2 = p_1 \]

Both Prices High

Customers near store 1 purchase the good there, while customers near store 2 purchase the good at store 2. Firms near the middle of town forego the good and do without. Purchasing firms near store 1 purchase the good as long as

\[ v - tx - p_1 > 0. \tag{1} \]
Therefore, we can look at store 1 as a local monopoly which is insulated from competition by transportation costs. The customer (firm) located at the point where \( v - tx - p_1 = 0 \) is the last or marginal customer for firm 1. We can represent this firm or location along the street by setting equation (1) to hold with equality and then solving for \( x \). This will give

\[
x(p_1) = \frac{v}{t} - p_1
\]

(2)

But since \( x \) is just a fraction representing how far along the street the one is located, the \( x \) value just tells us how far this is from the western edge of town. Since we assume that there are \( N \) customers or firms along the street, then the point \( x(p_1) \) represents the fraction of \( N \) that buy the good from store 1 when it charges \( p_1 \). The total number of units sold is \( Nx(p_1) \). As \( p \) falls, the monopolist expands its market and sells more units of the good. The inverse demand curve for the first store is then

\[
D^1(p_1) = x(p_1)N = \left[ \frac{v - p_1}{t} \right]N
\]

(3)

As \( N \) rises the demand rises and as \( t \) falls the demand rises. Similarly as \( v \) rises, demand will rise. For the second store we can construct a similar inverse demand function as

\[
D^2(p_2) = (1 - x(p_2))N = \left[ \frac{v - p_2}{t} \right]N
\]

(4)

In this situation, no store can take customers away from another store, or lose customers to the other store. For example, if only store 1 raises its price, store 2 does not get any additional sales, since all customers who could benefit from store 2 have already purchased there. The number of customers or firms buying from store 1 simply declines, as shown in the figure on the next page.
Surplus to Customers In Spatial Model

$p_2 < p_1$

Both Prices High

3. Equilibria in which all customers purchase the product

Now consider a situation in which both stores lower their prices as in the figure on the next page.

In this case, all customers (firms) buy the product. The marginal customer is now located where the upper curves cross. This customer is indifferent between buying from Store 1 and Store 2. At this point, the values of buying from the two stores are equal. This point is given by

\[ v - p_1 - tx = v - p_2 - t(1-x) \]

\[ x(p_1, p_2) = \frac{p_2 - p_1 + t}{2t} \]  \hspace{1cm} (5)

All customers to the left of \( x(p_1, p_2) \) buy one unit from Store 1 and all customers to the right buy one unit from Store 2. The indifferent customer buys one unit from one or the other. In this setting, neither store is a local monopoly. The demand for a store is based on the price charged by that store and the price charged by the other store. We can now find the demand curve for each store using the fact that there are \( N \) consumers.
\[ D^1(p_1, p_2) = x(p_1, p_2)N = \left[ \frac{p_2 - p_1 + t}{2t} \right]N \]

\[ D^2(p_1, p_2) = (1 - x(p_1, p_2))N = \left[ 1 - \frac{p_2 - p_1 + t}{2t} \right]N \]

4. Combining the two types of equilibria

We can construct the demand curve for Store 1 taking the price of Store 2 as given. If this price is high enough, the first store will be a local monopoly. If the price charged is higher still it may eventually choke off all demand for Store 1. This “choke” price is equal to \( v \). As the firm lowers its price, demand will rise according to

\[ D^1(p_1) = \left| \frac{v - p_1}{t} \right|N \]  

(7)

As a numeric example let \( v = 6 \), \( t=8 \), and \( N = 20 \). Also let \( p_2 = 5 \) so that few consumers will buy from the second store. In fact only consumers beyond the point 7/8 will buy from store 2 since their net price at 7/8 will be \([5 + 1/8 (8)] = 6\). A consumer at 7/8 will not buy from store 1 at any price since the net price is \( p+(8)(7/8) > 7 > 6 = v \). We are thus in a situation in which the market is not fully served and firm 1 is a local monopoly. Demand for the first store is then

\[ D^1(p_1) = \left| \frac{6 - p_1}{8} \right|(20) = 15 - 2.5p \]  

(8)

For example if \( p_1 = 4 \), then the marginal customer will be at the point 1/4 and the first store’s demand will be 5. If \( p_1 = 0 \) then the last customer will be at the 3/4 and demand will be 15. The second store will sell to 1/8 of the consumers or \((1/8)(20) = 2.5\) so that a total of 17.5 consumers are served.

If instead \( p_2 = 2 \) then customers beyond the point ½ will consider store 2, since at the point ½, the net price is \([2+(1/2)(8)] = 6\). Now if store 1 sets a price of 4, its marginal customer will still be at the point 1/4 and both stores will remain local monopolies. But if store 1 were now to consider a price of 1, its marginal customer, ignoring store 2, would be at the point 5/8. But since customers in the region (½ - 1) also consider store 2, the first store will get customers in this region only by taking them from the second store. Thus we must consider the second form of the demand curve.
\[ D^1(p_1, p_2) = \left[ \frac{p_2 - p_1 + t}{2t} \right] N \]

\[ = \left[ \frac{2 - p_1 + 8}{16} \right] 20 \]

\[ = 12.5 - 1.25 p, \]

For example, if \( p_1 = 2 \) then the marginal customer will be at \( \frac{1}{2} \) and demand for store 1 will be 10. The marginal customer of the second store will also be at \( \frac{1}{2} \) and the second store will have demand

\[ D^2(p_1, p_2) = (1 - x(p_1, p_2)) N = \left[ 1 - \frac{2 - 2 + 8}{16} \right] (20) \]

\[ = (1 - \frac{8}{16}) (20) = 10 \]

Total demand equals 20. Thus when the first store charges 2 and the second store charges 2, all consumers are served and they get one unit each at a price of 2 dollars per unit. If the first store sets its price at zero while the second store charges 2 dollars per unit, the marginal customer will be at \( \frac{5}{8} \) and demand will be 12.5. The second store will sell \((3/8)(20) = 7.5\) and the market will be fully served since customers at \( \frac{5}{8} \) and beyond pay a net price of \([2 + x(8)]\) which is less than 6 as long as \( x > \frac{1}{2} \). Rather than picking a price for the second store we find the two part demand curve for the first store as

\[ D^1(p_1, p_2) = \begin{cases} 
\left[ \frac{v - p_1}{t} \right] N & \text{if } \left[ \frac{v - p_1 + p_2 + t}{2} \right] < 0 \\
\left[ \frac{p_2 - p_1 + t}{2t} \right] N & \text{otherwise}
\end{cases} \]

The first curve has slope \(-1/t\) while the second has slope \(-1/2t\). We can then graph the demand curve for the first firm as follows
The demand curve has a kink at a price of 2 and a quantity of 10. With the second store charging 2 dollars per unit, the first store must lower the price more rapidly beyond 10 units in order to gain sales. In fact a decrease in $p_1$ brings in only $N/2t$ additional customers as is clear from equation 11.

We could develop a similar relationship for store 2.

5. **Best combination of $p_1$ and $p_2$ for the monopolist**

   a. No overlapping production

   The first important point is that a monopolist will not choose prices such that the stores compete for customers. If the stores are competing, then it is possible to raise the prices of both stores without affecting total demand at all. The monopolist will choose to operate the stores as two local monopolies. Thus the lowest price chosen will be $p_1 = p_2 = V - \frac{t}{2}$.

   b. $MR = MC$ pricing

   Each store will set marginal revenue (using the non-competing demand functions) equal to marginal cost. This may leave some of the market uncovered.

   c. Marginal customer

   If the whole market is covered, the marginal customer will be at the point $\frac{1}{2}$.
III. The Basic Spatial Model with a Duopoly

A. The Bertrand Model as a Pricing Game with Differentiated Products

Assume that, rather than there being one firm which chooses the prices at both stores, there are now two firms which compete in prices, \( p_1 \) and \( p_2 \), respectively. The prices are chosen simultaneously. Competition where firms compete by setting prices is called Bertrand competition.

If the firms’ cost per unit is less than \( v \) (and the same for both firms), one requirement for a Nash solution to the game is that both firms have a positive market share. If this condition were not satisfied it would mean that at least one firm’s price was so high that it had zero market share and, therefore, zero profits. Suppose that the firm with zero market share is firm 1. In such a situation, firm 2 should not set its price so low that the consumer at 0 (right next to firm 1) gets a positive payoff (pays a net price below \( v \)). If so, firm 2 could raise its price without losing any market share. But then, firm 1 can obtain positive profits by setting its price just below \( v \), thus selling to consumers very near 0. (If firm 1’s price is just below \( v \), then the consumer at 0 gets a positive payoff by purchasing from firm 1). Therefore, such a situation cannot be part of a Nash equilibrium.

To simplify, assume that the Nash equilibrium outcome is one in which the entire market is served in the spatial model. That is, assume the outcome involves a market configuration in which every customer buys the product from either Firm 1 or Firm 2. This assumption will be true so long as the reservation price, \( v \), is sufficiently large. When \( v \) is large, firms will have an incentive to sell to as many customers as possible because such a high willingness-to-pay will imply that each customer can be charged a price sufficiently high to make each such a sale profitable.

An important implication of the assumption that the entire market is served is that the marginal customer, \( x^m \), is indifferent between buying from either Firm 1 or Firm 2. That is, he enjoys the same surplus either way. Algebraically, we have:

\[
V - p_1 - tx^m = V - p_2 - t(1 - x^m)
\]  

Equation 12 may be solved to find the address of the marginal customer, \( x^m \). This is:

\[
x^m(p_1, p_2) = (p_2 - p_1 + t)/2t
\]  

At any set of prices, \( p_1 \) and \( p_2 \), all customers to the west or left of \( x^m \) buy from Firm 1. All those to the east or right of \( x^m \) buy from Firm 2. In other words, \( x^m \) is the fraction of the market buying from Firm 1 and \( (1 - x^m) \) is the fraction buying from Firm 2. If the total market size is \( N \), the demand function facing Firm 1 at any price combination \((p_1, p_2)\) in which the entire market is served is:

\[
D^1(p_1, p_2) = x^m(p_1, p_2)N = \left[ \frac{(p_2 - p_1 + t)}{2t} \right]N
\]  

Similarly, Firm 2’s demand function is:

\[
D^2(p_1, p_2) = \left[ 1 - x^m(p_1, p_2) \right]N = \left[ \frac{(p_1 - p_2 + t)}{2t} \right]N
\]
Unlike the original Bertrand duopoly model where any rise in price leads to a discontinuous drop in sales, the model here is one in which the demand function facing either firm is continuous in both $p_1$ and $p_2$. This is because when goods are differentiated, a decision by say Firm 1 to set $p_1$ a little higher than its rival’s price, $p_2$, does not cause Firm 1 to lose all of its customers. Some of its customers will still prefer to buy good 1 even at the higher price simply because they prefer that version of the good to the type (or location) marketed by Firm 2.\footnote{Our assumption that the equilibrium is one in which the entire market is served is critical to the continuity result.} As $t$ gets close to 0, this model approaches the original Bertrand model, and the slightest difference between $p_1$ and $p_2$ results in an enormous change in demand.

The continuity in demand functions carries over into the profit functions. Firm 1’s profit function is:

$$\Pi^1 (p_1, p_2) = N(p_1 - c) \left( \frac{p_2 - p_1 + t}{2t} \right) \quad (16)$$

Similarly, Firm 2’s profits are given by:

$$\Pi^2 (p_1, p_2) = N(p_2 - c) \left( \frac{p_1 - p_2 + t}{2t} \right) \quad (17)$$

We may derive Firm 1’s best response function by differentiating $\Pi^1$ with respect to $p_1$ taking $p_2$ as given. In this way, we will obtain Firm 1’s best price choice, $p_1^*$ as a function of $p_2$. Of course, differentiation for Firm 2’s profit function with respect to its price will, similarly, yield the best choice of $p_2$ conditional upon any choice of $p_1$. The firms are symmetrical so their best response functions are mirror images of each other. For firm 1 we obtain

$$d\Pi^1 (p_1, p_2) = \frac{d}{dp_1} \left( N(p_1 - c) \left( \frac{p_2 - p_1 + t}{2t} \right) \right) = 0$$

$$\Rightarrow N \left( \frac{p_2 - p_1 + t}{2t} \right) + N(p_1 - c) \left( \frac{-1}{2t} \right) = 0$$

$$\Rightarrow N \left( \frac{p_2 - p_1 + t - p_1 + c}{2t} \right) = 0$$

$$\Rightarrow p_2 - p_1 + t - p_1 + c = 0$$

$$\Rightarrow p_2 + t + c = 2p_1$$

$$\Rightarrow p_1 = \frac{p_2 + t + c}{2}$$

We can find a similar result for firm 2 so that the best response functions are:
\[ p_1^* = \left( \frac{p_2 + c + t}{2} \right) \]  
\[ p_2^* = \left( \frac{p_1 + c + t}{2} \right) \]  \hspace{1cm} (19)

The Nash equilibrium of the model is a pair of best response prices, \( p_1^* \), \( p_2^* \), such that \( p_1^* \) is Firm 1’s best response to \( p_2^* \), and \( p_2^* \) is Firm 2’s best response to \( p_1^* \). This means that we may replace \( p_1 \) and \( p_2 \) on the right-hand-side of 19 with \( p_1^* \) and \( p_2^* \), respectively. Then, solving jointly for the Nash equilibrium pair, \( p_1^* \), \( p_2^* \) yields:

\[ p_1^* = p_2^* = c + t \]  \hspace{1cm} (20)

In other words, the symmetric Nash equilibrium for the Bertrand duopoly model with differentiated products is one in which each firm charges the same price given by its unit cost \( c \) plus an amount, \( t \), the utility cost per unit of distance the consumer incurs in buying a good different from his own most preferred type. At these prices, the firms split the market. The marginal consumer is at \( x^m = \frac{1}{2} \). Demand for each firm is \( \frac{NT}{2} \).

Two points are worth making in connection with the foregoing analysis. First, note the role that the parameter, \( t \), plays. \( t \) is a measure of the value each customer places on obtaining his most preferred version of the product. The greater is \( t \), the more the customer is willing to pay a high price simply to avoid being “far away” from her favorite product location. That is, a high \( t \) value indicates that each firm has little to worry about in charging a high price, because the customers would prefer to pay that price rather than buy a low-price alternative that is “far away” from their preferred product. Thus, when \( t \) is large, the price competition between the two firms is softened. A large value of \( t \) implies then that the product differentiation matters. To the extent products are differentiated, price competition will be less intense.

However, as \( t \) falls the customers place less value on obtaining a preferred type of product and focus more on simply obtaining the best price. This intensifies price competition. In the limit, when \( t = 0 \), differentiation is of no value to the customers. They treat all goods as essentially identical. Price competition becomes fierce and, in the limit, forces prices to be set at marginal cost just as in the original Bertrand model.

The second point to be made in connection with the above analysis concerns the location of the firms. We simply assumed that the two firms were located at either end of town. It turns out that allowing the firms in the above model to choose \textit{both} their price and their location strategies makes the problem too complicated to work out here. Still, the intuition behind this indeterminancy is instructive. Two opposing forces make the combined choice of price and location difficult. On the one hand, the two firms will wish to avoid locating at the same point because to do so eliminates any gains from product diversification. Price competition in this case will be fierce as in the original Bertrand model. On the other hand, each firm has an incentive to locate near the center of town. This enables a firm to reach as large a market as possible. Evaluating the balance of these two forces is what makes determination of the ultimate equilibrium so difficult.