Lecture 5 & 6

- Material for lecture 5 & 6 will use the following transparencies and Table 1 of chapter 4 of the textbook
Terminology for Bonds and Loans

- **Principal** given to borrower when loan is made
- Simple loan: principal plus interest repaid at one date
- Fixed-payment loan: series of (often equal) repayments
- Bond is issued at some price
- **Face Value** is repayment at maturity date
- **Zero coupon bond** pays only face value at maturity
- **Coupon bond** also makes periodic coupon payments, equal to coupon rate times face value
Calculations for Bonds and Loans

Interest rate $i$ is yield to maturity.

$n$ is time to maturity

- Simple Loans: use lump sum formula; $PV = \text{principal}; FV = \text{principal plus interest}$.
- Zero Coupon Bonds: use lump sum formula; $PV = \text{price}; FV = \text{face value}$.
- Fixed-Payment Loans: use annuity formula; $pmt = \text{loan payment}; PV = \text{principal}$
- Coupon Bonds: combine annuity and lump sum formulas. With
face value $F$ and coupon payment $C$, the price $p$ is equal to

$$P = \frac{C}{1+i} + \frac{C}{(1+i)^2} + \ldots + \frac{C}{(1+i)^n} + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n}$$
Bond Yields and Bond Prices

- Yield to maturity on a coupon bond cannot be calculated analytically; need computer or financial calculator.

- YTM high if bond price is low.

- YTM equals coupon rate if bond price equals face value (bond trades at par).

- YTM higher (lower) than coupon rate if bond price lower (higher) than face value.

- Current yield (coupon rate/price) is approximation to YTM; works well if long maturity, bond trades close to par.
Compounding

- Assume that the interest rate is 10% p.a.
- What this means is that if you invest $1 for one year, you have been promised $1*(1+10/100) or $1.10 next year.
- Investing $1 for yet another year promises to produce 1.10*(1+10/100) or $1.21 in 2-years.
Value of $5 Invested

• More generally, with an investment of $5 at 10% we obtain

1 Year $5*(1+0.10) $5.5

2 years $5.5*(1+0.10) $6.05

3 years $6.05*(1+0.10) $6.655

4 Years $6.655*(1+0.10) $7.3205
Generalizing the method

- Generalizing the method requires some definitions. Let
  - \( i \) be the interest rate
  - \( n \) be the life of the lump sum investment
  - \( PV \) be the present value
  - \( FV \) be the future value
Future Value of a Lump Sum

\[ FV = PV \times (1 + i)^n \]

FV with growths from -6% to +6%

Years

Future Value of $1000

0  1000  1500  2000  2500  3000  3500

0  2  4  6  8  10  12  14  16  18  20

0%  2%  4%  6%  -2%  -4%  -6%
Example: Future Value of a Lump Sum

• Your bank offers a CD with an interest rate of 3% for a 5 year investment.

\[ FV = PV \times (1 + i)^n \]

\[ = $1500 \times (1 + 0.03)^5 \]

\[ = $1738.111145 \]

• You wish to invest $1,500 for 5 years, how much will your investment be worth?

\[ n \quad 5 \]

\[ i \quad 3\% \]

\[ PV \quad 1,500 \]

\[ FV \quad ? \]

Result 1738.911111
Present Value of a Lump Sum

\[ FV = PV \times (1 + i)^n \]

Divide both sides by \((1 + i)^n\) to obtain:

\[ PV = \frac{FV}{(1 + i)^n} = FV \times (1 + i)^{-n} \]
Example: Present Value of a Lump Sum

- You have been offered $40,000 for your printing business, payable in 2 years. Suppose the interest rate is 8%. What is the present value of the offer?

\[
PV = \frac{FV}{(1+i)^n}
\]

\[
= \frac{40,000}{(1+0.08)^2}
\]

\[
= 34293.55281
\]

\[
\approx \$34,293.55 \text{ today}
\]
Solving Lump Sum Cash Flow for Interest Rate

\[ FV = PV \times (1+i)^n \]

\[ \frac{FV}{PV} = (1+i)^n \]

\[ (1+i) = \sqrt[n]{\frac{FV}{PV}} \]

\[ i = \sqrt[n]{\frac{FV}{PV}} - 1 \]
Example: Interest Rate on a Lump Sum Investment

- If you invest $15,000 for ten years, you receive $30,000. What is your annual return?

\[ i = \sqrt[n]{\frac{FV}{PV}} - 1 \]

\[ = \sqrt[10]{\frac{30000}{15000}} - 1 = \frac{\sqrt[10]{2}}{2} - 1 = 2^{\frac{1}{10}} - 1 \]

\[ = 0.071773463 \]

- Other interpretation: if price of 10 year bond with face value 30,000 is 15,000, then interest rate (yield to maturity) on bond is 7.18% (to the nearest basis point)
The Problem

How much will I have available in my retirement account if I deposit $2,000 per year into an IRA that pays on average 16% per year if I plan to retire in 40 years time?
Annuity = Stream of Cash Flows, such that

- the first cash flow will occur exactly one period from now
- all subsequent cash flows are separated by exactly one period
- all periods are of equal length
- the interest rate is constant
- all cash flows have the same value
Annuity Formula Notation

- $PV = \text{the present value of the annuity}$
- $i = \text{interest rate to be earned over the life of the annuity}$
- $n = \text{the number of payments}$
- $pmt = \text{the periodic payment (cash flow)}$
Derivation of PV of Annuity Formula

\[ PV = \frac{pmt}{(1+i)^1} + \frac{pmt}{(1+i)^2} + \frac{pmt}{(1+i)^3} + \Lambda + \frac{pmt}{(1+i)^{n-1}} + \frac{pmt}{(1+i)^n} \]
PV of Annuity Formula

\[ PV = \frac{pmt \times \left\{1 - \frac{1}{(1+i)^n}\right\}}{i} \]

\[ = \frac{pmt}{i} \times \left(1 - \frac{1}{(1+i)^n}\right) \]
PV Annuity Formula: Payment

\[ PV = \frac{pmt}{i} \times \left( 1 - \frac{1}{(1+i)^n} \right) \]

\[ = \frac{pmt}{i} \times \left( 1 - (1+i)^{-n} \right) \]

\[ pmt = \frac{PV \times i}{(1-(1+i)^{-n})} \]
Derivation of FV of Annuity Formula: Algebra

\[ PV = \frac{pmt}{i} \left( 1 - \frac{1}{(1+i)^n} \right) \] (reg. annuity)

\[ FV = PV \times (1+i)^n \] (lump sum)

\[ FV = \frac{pmt}{i} \left( 1 - \frac{1}{(1+i)^n} \right) \times (1+i)^n \]

\[ = \frac{pmt}{i} \left( (1+i)^n - 1 \right) \]
FV Annuity Formula: Payment

\[ FV = \frac{pmt}{i} \left( (1+i)^n - 1 \right) \]

\[ pmt = \frac{FV \times i}{(1+i)^n - 1} \]
Data for Problem

- \( I = 16\% \)
- \( n = 40 \)
- \( Pmt = $2,000 \)
- \( FV = ? \)
Solution

\[ F = \frac{pmt}{i} \left( (1 + i)^n - 1 \right) \]

\[ = \frac{2000}{0.16} \left( (1 + 0.16)^{40} - 1 \right) \]

\[ = \$4,721,514.481 \]