(1) Stock and Watson 2.2: (a) First,

\[ E(W) = 3 + 6E(X) \text{ and } E(V) = 20 - 7E(Y). \]

From table 2.2,

\[ E(X) = 0(.3) + 1(.7) = .7 \]

and

\[ E(Y) = 0(.22) + 1(.78) = .78. \]

Therefore,

\[ E(W) = 3 + 6(.7) = 7.2 \]

and

\[ E(V) = 20 - 7(.78) = 14.54. \]

(b) First,

\[ \sigma_W^2 = \text{Var}(W) = 36\text{Var}(X) \text{ and } \sigma_V^2 = \text{Var}(V) = 49\text{Var}(Y). \]

Now, again from table 2.2,

\[ E(X^2) = 0(.3) + 1(.7) = .7 \text{ and } E(Y^2) = 0(.22) + 1(.78) = .78. \]

Thus,

\[ \text{Var}(X) = E(X^2) - E^2(X) = .7 - .7(.7) = .21. \]

Similarly,

\[ \text{Var}(Y) = E(Y^2) - E^2(Y) = .78 - (.78)(.78) = .1716. \]

Thus,

\[ \text{Var}(W) = 36(.21) = 7.56 \text{ and } \text{Var}(V) = 49(.1716) = 8.41. \]

(c) Note that

\[ \sigma_{WV} = \text{Cov}(W, V) \]

\[ = E(WV) - E(W)E(V) \]

\[ = E[(3 + 6X)(20 - 7Y)] - E(W)E(V) \]

\[ = E(60 + 120X - 21Y - 42XY) - E(W)E(V) \]

\[ = 60 + 120E(X) - 21E(Y) - 42E(XY) - E(W)E(V) \]

From previous parts of this exercise, we know all of the above quantities, except for \( E(XY) \). We can, however, calculate this from table 2.2:

\[ E(XY) = 0(.15) + 0(.07) + 0(.15) + 1(.63) = .63. \]
Therefore,
\[
\text{Cov}(W, V) = 60 + 120(.7) - 42(.63) - 72(.78) - 21(.78) - 42(.63) - 72(1.54) = 60 + 84 - 16.38 - 26.46 - 104.688 = -3.528
\]

It follows that \( \text{Corr}(W, V) = \frac{\text{Cov}(W, V)}{\sqrt{\text{Var}(W)\text{Var}(V)}} = -3.528/\sqrt{7.56(8.41)} \approx -0.44 \). It is reasonable to expect a negative correlation between rain and the shorness of the commute.

(2). Note that
\[
E(Z) = E[(1/2)(Y - 1)] = (1/2)[E(Y) - 1] = (1/2)[1 - 1] = 0.
\]
Similarly,
\[
\text{Var}(Z) = \text{Var}((1/2)(Y - 1)) = (1/4)\text{Var}(Y - 1) = (1/4)\text{Var}(Y) = (1/4)4 = 1.
\]

(3 a) By definition,
\[
\text{Pr}(Y = y_j) = \sum_{i=1}^{l} \text{Pr}(Y = y_j, X = x_i)
\]
\[
= \sum_{i=1}^{l} \text{Pr}(Y = y_j|X = x_i)\text{Pr}(X = x_i),
\]
where the last line follows from the definition of conditional probability.

(b) Note
\[
E(Y) = \sum_{j=1}^{k} y_j \text{Pr}(Y = y_j)
\]
\[
= \sum_{j=1}^{k} y_j \sum_{i=1}^{l} \text{Pr}(Y = y_j, X = x_i)\text{Pr}(X = x_i)
\]
\[
= \sum_{j=1}^{k} \sum_{i=1}^{l} y_j \text{Pr}(Y = y_j|X = x_i)\text{Pr}(X = x_i)
\]
\[
= \sum_{i=1}^{l} \left[ \sum_{j=1}^{k} y_j \text{Pr}(Y = y_j|X = x_i) \right] \text{Pr}(X = x_i)
\]
\[
= \sum_{i=1}^{l} E(Y|X = x_i)\text{Pr}(X = x_i).
\]
Note that the second line substitutes the result in part (a), we can interchange the order of the summations, and the last line applies the definition of the conditional expectation function.

(c) If \( X \) and \( Y \) are independent, then, as shown in class, \( E(XY) = E(X)E(Y) \). Since, by definition, \( \sigma_{XY} = \text{Cov}(X, Y) = E(XY) - E(X)E(Y) \), independence implies \( \sigma_{XY} = 0 \). Similarly, since \( \text{Corr}(X, Y) = \sigma_{XY}/\sqrt{\text{Var}(X)\text{Var}(Y)} \), the correlation must also be zero.
(4) There are several different strategies for proving this, using different, though equivalent, ways to calculate the covariance.

Let $\mu_x = E(X)$ and $\mu_y = E(Y)$. Note that

$$E[aX + b] = a\mu_x + b, \quad \text{and} \quad E(cY + d) = c\mu_y + d.$$  

Using the first definition of the covariance, as discussed in class, we obtain

$$\text{Cov}(aX + b, cY + d) = E\left[(aX + b - (a\mu_x + b) \left[(cY + d) - (c\mu_y + d)\right]\right]$$

$$= E\left[(a(X - \mu_x)) [c(Y - \mu_y)]\right]$$

$$= acE[(X - \mu_x)(Y - \mu_y)]$$

$$= ac\text{Cov}(X, Y).$$

(5) Note that, using the definition of the covariance,

$$\text{Cov}(X, X) = E(X^2) - E(X)^2 = \text{Var}(X).$$

(6)

$$\text{Var}(aX + bY) = E[(aX + bY)^2] - [E(aX + bY)]^2$$

$$= E\left[a^2X^2 + 2abXY + b^2Y^2\right] - (a\mu_x + b\mu_y)^2$$

$$= a^2E(X^2) + 2abE(XY) + b^2E(Y^2) - \left[a^2\mu_x^2 + 2ab\mu_x\mu_y + b^2\mu_y^2\right]$$

$$= a^2\left[E(X^2) - \mu_x^2\right] + 2ab\left[E(XY) - \mu_x\mu_y\right] + b^2\left[E(Y^2) - \mu_y^2\right]$$

$$= a^2\text{Var}(X) + 2ab\text{Cov}(X, Y) + b^2\text{Var}(Y).$$