Solutions: Problem Set #1

(1) The following table gives the joint probability distribution $p(X, Y)$ of random variables X and Y.

	X		
Y	1	2	3
1	.02	.04	.08
2	.03	.18	.04
3	.04	.04	.08
4	.09	.18	.18

Determine the following:

(a) Do the entries of the table satisfy the conditions for a bivariate density function?

ANSWER: Yes, since all the entries are non-negative, and they sum to unity (one).

(b) The marginal (or unconditional) probability distributions of X and Y. [Note: These will be a collection of probabilities: the probabilities associated with the 3 values of X and the probabilities associated with the 4 values of Y]. ANSWER

$$\Pr(X = 1) = .18, \ \Pr(X = 2) = .44, \ \Pr(X = 3) = .38$$

and

$$\Pr(Y = 1) = .14, \ \Pr(Y = 2) = .25, \ \Pr(Y = 3) = .16, \ \Pr(Y = 4) = .45.$$

(c) The conditional probability distributions $p(X|Y = 3)$ and $p(Y|X = 1)$. (Note: The first conditional probability distribution is the collection of three numbers, $\Pr(X = 1|Y = 3), \Pr(X = 2|Y = 3), \Pr(X = 3|Y = 3)$.)

ANSWER:Applying our formulas for calculating a conditional from a joint:

$$\Pr(X = 1|Y = 3) = .04/.16 = 1/4, \ \Pr(X = 2|y = 3) = .04/.16 = 1/4,$$

and

$$\Pr(X = 3|Y = 3) = .08/.16 = 1/2.$$

As for the remaining conditional

$$\Pr(Y = 1|X = 1) = .02/.18 = 1/9,\ \Pr(Y = 2|X = 1) = .03/.18 = 1/6$$

and

$$\Pr(Y = 3|X = 1) = .04/.18 = 2/9,\ \Pr(Y = 4|X = 1) = .09/.18 = 1/2.$$

(2) Using straightforward manipulations of the conditional probability (see, e.g., Down's Syndrome example):

$$\begin{aligned}
\Pr(D = 1|S = 0) &= \frac{\Pr(S = 0|D = 1)\Pr(D = 1)}{\Pr(S = 0)} \\
&= \frac{\Pr(S = 0|D = 1)\Pr(D = 1)}{\Pr(S = 0, D = 1) + \Pr(S = 0, D = 0)} \\
&= \frac{\Pr(S = 0|D = 1)\Pr(D = 1)}{\Pr(S = 0|D = 1)\Pr(D = 1) + \Pr(S = 0|D = 0)\Pr(D = 0).}
\end{aligned}$$

The second line simply notes that the marginal probability can be obtained by summing over all of the corresponding joint probabilities.

Base on what is given in the problem, $\Pr(S = 1|D = 1) = .5$, and therefore $\Pr(S = 0|D = 1) = .5$, since these two numbers have to add up to one. Similarly, $\Pr(D = 1) = .2$ and thus $\Pr(D = 0) = .8$. Finally, $\Pr(S = 0|D = 0) = 1$, since if the father does not have the disease, the son can not have the disease either.

Putting these numbers into the expression above gives

$$\Pr(D = 1|S = 0) = .5(.2)/[.5(.2) + 1(.8)] = 1/9.$$

Extra Credit

Using similar manipultions, we obtain

$$\Pr(D = 1|S_1 = 0, S_2 = 0) = \frac{\Pr(S_1 = 0, S_2 = 0|D = 1)\Pr(D = 1)}{\Pr(S_1 = 0, S_2 = 0)}.$$

What is given in the problem is that the outcome of each son is independent given the disease status of the father. Thus,

$$\Pr(S_1 = 0, S_2 = 0|D = 1) = \Pr(S_1 = 0|D = 1)\Pr(S_2 = 0|D = 1).$$

We thus substitute this into the numerator of our expression above, and continue to simplify the denominator, to obtain

$$\Pr(D = 1|S_1 = 0, S_2 = 0) \quad = \quad \frac{\Pr(S_1 = 0|D = 1)\Pr(S_2 = 0|D = 1)\Pr(D = 1)}{\Pr(S_1 = 0, S_2 = 0, D = 1) + \Pr(S_1 = 0, S_2 = 0, D = 0)}$$

which simplifies to

$$\frac{\Pr(S_1 = 0|D = 1)\Pr(S_2 = 0|D = 1)\Pr(D = 1)}{\Pr(S_1 = 0, S_2 = 0|D = 1)\Pr(D = 1) + \Pr(S_1 = 0, S_2 = 0|D = 0)\Pr(D = 0)}$$

$$= \frac{\Pr(S_1 = 0|D = 1)\Pr(S_2 = 0|D = 1)\Pr(D = 1)}{\Pr(S_1 = 0|D = 1)\Pr(S_2 = 0|D = 1)\Pr(D = 1) + \Pr(S_1 = 0|D = 0)\Pr(S_2 = 0|D = 0)\Pr(D = 0)}$$

$$\frac{.5(.5)(.2)}{.5(.5)(.2) + (1)(1)(.8)},$$

which equals .059.