1. Consider a firm that produces bicycles using capital (machines) and labor. Its production function is given by:

\[ Q = AK^\alpha L^\beta, \quad \alpha > 0, \quad 1 > \beta > 0 \]

where \( Q \) measures output (of bicycles), while \( K \) and \( L \) measure the amount of capital and labor, respectively, used to produce bicycles. The parameter \( A \) is a measure of the firm’s productivity (“know-how”); increases in \( A \) represent improved productivity (the function is a “generalized Cobb-Douglas” and this type of technical progress represents what is called Hicks-neutral technical progress).

Let \( W \) denote the wage the firm pays for each unit of labor, and let \( R \) denote the per unit cost of capital (machines). The firm’s costs are \( \{WL + RK\} \). **In the short run the amount of capital the firm can use is fixed, while in the long run both labor and capital are choice variables.**

a) A function \( f(x_1, x_2, ..., x_n) \) is said to be homogeneous of degree \( h \) if

\[ f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^h f(x_1, x_2, ..., x_n) \text{ for all } \lambda > 0, \text{ for all } x_i > 0. \]

Is the production function given above homogeneous of any degree? If so, what is \( h \) for this function?

b) Find the firm’s short run cost curve (that is, express costs as a function of output (\( Q \)), input prices (\( W, R \)), technology (\( A \)), and the fixed amount of capital (\( K \)).

c) How do the following affect the short run marginal cost curve: (i) an increase in productivity (in \( A \)); (ii) an increase in \( W \); (iii) an increase in \( R \)?

d) In the long run, the amount of capital (as well as labor) can be varied and inputs are chosen to minimize the cost of producing a given level of output:

\[ \text{Min}(WL + RK) \quad \text{such that: } \left( AK^\alpha L^\beta \right) \geq Q. \]

**Find:**

i. The input demands that minimize cost (these are called conditional factor demands). Express your answers as functions of \((Q, W, R)\)

ii. The minimum cost (or cost function, \( C(Q, W, R) \)) as a function of \((Q, W, R)\). This is the firm’s long run cost curve.

iii. Relate the slope of the long run marginal cost curve to the degree of homogeneity (\( h \)) of the production function.

e) Compare the elasticity (with respect to \( Q \)) of the long run marginal cost curve to the elasticity of the short run marginal cost curve.

f) Verify the following properties of the cost function: \( C(Q, W, R) \)
i. It is increasing and homogeneous of degree one in \((W, R)\): this implies that if you
double input prices, you double production costs.

ii. The (partial) derivate of the cost function with respect to \(W\) gives the (conditional) factor
demand for \(L\) and the partial derivate of the production function with respect to \(R\) gives
the (conditional) factor demand for \(K\).

iii. The cost function is concave in factor prices (don’t worry if do don’t know what this
means)

2. Let \(P\) denote the (constant) price at which the firm can sell its bicycles. Thus, the firm’s profits
can be written as:

\[
\pi = PQ - C(Q, A, W, K, R)
\]

where \(C(Q, A, W, K, R)\) denotes the firm’s short run cost curve (which you derived in problem 1). For
simplicity, assume: \(\beta = (1/2), K=4\ and R=5\), and that the firm chooses its (short run) output level to
maximize profits:

a) Find the firm’s short run supply curve, its demand for labor and the maximum profits of the
firm (which depend on prices, capital, and productivity).

b) How do increases in productivity affect the firm’s profits, its demand for labor and its output
level (i.e., how do productivity increases shift the labor demand and output supply curve)?
Explain why this happens.

c) Use the expression (function) for maximum profits derived in part (a) to calculate how an
increase in price from \(P = 10\) to \(P = 20\) affects the firm’s profits. Show that your answer is
the same as the increase in “producer surplus” that is calculated using the area next to the
supply curve between the two prices.

d) For this part, let \(A=1, W=1\ (and K=4, R=5)\). Use the firm’s short run cost curve to find the
increase in costs when the firm increases output from \(Q = 10\) to \(Q = 20\). Calculate the area
under the firm’s short run marginal cost curve (short run supply curve) between \(Q = 10\) and
\(Q = 20\). Why is this area the same as the increase in costs you first calculated? Explain.

e) Find the firm’s long run supply curve, if it exists (i.e., its long run profit maximizing
decision). Be careful to consider separately the following three cases:

(i) \((\alpha + \beta) > 1\); (ii) \((\alpha + \beta) = 1\); and (iii) \((\alpha + \beta) < 1\)

Explain the meaning and economic implication of your results for each case.

3. Consider a consumer who has the following (quasi-linear) utility function:

\[
U(x, y) = x + \left(25y - \left(y^2 / 2\right)\right)
\]

where \((x, y)\) denotes the goods consumed by the individual. Let \(I\) denote the individual’s income, and
\((P_x, P_y)\) denote the prices the individual pays for goods \(x\) and \(y\), respectively.
a) Derive the individual’s demand functions for the two goods (that is, maximize utility subject to the budget constraint).

b) Find the individual’s maximized utility as a function of income and prices (substitute the solution from part (a) back into the utility function). This result is called the indirect utility function.

c) Suppose initially \( I = 1,000; \ P_x = 1; \ P_y = 6 \); using your result from part (b), calculate the individual’s maximized utility.

d) Suppose the government imposes a tax of 4 units on good \( y \), which raises the price to \( P_y = 10 \) (the producer price does not change). Calculate:

i. How this tax affects the person’s utility, consumption of each good and how much tax revenue is raised (call the tax revenue \( T_t \)).

ii. Assume, instead of a tax on good \( y \), the government imposes a “flat tax” (economists call it a lump sum tax) on the individual. Call this “flat tax” \( F \). Find the flat tax that has the same effect on the person’s utility as the tax on good \( y \) from part i (i.e., find \( F \) so that the consumer’s maximized utility is the same in the situation \( (I = 1,000; P_x = 1, P_y = 10) \) as it is in the situation \( (I = [1,000 − F]; P_x = 1, P_y = 2) \)).

iii. Compare the tax revenue from the flat tax to that from the tax on good \( y \). What is the deadweight loss (inefficiency) due to the tax on good \( y \)?

e) Using the person’s demand curve for good \( y \), calculate the change in consumer surplus associated with an increase in price from \( P_y = 6 \) to \( P_y = 10 \). Compare your answers for parts (d) and (e).

4. Suppose the US has the following market supply (S) and demand (D) curves for coffee:

\[
S = 30P^d, \quad D = 8,000 − 20P^d
\]

where \( P^d \) denotes the domestic price of coffee (in cents/pound)

a) Assuming there is no trade, find the equilibrium price of coffee and the quantity transacted.

b) Suppose that the world price of coffee is $1/pound (i.e., the world price \( P^w = 100 \)). Assuming the country can freely import or export coffee, how does free trade affect domestic price, production and consumption? Find the change in consumer surplus and producer surplus due to this move to free trade (a quantitative answer is required). Overall, does the country gain from free trade? Does everyone gain? Explain.

c) Finally, assume that the country limits imports of coffee to 1500 (i.e., it imposes an import quota of 1500). How would this quota affect domestic price, output and consumption? Who gains and who loses from this policy?