1. A consumer has the (quasi-linear) utility function: \( U(x, y) = x + \left(50y - \frac{y^2}{2}\right) \). \( I \) is income, and \((P_x, P_y)\) denote prices the individual pays for goods \( x \) and \( y \), respectively.

**a) Derive the individual’s demand functions (maximize utility subject to the budget constraint).**

There are several ways to derive the demand curves. (1) Set up the constrained optimization problem (if you know how to do that) and solve; (2) since the constraint binds as an equality, use it to eliminate one variable and solve; or (3) use your economic knowledge that optimality requires (for an interior solution) that the MRS (the ratio of marginal utilities) equals the price ratio, and substitute back into the budget constraint. All give the same result. Using substitution: \( x = \left(I - P_y y\right)/P_x \), we get:

\[
U(x, y) = \frac{I - P_y y}{P_x} + \left(50y - \frac{y^2}{2}\right); \quad \text{thus: } \frac{dU}{dy} = \frac{-P_y}{P_x} + (50 - y) = 0.
\]

Solving for \( y \) yields:

\[
y^* = (50 - \rho) \rightarrow x^* = \left(I/P_x\right) - \rho[50 - \rho]; \quad \rho = \left(P_y/P_x\right) \text{ where } \rho \text{ is the relative price of } y \text{ (in terms of } x).\]

To guarantee both \( x \) and \( y \) are non-negative we require \( \rho \leq 50, \frac{I}{P_x} \geq \rho[50 - \rho]. \) We assume both conditions hold. Note that, with this solution, maximized utility is:

\[
U^*(I, P_x, P_y) = \frac{I}{P_x} - \rho y^* + 50 y^* - \frac{(y^*)^2}{2} = \frac{I}{P_x} + \frac{(50 - \rho)^2}{2} \text{ since } y^* = (50 - \rho)
\]

The function above is called the “indirect utility function” and it calculates the individual’s utility as a function of prices and income (assuming the person chooses the utility maximizing consumption bundle).

*An alternative way to get the same demand curves is to set the marginal rate of substitution equal to the price ratio:*

\[
\left(\frac{MU_y}{MU_x}\right) = \left(\frac{P_y}{P_x}\right) \rightarrow (50 - y) = \left(\frac{P_y}{P_x}\right) = \rho, \text{ as above.}
\]

**b) Currently the consumer faces income and prices: } I = 3000, P_x = 1, P_y = 20. \text{ Hence, her consumption bundle and utility are: } y^* = 30, x^* = 3000 - 20 \cdot 30 = 2400; \text{ hence, his maximized utility is:}

\[
U^*(I, P_x, P_y) = \frac{I}{P_x} + \left(\frac{50 - \rho}{2}\right)^2 = 3000 + \frac{30^2}{2} = 3450
\]

i. The new store allows her to buy good \( y \) at a price \( P_y = 15 \). Hence, her purchases will be:

\[
y = (50 - 15) = 35; \quad x = I - P_y y = I - 525.
\]

Clearly the person is better off if she can buy at a lower price.
ii. Using the indirect utility function, if the person pays a fee $F$ to join the store, and buys good $Y$ at the price $P_y = 15$ her utility will be:

$$U^*(I-F,P_x,P_y) = \frac{I-F}{P_x} + \frac{(50-\rho)^2}{2} = 3000 - F + \frac{(50-15)^2}{2} = 3612.5 - F$$

Since, before the discount store, she paid a price of 20 and had utility of 3450, the maximum the person would pay: $U^*(I-F,1,15) = 3612.5 - F = 3450 = U^*(I,1,20)$ or $F = 162.5$

iii. Use the demand curve to show how this area can be calculated.

There are two ways to calculate this gain. The simple way is to “remember” that the area next to the demand curve, between two prices, represents the “consumer surplus” which – assuming no income effects – represents the individual’s maximum willingness to pay for a price decrease. Hence, for this example, the consumer surplus is the area of the trapezoid $\{20,B,A,15\}$ or

**Consumer surplus** = $\frac{5 \cdot (30 + 35)}{2} = 162.5$, the same amount as calculated above.

The other way to calculate this gain (which is the way to prove that the area next to the demand curve actually measures consumer surplus) is to note the following:

The consumer gains from the price decrease in two ways:

1. The original quantity purchased (30) costs less, for a saving of $(20 - 15) \cdot 30 = 150$. This is the gain to the consumer if she does not change the amount purchased.

2. The other gain is due to the increased purchases. The consumer increases purchases from 30 to 35 units; since the height of the demand curve (the marginal rate of substitution) gives the value to the consumer of each additional unit, the area under the demand curve, corresponding to the quantity change, represents the maximum amount the consumer would pay for that additional quantity. Hence:
Gross value additional consumption = Area of \{30,B,A,35\}= 5 \cdot \left(\frac{20+15}{2}\right) = 87.5 \\
Cost of additional consumption = 5 \cdot 15 = 75 \\
Net gain from additional consumption = 87.5 - 75 = 12.5 \\

Thus, overall the consumer gains two ways: (1) by purchasing the original quantity at a lower price, and (2) by adjusting her purchases to reflect this new price. The total gain is: \(150 + 12.5 = 162.5\) which is, of course, the same answer reached above.

c) Return to the original situation: \(I = 3000, P_x = 1, P_y = 20\). Now the consumer has the opportunity to join an online club which sells good \(y\) at the price of 15, but requires the person to be at least 45 units.

   i. As we saw in (b), if there were no minimum purchase requirement, the person would only want to buy 35 units. Thus, the minimum purchase constraint is binding (i.e., forces her to buy more than she wants).

   ii. Since she is forced to buy more than she wants, the value of joining this club is lower than in part (b), where there was no minimum requirement. In fact, as the picture below shows, the value of the 45th unit to her is only 5, less than the price of 15 which she has to pay (if she joins the club)

Thus, the (gross) value to her of expanding consumption from 35 to 45 units is the area under the demand curve between those two quantities, i.e., the area of the trapezoid \{A,G,45,35\}=10 \cdot \left(\frac{15 + 5}{2}\right) = 100

But the cost of buying those additional 10 units is 150. **Hence, the loss she incurs because she is force to buy 45 units is (100-150)=-50**

Overall, then, the net gain from joining the club is: \([162.5-50]=112.5\)
So, she is still better off from joining the club then she would be in shopping locally and paying 20, but the minimum buying constraint – by forcing her to buy more than she wants – lowers the amount she would pay to join the club.

2. A consumer has the utility function: \( U(x, y) = x \cdot y \). Given income, \( I \), and prices \( (P_x, P_y) \):

a) Again, the simplest way to find the demand curves is to use the budget constraint to eliminate one good and then maximize over the remaining good:

\[
P_x x + P_y y = I \Rightarrow x = \left( \frac{I}{P_x} \right) - \rho y; \quad \rho = \left( \frac{P_y}{P_x} \right)
\]

Substituting into the utility function:

\[
U = xy = \left[ \left( \frac{I}{P_x} \right) - \rho y \right] y.
\]

Maximizing over \( y \) yields the demand function:

\[
\left( \frac{dU}{dy} \right) = \left[ \left( \frac{I}{P_x} \right) - 2 \rho y \right] = 0 \Rightarrow y^* = \left( \frac{I}{2P_y} \right) \Rightarrow x^* = \left( \frac{I}{2P_x} \right)
\]

Since the utility function is “Cobb-Douglas”, the demand curves have the property that the amount spent on each good is independent of price (a constant share of income is spent on each good). If you substitute these demands back into the utility function you get the “indirect utility function”:

\[
U^* (I, P_x, P_y) = x^* \cdot y^* = \frac{I^2}{4P_x P_y}
\]

i. Clearly, for these preferences, an increase in income increases the demands for both goods. In problem 1 (with “quasi-linear” utility) the demand for good \( y \) was independent of income.

b) Assume originally \( I = 3000, P_x = 1, P_y = 20 \), so

\[
y^* = 75, x^* = 1500 \quad \text{and} \quad U^* = \frac{I^2}{4P_x P_y} = \frac{(3000)^2}{80}
\]

If Costco opens and the person can buy at \( P_y = 15 \) then at these prices:

i. The new demands will be \( y^{**} = 100, x^{**} = 1500 \) and clearly the person is better off.

ii. Let \( F \) denote the maximum amount the person is willing to pay to join Costco. If she pays that amount, her income (left to spend on goods) is \( I' = (3000 - F) \), and at the new prices her purchases are:

\[
x' = \frac{3000 - F}{2P_x} = \left[ 1500 - \left( \frac{F}{2} \right) \right]; \quad y' = \frac{3000 - F}{2P_y} = \left[ 100 - \left( \frac{F}{30} \right) \right]
\]

Maximized utility is:

\[
U' = \frac{(I - F)^2}{4P_x P_y} = \frac{(3000 - F)^2}{60}
\]

Setting that equal to the utility obtained in the original situation gives:

\[
\frac{(3000 - F)^2}{60} = \frac{(3000)^2}{80} \Rightarrow (3000 - F) = (3000)\sqrt{\frac{3}{4}} \quad \text{or} \quad F = 3000 - 1500 \cdot \sqrt{3} \approx 402
\]
iii. The amount, from (ii), is not the same as the area next to the demand curve (if you are using the usual Marshallian demand curve). The reason for this is that as you move along this demand the amount the person buys – as price falls – increases both because of substitution and income effects. But, if the person actually had to pay to join the club, this would reduce her income and hence reduce the amount of \( y \) she would buy. Hence, if the demand for a good varies with income, you cannot use the Marshallian demand curve to calculate the true willingness to pay – you have to use what is called a Hicksian demand curve – in which utility is held constant – to calculate the willingness to pay. From the actual demand curve:

![Diagram of demand curve](image)

The area next to the demand curve – i.e., the area \( \{20, A, B, 15\} \) can be shown to be: 

\[
\text{Area} = 1500 \cdot \left( \ln(20) - \ln(15) \right) = 1500 \cdot \ln\left(\frac{4}{3}\right) \approx 431
\]

which is larger than the amount calculated in (ii). It is a general result that measuring the “consumer surplus” in this way will lead to an overcalculation of the value of a price decrease for a normal good (because if the consumer actually pays to join the club consumption will fall).

3. Consider a firm with total cost curve: 
\[
TC(q) = 9 + 4q + \left(\frac{q^2}{4}\right)
\]
where 9 is fixed cost.

(a) Find the firm’s marginal cost curve and, given price, \( p \), the firm’s short run supply curve.

The firm’s marginal cost is the derivative of the total cost curve: 
\[
MC = \frac{dTC}{dq} = 4 + \left(\frac{q}{2}\right)
\]

The firm’s profit function is: 
\[
\pi = pq - TC(q) = pq - \left( 9 + 4q + \left(\frac{q^2}{4}\right) \right)
\]

Maximizing profits yields: 
\[
\left( \frac{d\pi}{dq} \right) = p - MC = p - \left( 4 + \left(\frac{q}{2}\right) \right)
\]

Since \( q < 0 \) makes no sense, if \( p < MC \) at \( q=0 \), the firm should not produce. Otherwise, output level is determined by choosing \( q \) so that \( p = MC \).
Hence, the supply curve is: 
\[ p \leq 4, \quad q^* = 0; \quad p \geq 4, \quad q^* = 2 \cdot (p - 4) \]

And maximized profits are:
\[ \pi^*(p) = \pi(q^*) = p q^* - 9 - 4q^* - (1/4)(q^*)^2 = [p - 4]^2 - 9; \quad p \geq 4 \]

(b) Find the firm’s long run supply curve.

To produce in the long run, we need non-negative profits; this is the same thing as saying that price must be greater than, or equal to, minimum average cost (which occurs at \( MC = AC \)). From the cost function:

\[ AC(q) = (TC/q) = (9/q) + 4 + (q/4) \quad \text{and} \quad MC(q) = 4 + (q/2); \quad \text{hence,} \quad MC = AC \implies \quad q^2 = 36 \quad \text{or} \quad q = 6. \quad \text{At} \quad q = 6, \quad AC = 7. \quad \text{So, for} \quad p < 7, \quad \text{the firm would lose money in the long run.} \]

The same result can be seen from the profit function above: 
\[ \pi^*(p) = [p - 4]^2 - 9 \geq 0 \Rightarrow p \geq 7. \]

Hence, the long run supply curve is: 
\[ p \leq 7, \quad q^*(p) = 0; \quad p \geq 7, \quad q^*(p) = 2 \cdot (p - 4). \]

(If you look carefully, you will note that there are two different output levels at \( p = 7 \); the firm is indifferent between these two output levels at that price.

(c) Using the firm’s supply curve, show how much costs increase as \( q \) increases from 12 to 22.

Since the supply curve is the marginal cost curve, the area under the supply curve, between the two output levels, measures the change in total cost due to the change in output. Thus, the relevant area is the area of the trapezoid \{12,A,B,22\}, which has area 125 (base is 10, average height is 12.5). This answer can be
verified by plugging the output levels back into the total cost curve:

\[ TC(12) = 9 + 4 \cdot 12 + \left(12^2/4\right) = 93; \quad TC(22) = 9 + 4 \cdot 22 + \left(22^2/4\right) = 218. \]

Thus,

\[ TC(22) - TC(12) = 218 - 93 = 125. \]

(d) The firm sells output at \( p = 10 \). How much do profits increase if price increases to 15?

Note that (not coincidentally!) at \( p = 10 \) the profit-maximizing output is 12, and at \( p = 15 \) the profit-maximizing output is 22. The increase in producer surplus (or profits) due to this price increase is the area next to the supply curve between the two prices; in the figure above, this is the area of the trapezoid \{10,A,B,15\}. This area is: \((1/2) \cdot 5 \cdot (12 + 22) = 85\). This answer can be verified from the profit function: \( \pi^*(p) = [p - 4]^2 - 9; \rightarrow \pi^*(10) = 27; \quad \pi^*(15) = (121 - 9) = 112; \) hence, the change in profits is 85, as measured by the area next to the supply curve between the two prices.

4. Nicaragua has the following market supply (S) and demand (D) curves for sugar:

\[ S = 175P^d; \quad D = 1000 - 25P^d \quad \text{where} \ P^d \ \text{denotes the domestic price of sugar (in cents/pound)}\]

a) Assuming there is no trade, find the equilibrium price of sugar and the quantity transacted.

For Nicaragua, excess supply (ES) is: \( ES = S - D = 200P^d - 1000 \). Without trade, we must have \( ES = 0 \), which implies: \( P^e = 5, q^e = 875 \).

b) Suppose world price is $0.20/pound (\( P^w = 20 \)). Assuming free trade, what are domestic price, production and consumption? Find the change in consumer surplus and producer surplus.

Due to the price increase, domestic output increases to 3500, domestic demand falls to 500, and 3000 units are exported.
Consumer surplus (CS): the **decrease** in CS is area \( \{5,A,C,20\} = 10,312.5 \).
Producer surplus (PS): the **increase** in PS is area \( \{5,A,B,20\} = 32,812.5 \).

Not everybody gains, but due to trade the benefits to the winners (the producers) exceed the costs to the losers (the consumers) by 22,500 – which is the same as the area of triangle \( \{ABC\} \). Why this is so will be explained later in the semester.

c) What is the impact of an export subsidy of 10, assuming the world price does not change?

If the government gives an export subsidy of 10, then any exporter earns the world price, 20, (which he gets from foreign buyers) plus the export subsidy, 10, which he gets from the government. Hence, if any sugar is to be sold domestically, it must yield the same price. Let \( P^d \) be the domestic price, \( P^w \) the world price, and \( s \) the export subsidy. Then:

\[
P^d = P^w + s = 20 + 10 = 30
\]

So, the export subsidy raises domestic prices, **benefits producers, hurts consumers and hurts taxpayers (who have to pay taxes to offset the subsidy)**. The quantitative impact is obtained as follows:

The subsidy increases domestic price from 20 to 30. **Compared to free trade:**

The loss in consumer surplus, due to the price rise, is the area: \( \{30,G,C,20\} = 10 \times (750/2) = 3,750 \)

The increase in producer surplus, due to the price rise, is the area: \( \{30,H,B,20\} = 10 \times (8750/2) = 43,750 \)

The cost of the subsidies, paid by the taxpayer, is the area: \( \{G,H,K,J\} = 10 \times 5000 = 50,000 \)

*The deadweight loss is \( 10,000 \) – which is the sum of the area of triangles 1 (GCJ) and 2 (HBK).*