1. Consider a specific factor model with the following production functions:

\[ Q_c = \theta \left( L_c \right)^{2/3} \left( K \right)^{1/3}; \quad Q_f = \left( L_f \right)^{2/3} \left( T \right)^{1/3}; \quad \left[ L_c + L_f \right] \leq \overline{L} \]

a) To derive the labor demand curves, set the marginal value product of labor equal to the wage:

\[ Q_c = \theta \left( L_c \right)^{2/3} K^{1/3} \rightarrow P_c \left( \frac{\partial Q_c}{\partial L_c} \right) = \left( 2 \theta P_c / 3 \right) \left( L_c \right)^{-1/3} K^{1/3} = W \rightarrow \dot{L}_c = \dot{K} = K \left( 2 \theta P_c / 3 W \right)^3; \]

\[ Q_f = \left( L_f \right)^{2/3} T^{1/3} \rightarrow T \left( \frac{\partial Q_f}{\partial L_f} \right) = \left( 2 P_f / 3 \right) \left( L_f \right)^{1/3} T^{1/3} = W \rightarrow \dot{L}_f = T \left( 2 P_f / 3 W \right)^3 \]

Adding labor demands and setting it equal to supply yields the equilibrium wage:

\[ \dot{L}_f + \dot{L}_c = \dot{L} \rightarrow T \left( 2 P_f / 3 W \right)^3 + K \left( 2 \theta P_c / 3 W \right)^3 = \overline{L} \rightarrow W^* = \left( 2 / 3 \right) \left[ \left( T / \overline{L} \right) P_f^3 + \left( K / \overline{L} \right) \left( \theta P_c \right)^3 \right]^{1/3} \]

Note that: doubling prices causes wages to double (again, the pure inflation effect); and that doubling all inputs has no effect on wages (because of “constant returns to scale”).

b) Find the supply curves. Using the results for labor demands:

\[ Q_c = \left( \theta \left( K \right)^{1/3} \right) \left( \dot{L}_c \right)^{2/3} = \left[ \theta \left( K \right)^{1/3} \right] \left[ K \left( 2 \theta P_c / 3 W \right)^3 \right]^{2/3} = \left( 2 / 3 \right)^2 \theta^3 K \left( P_c / W \right)^2 \]

\[ Q_f = \left( T \right)^{1/3} \left( \dot{L}_f \right)^{2/3} = \left( T \right)^{1/3} \left[ \left( 2 P_f / 3 W \right)^3 \right]^{2/3} = \left( 2 / 3 \right)^2 T \left( P_f / W \right)^2 \]

Note that supplies depend on the ratio of prices. Substituting for the equilibrium wage from (a):

\[ Q_c = \theta K \left[ \left( T / \overline{L} \right) \left( \theta \rho \right)^3 + \left( K / \overline{L} \right) \right]^{-2/3}; \quad Q_f = T \left[ \left( T / \overline{L} \right) + \left( K / \overline{L} \right) \left( \theta \rho \right)^3 \right]^{-2/3}; \quad \text{where:} \]

\[ \rho \equiv \left( P_c / P_f \right) \]

Note that, from above, relative supplies are less complicated: \( Q_c^* / Q_f^* = \left( \theta^3 K \rho^2 / T \right) \)

c) Given prices, find how increases in resources affects supplies and factor prices. Supplies are given above, as is the equilibrium wage. To find the returns to capital \( R_K \) and land \( R_T \):

\[ R_K = P_c \left( \frac{\partial Q_c}{\partial K} \right) = \left( 2 \theta^2 P_c^2 / 9 W \right) \]

\[ R_T = P_f \left( \frac{\partial Q_f}{\partial T} \right) = \left( 2 P_f^2 / 9 W \right) \]

Substituting for the equilibrium wage:

\[ R_K = \left( 2 \theta^2 P_c^2 / 9 \right) \left[ \left( T / \overline{L} \right) P_f^3 + \left( K / \overline{L} \right) \left( \theta P_c \right)^3 \right]^{-1/3}; \quad R_T = \left( P_f^2 / 9 \right) \left[ \left( T / \overline{L} \right) P_f^3 + \left( K / \overline{L} \right) \left( \theta P_c \right)^3 \right]^{-1/3} \]

Now we can answer the questions asked.
(i) An increase in $T$: from the supply functions in (b) and the factor prices in (a) and (c) we see:

As $T$ (the factor specific to good $F$) increases, output of good $F$ increases, output of good $C$ falls, the equilibrium $W$ increases (more land per worker), and the returns to both capital and land fall (higher wage, plus for land the diminishing marginal productivity).

(ii) An increase in $L$: since $L$ is not specific to either sector, we (correctly) expect output of both goods to increase. The wage falls (more supply, same demand), which raises the returns to the other factors ($K$ and $T$). This is pretty much what you would “normally” expect.

(iii) The increase in $\theta$ (productivity in sector $C$), given output prices, increases output of $C$ and attracts more labor to $C$. Hence, output of $F$ falls, while output of $C$ obviously increases. The equilibrium wage rises (higher labor productivity), as does the return to capital (more tedious to show; the productivity increase raises the return to capital, higher wages lowers that return – but overall the return must rise). Though technology in sector $F$ is unaffected, the higher wage lowers the return to land.

d) Intuitively, the increase in $P_c$ should benefit factors specific to that sector ($K$), hurt factors specific to the other sector ($T$) and have an ambiguous effect on the mobile factor ($L$). Formally, from the results above:

$$\frac{R_k}{P_r} = \left( \frac{\theta^2 P_c}{3} \left[ \frac{T}{L} \right] P_r^3 + \left( \frac{K}{L} \right) \left( \frac{\theta P_c}{T} \right)^{1/3} \right) = \left( \frac{\theta^2}{3} \left[ \frac{T}{L} \rho^3 + \left( \frac{K}{L} \right) (\theta)^3 \right]^{1/3} \right)$$

Since the relative price of $C$ increases ($\rho$ increases), the real return to $K$ in terms of $C$ rises, which of course implies $\left( \frac{R_k}{P_r} \right)$. From (c), it is easy to see that $R_r$ decreases as $P_r$ rises, so the real return on land falls. Finally, from (a) you can see that $W$ (and hence $\left( \frac{W}{P_r} \right)$) increases as $P_r$ increases but: $\left( \frac{W}{P_r} \right) = \left( \frac{2}{3} \right) \left[ \left( \frac{T}{L} \right) \left( \frac{P_r}{P_c} \right)^3 + \left( \frac{K}{L} \right) (\theta)^3 \right]^{1/3}$ falls as $P_r$ increases, so the impact on the real return to labor is ambiguous.

e) We found relative supply in part (b); hence, setting relative supply equal to relative demand:

$$\left( \frac{Q_c}{Q_f} \right) = \left( \frac{D_c}{D_f} \right) \rightarrow \left( \theta^3 K \rho^3 / T \right) = \left( P_r / P_c \right) \rightarrow \rho^3 = \left( T / \theta^3 K \right) \rightarrow \rho^* = \theta^{-1} \left( T / K \right)^{1/3}$$

f) Given $K = 20, L = 20, T = 10, \theta = 1$, from (e), the equilibrium relative price is: $\rho^* = (1/2)^{1/3} \approx 0.7937$

Hence, if world prices are $\rho^* = 2$, the country will export $C$ and import $F$. From (d) we know that “producers” in sector $C$ – i.e., owners of capital, gain while landowners lose. The impact on labor is ambiguous. “Overall”, the country gains from trade, but not everybody gains.
Graphically, the gain would be shown as follows:

In the figure, AMQB is the production possibility frontier, point M is the autarky production (and consumption) point, and the slope of the line RR represents the autarky relative price of good C. The slope of line TT represents the world relative price of C, at which the country can import or export C. Since the world relative price of C is higher than the autarky price, domestic output of C increases and output of F falls (production moves to point Q, where the slope of the ppf equals the world relative price of C). Given production at Q, the budget constraint (balance of trade constraint) facing the economy is given by the line TT; along this line, utility is maximized at V, and hence this is where domestic consumption occurs. The difference between consumption at V and production at Q shows net imports of each good (i.e., imports of food and exports of cloth). Since the indifference curve through V must be higher than that through the autarky equilibrium, this shows that the “country as a whole” gains from trade, even though not everybody need gain.

g) If the world price is $\rho^w = (1/2)$, then the discussion above is reversed, since the autarky price of C is higher than the world price. The country will export F, and landowners will gain from trade, capitalists will lose and the impact on workers is ambiguous. Due to trade, the domestic relative price of C falls, output of C falls and output of F increases. Overall, the country will benefit but not all groups will favor free trade.
2. We have two countries and we want to show that, as in the Ricardian model, starting from autarky it is possible to increase output of both goods. Technology is given by:

\[
\begin{align*}
\text{US: } Q_c &= (L_c)^{2/3} (K)^{1/3}; \quad Q_f = (L_f)^{2/3} (T)^{1/3}; \quad \left[ L_c + L_f \right] \leq L^{US} = 20, K = 20, T = 10 \\
\text{UK: } Q_c^* &= \left( L_c^* \right)^{2/3} \left( K^* \right)^{1/3}; \quad Q_f^* = \left( L_f^* \right)^{2/3} \left( T^* \right)^{1/3}; \quad \left[ L_c^* + L_f^* \right] \leq L^{uk} = 20; K = 10, T = 20
\end{align*}
\]

Use the same demands from problem 1, so it is not necessary to resolve.

(a) From problem (1e) \( \rho^c = \theta^{-1} (T/K)^{1/3} \) hence: \( \rho^{US} = \left(1/2\right)^{1/3} < 2^{1/3} = \rho^{uk} \) since \( \theta = 1 \).

From (1b) we have supplies, so substituting:

\[
\begin{align*}
\text{US: } Q_c^* &= K \left[ \left( T/L \right) (\theta \rho)^{-3} + \left( K/L \right) \right]^{-2/3} = 10 \cdot 2^{1/3}; \quad Q_f^* = T \left[ \left( T/L \right) + \left( K/L \right) (\theta \rho)^{-3} \right]^{-2/3} = 10; \\
\text{by symmetry (or substituting):} \\
\text{UK: } Q_c^* &= 10; \quad Q_f^* = 10 \cdot 2^{1/3}
\end{align*}
\]

Note that the labor allocation for both countries is such that: \( L_c = L_f = 10 \).

(b) Since the MRT measures the opportunity cost of production in each country, and since these are different between the US and UK, it follows that world output of both goods can be increased by a proper rearrangement of resources. Starting from autarky, the MRT \( \left( dQ_f/dQ_c \right) \) in the UK is \( \rho^{uk} = 2^{1/3} \approx 1.26 \), whereas in the US it originally is \( \rho^{US} = 2^{-1/3} \approx .79 \). Thus, if output of good C is decreased by 1 unit in the UK, output of good F would increase by approximately 1.26 units, reflecting the autarky price (and opportunity cost) of good C. In the US, if output of good C is increased by (1.25) units, output of good F would decrease by approximately \( (1.25) \cdot (2^{-1/3}) \approx .99 \) units, reflecting the (autarky) MRT in the US. This change would result in an increase in world output of approximately \( \frac{1}{4} \) unit of cloth and .27 units of food.

To see numerically, initially world output of each good is given by:

\[
\begin{align*}
S_c^{aw} + S_c^{uk} &= 10 + 10 \cdot 2^{1/3} \approx 22.60; \quad S_f^{aw} + S_f^{uk} &= 10 + 10 \cdot 2^{1/3} = 22.60
\end{align*}
\]

We should be able to increase this by increasing food production in the US and cloth production in UK. If we try \( L_c^{uk} = 5, L_f^{uk} = 15 \) and \( L_c^{aw} = 15, L_f^{aw} = 5 \), for example, then:

\[
\begin{align*}
Q_c^{uk} &= \left(10 \right)^{1/3} \left(5 \right)^{2/3} \approx 6.3; \quad Q_f^{uk} = \left(20 \right)^{1/3} \left(15 \right)^{2/3} \approx 16.5; \quad \text{by symmetry, } Q_f^{aw} \approx 6.3; \quad Q_c^{aw} \approx 16.5
\end{align*}
\]

So, world output of both goods is now 22.8, and hence output of both goods. (this is very tedious and you do not have to do the computation – just explain the logic).
World efficiency in production would require – if both goods are produced in both countries - that the MRT be equal in the two countries. Free trade guarantees this condition is satisfied. If, for example, \[ \frac{dQ_f}{dQ_c}^{uk} > \frac{dQ_f}{dQ_c}^{us} \], this means that it is relatively cheaper to produce cloth in the US, and a small increase in cloth output in the US, coupled with a decrease in the UK, could increase world output of both goods.

No, the condition for efficiency with an interior solution (both countries producing both goods) requires the MRT be equal across countries. However, if for example – \[ \frac{dQ_f}{dQ_c}^{uk} > \frac{dQ_f}{dQ_c}^{us} \] for all levels of output, then it is always cheaper to produce C in the US than in the UK, and you will not have both countries produce both goods. Thus, the difference between the two models is that in the Ricardian model the MRT is constant so MRTs can never be equal across countries - whereas in most other models the MRT changes with the level of output. So, specialization is much less likely in a model with increasing MRT.

In this example, the US will export C (it has more capital) and import food. Trade will achieve efficient production, given inputs (i.e., the MRT will be equalized across countries). However, it need not be the case that the wage will be equalized across countries and hence labor migration could possibly further increase efficiency.

If US productivity doubles in both goods, the US production possibility frontier shifts out and, because of the way in which productivity has increased, the slope of the ppf will be the same along a ray from the origin. Given the supply and demand structure, the autarky equilibrium relative price in the US won’t change, but real income (and the real return to all factors) increases due to this technological change. Nevertheless, of course, both countries would still gain from trade (in the Ricardian context, neither country has a “comparative advantage” due to technological considerations) and specialization would still be unlikely (it will, in fact, not occur given the functional forms).

Consider a simplified version of the Heckscher-Ohlin model with the following technology:

To produce cloth: two units of labor and one unit of capital are required per unit output of cloth. To produce food: one unit of labor and two units of capital are required for each unit of food.

Let \( Q_c \), \( Q_f \) denote the outputs of good \( C \) and \( F \), respectively. The resource constraints are:

**Labor:** \( 2Q_c + Q_f \leq L \) since the technology implies: \( L_c = 2Q_c \) and \( L_f = Q_f \)

**Capital:** \( Q_c + 2Q_f \leq K \) since the technology implies: \( K_c = Q_c \) and \( K_f = 2Q_f \)

The following figure shows the production possibility frontier for this economy; the points on, or below, the line labeled “labor constraint” insure that labor employed is no larger than available labor (with full employment on that line), while the line labeled capital constraint has the same interpretation. For this
simplified economy, the only output level where both inputs are fully employed is where the two lines intersect, at point V, where output is:

\[ Q_c = \frac{2L - K}{3}; \quad Q_f = \frac{2K - L}{3} \]

The feasible production set is the region bounded by: \( \{0,(K/2), V, (L/2)\} \), and the production possibility frontier is the line segments described by: \( \{(K/2), V, (L/2)\} \).

(i) **Show how an increase in the supply of capital shifts the ppf.**

An increase in capital shifts the capital constraint outward, as shown by the dotted line below. The point \( W \) represents the new output level where both factors are fully employed (in this simple version, there is a unique production point that represents full employment of both inputs).

**Note that an increase in \( K \) leads to an increase in output of the capital intensive good (F) and a decrease in output of the labor intensive good – much as in the specific factor model, where an increase in the amount of a specific factor leads to an increase in output of the good that uses that specific factor and a decrease in output of the other good.**

b) **Find input prices \( \{W, R\} \) in terms of output prices, assuming both goods are produced and both factors fully used.**
Using the technology, since each unit of $C$ requires two units of labor and 1 unit of capital, costs are:

$$MC_c = 2W + R.$$ 
Similarly, for good $F$, which uses two units of capital and 1 unit of labor:

$$MC_f = 2R + W.$$ If goods are produced, price must equal marginal cost; thus:

$$P_c = 2W + R; \quad P_f = 2R + W.$$ Solving for $R, W$ in terms of output prices yields:

$$W = \frac{2P_c - P_f}{3}; \quad R = \frac{2P_f - P_c}{3}$$

Note that an increase in the price of good $F$ (the capital intensive good) leads to an increase in the real return on capital (i.e., $\left(\frac{R}{P_f}\right)$ increases as $P_f$ increases) and a decrease in the real wage; compare this to the results of the specific factor model. Similarly, an increase in $P_c$ would cause $R$ to fall and $\left(\frac{W}{P_c}\right)$ to increase).

(i) Note that, as long as the country produces both goods, then – given output prices – the factor prices are independent of the domestic supplies of labor and capital. **Hence**, given output prices, changes in factor supplies do not affect factor prices. As we saw in Question 2, in the specific factor model, factor prices depend on factor supplies and output prices. {Mathematically, these results depend on whether there are at least as many goods as there are factors}.

c) **Assuming the US is capital abundant and Mexico is labor abundant (but they have identical tastes and technology), find the pattern of trade and discuss its consequences.**

As shown above, given prices, an increase in the supply of capital increases output of the capital intensive good ($F$) and decreases output of the labor intensive good ($C$). At given prices, this will create an excess supply of good $F$ and an excess demand for good $C$. Hence, as the stock of capital increases within an economy, the equilibrium price of the labor-intensive good increases.

Thus, the autarky relative price of good $C$ will be higher in the US than in Mexico. This, from the previous part, implies that the wage rate will be higher in the US and the return on capital will be higher in Mexico (i.e., in autarky $\left(\frac{P_c}{P_f}\right)_\text{us} > \left(\frac{P_c}{P_f}\right)_\text{mex}$)

Thus, with trade, the US will export $F$ (the capital-intensive good) and import $C$ (the labor-intensive good). **As a result of trade,** $\left(\frac{P_c}{P_f}\right)$ falls in the US and increases in Mexico.

But, from (b), this implies that the wage rate falls in the US and rises in Mexico, while the return on capital ($R$) rises in the US and falls in Mexico.

Finally, if free trade equalizes commodity prices and both goods are produced in both countries, it must equalize factor prices (see equations determining factor prices in (b)), **provided technology is the same in the two countries.** This is the “factor price equalization theorem”.

d) Suppose US productivity in the food sector doubles so:

**To produce food requires one-half unit of labor and one unit of capital per unit output of food.**

Given output prices, this productivity increase will increase output of food, not affect output of cloth (because of the special technology; typically output of cloth would **decrease**), raise the return to capital
(the factor used intensively in food) and lower the return to labor. Because of the technological
difference free trade will no longer equalize factor prices. If you want to see this mathematically, suppose
technology is such that:

**To produce cloth:** two units of labor and one unit of capital are required per unit output of cloth.
**To produce food:** \((1/\theta)\) unit of labor and \((2/\theta)\) units of capital are required for each unit of food.

(Originally, \(\theta = 1\), then with improved technology \(\theta = 2\)). The resource constraints look like:

\[
2Q_c + (1/\theta)Q_f \leq \bar{L}; \quad Q_c + (2/\theta)Q_f \leq \bar{K}.
\]

Assuming both are equalities and solving implies:

\[
Q_f = \left[(2\theta K - \theta L)/3\right]; \quad Q_c = \left[(2L - K)/3\right].
\]

Similarly, for factor prices:

\[
P_c = MC_c = 2W + R; \quad P_f = MC_f = (W/\theta) + (2R/\theta).
\]

Solving gives:

\[
W = \frac{2P_c - \theta P_f}{3}; \quad R = \frac{2\theta P_f - P_c}{3}
\]

so the productivity increase in sector \(F\) raises the real return to the
factor used intensively in that sector \((K)\) and lowers the real return to the other factor \((L)\).

4. **(10 point extra credit)** (More sophisticated version of Heckscher-Ohlin model). There are two goods
(C and F) and two inputs (K and L), with the following production functions:

\[
Q_c = K_c^{1/3}L_c^{2/3}; \quad Q_f = K_f^{2/3}L_f^{1/3}
\]

where \(\{K_c, L_c\}\) are the inputs (capital, labor) used in sector \(C\) and \(\{K_f, L_f\}\) are the inputs used in
sector \(F\). Let \(W\) denote the wage rate (price of L) and \(R\) the rental rate (cost of using K, capital).

Finally, let \(P_c, P_f\) denote the output prices of goods C and F, respectively.

(a) There are two ways to derive the cost function – one by substitution, and one by using non-linear
programming (which involves using the Lagrangean function).

By substitution:

Let \(Q_i = K_i^{\alpha}L_i^{\beta}\). Solving for capital yields:

\[
K_i = \left(\frac{Q_iL_i^{-\beta}}{L_i^{1-\beta}}\right)^{1/\alpha} = \left(Q_i\right)^{1/\alpha} \left(L_i\right)^{-\beta/\alpha}.
\]

Total costs are:

\[
TC = WL_i + RK_i = WL_i + R\left(Q_i\right)^{1/\alpha} \left(L_i\right)^{-\beta/\alpha}
\]

Equation (1) expresses total costs as a function of factor prices, output and labor inputs. Cost
minimization means choose the labor input that minimizes this expression. Thus:

\[
\frac{d(TC)}{dL_i} = W - \left(\frac{\beta}{\alpha}\right)R\left(Q_i\right)^{1/\alpha} \left(L_i\right)^{-\beta/\alpha - 1} = 0
\]

since:

\[
\frac{d\left(L_i\right)^{\alpha}}{dL_i^{\alpha-1}} = \alpha \left(L_i\right)^{\alpha-1}
\]

A sufficient condition for an interior minimum is that the first derivative of the function be zero
and that the second derivative be positive; it is readily seen that the second derivative is positive.

Solving (2) for \(L_i\) yields:
where \( L_i^* \) denotes the solution. Substituting back for \( K_i \) yields:

\[
K_i^* = \left( \frac{\alpha W}{\beta R} \right)^{\beta} Q_i \left( \frac{\beta R}{\alpha W} \right)^{\alpha} \quad (4)
\]

Note that the choice of inputs depends on relative factor prices, not absolute factor prices. Also, note that when \((\alpha + \beta) = 1\), the input use is proportional to output. Finally, substituting back into the cost function (1) yields minimum costs:

\[
C^* (Q, W, R) = WL_i^* + RK_i^* = \lambda \left( W^\beta R^\alpha Q_i \right)^{\frac{1}{\alpha + \beta}} \quad \text{where:} \quad \lambda = \frac{(\alpha + \beta)}{\left( \beta^\beta \alpha^\alpha \right)^{\frac{1}{\alpha + \beta}}}
\]

(you need to substitute and then simplify the expression; it is a bit tedious, but you should get the result above):

This result can be checked because the derivative of the cost function with respect to input price should give you back the optimal input use. Hence:

\[
\frac{dC^*}{dW} = \frac{\beta}{\alpha + \beta} \left( W^\beta R^\alpha Q_i \right)^{\frac{1}{\alpha + \beta}} R^\beta \left( \frac{\beta R}{\alpha W} \right)^{\alpha} \left( \frac{\alpha W}{\beta R} \right)^{\beta} = \left( \frac{\beta R}{\alpha W} \right)^{\alpha} Q_i \left( \frac{\beta R}{\alpha W} \right)^{\beta} \quad (3)
\]

which is \( L_i^* \). Similarly, differentiating with respect to \( R \) gives you \( K_i^* \):

\[
\frac{dC^*}{dR} = \frac{\alpha}{\alpha + \beta} \left( W^\beta R^\alpha Q_i \right)^{\frac{1}{\alpha + \beta}} W^\beta \left( \frac{\alpha W}{\beta R} \right)^{\alpha} \left( \frac{\beta R}{\alpha W} \right)^{\beta} = \left( \frac{\alpha W}{\beta R} \right)^{\alpha} Q_i \left( \frac{\alpha W}{\beta R} \right)^{\beta} \quad (3)
\]

Use of the Lagrangean function gives the same results, of course. Briefly, the Lagrangean is:

\[
H \equiv WL_i + RK_i + \theta \left( Q_i - K_i^* L_i^\beta \right)
\]

where \( \theta \) is the Lagrangean multiplier. Partially differentiating yields, for an interior solution:

\[
\frac{\partial H}{\partial L_i} = W - \theta \beta L_i^{\beta - 1} K_i^\alpha = 0 \quad (1a)
\]

\[
\frac{\partial H}{\partial K_i} = R - \theta \alpha L_i^{\beta} K_i^{\alpha - 1} = 0 \quad (2a)
\]

\[
\frac{\partial H}{\partial \theta} = \left( Q_i - K_i^* L_i^\beta \right) = 0 \quad (3a)
\]
Taking the ratio of (1a) to (2a) yields:

\[
\frac{\beta K_i}{\alpha L_i} = \frac{W}{R} \Rightarrow K_i = \left(\frac{\alpha}{\beta}\right)\left(\frac{W}{R}\right) L_i \quad (4a)
\]

Hence, the capital intensity depends on factor prices and – in terms of the original production function – is increasing in the parameter on “K” and decreasing in the parameter on “L”.

Using (4a) to solve for \( K_i \) in terms of \( L_i \), and then substituting this into (3a) yields the optimum labor input, which will be the same as above. Then, using this solution for labor, the solution for capital is found from (4a), and the cost curve by plugging back into the objective function.

You do not need to solve for \( \theta \), but if you do you get the following from (1a)

\[
\theta^* = \frac{W}{\beta} \left(\frac{L_i}{K_i}\right)^{1-\beta} = \frac{W}{\beta} \left(\frac{\beta R}{\alpha W} Q\right)^{(1-\beta)/(\alpha+\beta)} \left(\frac{\alpha W}{\beta R} Q\right)^{-\alpha/(\alpha+\beta)} \quad (5a)
\]

Simplifying (5a) yields:

\[
\theta^* = \left(\frac{W}{\beta} \left(\frac{\alpha W}{\beta R} Q\right)^{-\alpha/(\alpha+\beta)} \right)^{(1/(\alpha+\beta))} \left(Q_i\right)^{(1/(\alpha+\beta))-1} \quad (6a)
\]

Looking back at the cost function derived above and comparing to (6a), we see that (6a) represents the marginal cost function. This is no coincidence; the Lagrangean multiplier – in this problem – will always yields the marginal cost function.

Finally, for the specific functions given:

\[
Q_e = K_c^{1/3} L_c^{2/3} \Rightarrow \alpha = \frac{1}{3}, \beta = \frac{2}{3}, (\alpha + \beta) = 1 \text{ so: } TC_e^* (Q_e, W, R) = \lambda W^{2/3} R^{1/3} Q_e
\]

\[
Q_f = K_f^{2/3} L_f^{1/3} \Rightarrow \alpha = \frac{2}{3}, \beta = \frac{1}{3}, (\alpha + \beta) = 1 \text{ so: } TC_f^* (Q_f, W, R) = \lambda W^{1/3} R^{2/3} Q_f
\]

where: \( \lambda = \left(\frac{2}{3}\right)^{(2/3)} \left(\frac{1}{3}\right)^{(1/3)} = \frac{3}{2^{(2/3)}} \)

Clearly, good C is labor intensive and F is capital intensive as we have from (4a) above:

\[
\frac{K_i}{L_i} = \left(\frac{\alpha}{\beta}\right)\left(\frac{W}{R}\right) \Rightarrow K_c = \frac{\omega}{2}, \quad K_f = 2\omega \text{ where: } \omega = \frac{W}{R}
\]
(b) **Given output prices**, show how an increase in the available supply of labor will increase output.

From the costs curves above we have:

\[ MC_c = \lambda W^{2/3} R^{1/3} = P_c \quad \text{and} \quad MC_f = \lambda W^{1/3} R^{2/3} = P_f \quad (1b) \]

We can use these two equations to solve for factor prices in terms of output price. Taking the ratio of marginal costs and setting equal to the price ratio (relative prices) yields:

\[
\frac{MC_f}{MC_c} = \frac{\lambda W^{1/3} R^{2/3}}{\lambda W^{2/3} R^{1/3}} = \left( \frac{W}{R} \right)^{-1/3} \Rightarrow \omega = \rho^3 \quad \text{where:} \quad \rho = \frac{P_c}{P_f}; \quad \omega = \frac{W}{R} \quad (2b)
\]

Plugging this back into (1b) and solving gives the level of factor prices:

\[
R = \frac{P_f^2}{\lambda P_c} \quad W = \frac{P_c^2}{\lambda P_f} \quad (3b)
\]

From (3) and (4) in part (a) you have the optimal amount of inputs in each sector:

\[
L_c^* = \left( \frac{\beta R}{\alpha W} \right)^{\alpha/(\alpha + \beta)} Q_i, \quad \rightarrow L_c^* = \left( \frac{2R}{W} \right)^{\alpha/(\alpha + \beta)} Q_c, \quad \text{and} \quad L_f^* = \left( \frac{R}{2W} \right)^{2/3} Q_f \quad (4b)
\]

\[
K_c^* = \left( \frac{\alpha W}{\beta R} \right)^{\beta/(\alpha + \beta)} Q_i, \quad \rightarrow K_c^* = \left( \frac{W}{2R} \right)^{2/3} Q_c, \quad K_f^* = \left( \frac{2W}{R} \right)^{1/3} Q_f \quad (5b)
\]

You can express these input demands in terms of output price by substituting for \( \frac{W}{R} \) in terms of \( \frac{P_c}{P_f} \). Doing so and writing the resource constraints yields:

\[
L_c + L_f = L \rightarrow L_c = \frac{2^{1/3}}{\rho} Q_c + \frac{1}{2^{2/3} \rho^2} Q_f = L \quad (6b)
\]

\[
K_c + K_f = K \rightarrow \left( \frac{\rho^2}{2^{2/3}} \right) Q_c + 2^{1/3} \rho Q_f = K \quad (7b)
\]

Given prices, equations (6b) and (7b) are just like problem #3 (i.e., the labor and capital use per unit output are fixed) and can be solved for output levels. Doing so yields:

\[
Q_c = 2^{2/3} \left( 2\rho L - \rho^2 K \right); \quad Q_f = 2^{1/3} \left( 2\rho K - \rho^2 L \right)
\]

Thus, given prices, an increase in \( L \) will increase the output of good C, the labor intensive good,
and decrease the output of good F. Also, note that if output prices do not change, input prices do not change since, from equation (3b) above, factor prices can be solved for in terms of only output prices.

Thus, given prices, an increase in L causes the supply of good C to increase and that of good F to decrease. But, total income increases as L increases, so demand for both goods increases. Hence, to restore equilibrium, the price of good C must decrease (relative price of C decreases). Thus, the autarky (relative) price of good C is a decreasing function of the country’s relative labor abundance. Consequently, given the same demands and technology, the labor abundant country will export the labor intensive good.

Further, since the wage rate decreases, and the rental rate increases, as the price of good C decreases (i.e., the relative price of good F increases), this means that the autarky wage rate will be lower, and the autarky rental rate on capital higher, in the labor abundant country. Hence, differences in factor supplies lead to differences in autarky factor prices, as one would expect, and to differences in output prices.

(c) To show how factor prices change with output prices, look back at equation (3b).

\[
R = \frac{P_f^2}{\lambda P_c} \rightarrow \left( \frac{\partial R}{\partial P_f} \right) = \frac{2 P_f}{\lambda P_c} = \frac{2 R}{P_f} \quad \text{so:} \quad \left( \frac{\partial R}{\partial P_f} \right) \left( \frac{P_f}{R} \right) = 2 > 1
\]

and:

\[
\left( \frac{\partial R}{\partial P_c} \right) = -\frac{P_f^2}{\lambda P_c^2} = -\left( \frac{R}{P_c} \right) < 0.
\]

Similarly:

\[
W = \frac{P_c^2}{\lambda P_f} \rightarrow \left( \frac{\partial W}{\partial P_f} \right) = -\frac{P_c^2}{\lambda P_f^2} < 0, \quad \left( \frac{\partial W}{\partial P_c} \right) = \frac{2 P_c}{\lambda P_f} = \frac{2 W}{P_c} \rightarrow \left( \frac{\partial W}{\partial P_c} \right) \left( \frac{P_c}{W} \right) = 2 > 1
\]

These results imply that an increase in \( P_c \), the labor intensive good, lowers the real return to capital and raises the real return to labor in terms of either good (the second result says for a 1% increase in \( P_c \), increases the wage \( W \) by 2%, and hence \( W/P_c \) increases as \( P_c \) increases). Similarly, an increase in the price of the capital intensive good (\( F \)) lowers the real return to labor and raises the real return to capital in terms of either good.

The results show that, as \( P_f \) increases, \( R \) increases proportionately more than \( P_f \) (i.e., \( R/P_f \) increases with \( P_f \)). This is the Stolper-Samuelson result.

(d) From part (c), it is apparent that the impact of trade on the distribution of income depends upon how trade affects the relative prices of goods.

If a country exports good F, then trade causes the relative price of F to increase, and thus trade increases the real return to capital and lowers the real return to labor.

If a country exports good C, then trade raises the relative price of good C, and hence trade increases the real return to labor and lowers the real return to capital.

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From part (b), assuming countries have identical technologies and tastes, then the autarky relative price of good C will be higher in the capital-abundant country (equivalently, the autarky relative price of good F will be higher in the labor-abundant country). Thus, from part (b), we predict that the capital-abundant country (the country with more capital per worker) will export good F and import good C, and that the labor-abundant country (the one with less capital per worker) will export C and import F.

Thus, putting the above together, we predict that trade will lower the real return to labor and raise the real return to capital in the capital-abundant country; and that trade will raise the real return to labor and lower the real return to capital in the labor-abundant country. This is the key prediction of the Heckscher-Ohlin model. Not everybody gains from trade – and hence there will be groups opposing trade liberalization.

Finally, note that free trade – assuming technology is the same throughout the world - will lead to equal post-trade factor prices across the world (i.e., the wage rate will be equalized between the capital-abundant and labor-abundant countries, and the same for the return on capital) provided that (i) free trade equalizes the prices of goods – i.e., ignore transportation costs and tariffs; and (ii) both goods are produced in both countries.

(e) If the U.S. is capital-abundant and the Heckscher-Ohlin model can be applied to the world economy, the U.S. will export the capital-intensive good (F) and import the labor-intensive good (C). Trade restrictions (tariffs or quotas) will raise the domestic price of the import good (C) and thus will increase the real return to labor and decrease the real return to capital. Hence, we would predict that groups representing labor (Unions, for example) would oppose free trade and groups representing “capitalists” – firms – would favor free trade. This is a reasonably accurate description of actual political positions (though the reality is a bit more complicated, of course).

(f) The specific answer depends upon how the technology differs between countries, and in which goods. To give you an idea, assume what is called factor-neutral technology, which increases productivity of inputs proportionately. Mathematically, this means the US production functions, and hence the cost curves, are modified to:

\[ Q_c = A_c K_c^{1/3} L_c^{2/3}; \quad Q_f = A_f K_f^{1/3} L_f^{2/3}; \quad A_c \geq 1, A_f \geq 1. \]

\[ TC^*_c = \lambda W^{2/3} R^{1/3} (Q_c / A_c); \quad TC^*_f = \lambda W^{1/3} R^{2/3} (Q_f / A_f). \]

As a result, factor prices (in terms of output prices) are (follow the same steps as earlier):

\[ P_c = MC_c = \lambda W^{2/3} R^{1/3} \left( 1/A_c \right) \quad P_f = MC^*_f = \lambda W^{1/3} R^{2/3} \left( 1/A_f \right) \]

implies:

\[ R = \frac{(A_f P_f)^2}{\lambda \left( A_c P_c \right)}; \quad W = \frac{(A_f P_f)^2}{\lambda \left( A_c P_c \right)} \]

(note that it is the product of price times technology that determines factor prices). Again, following the same steps as earlier to establish factor market equilibrium yields:

\[ K_c + K_f = K \rightarrow \left( \frac{\rho^2}{2^{2/3}} \right) (Q_c / A_c) + 2^{1/3} \hat{\rho} (Q_f / A_f) = K; \quad \omega = \left( A_c P_c / A_f P_f \right)^3 \]

\[ L_c + L_f = L \rightarrow 2^{1/3} \hat{\rho} (Q_c / A_c) + \frac{1}{2^{2/3} \rho^2} (Q_f / A_f) = L \quad \hat{\rho} = \left( A_c P_c / A_f P_f \right) = \omega^{1/3} \]
Finally, solving for outputs yields:

\[ Q_c = A_i \left( 2^{2/3} / 3 \right) \left( 2 \hat{\rho} L - \hat{\rho}^{-1} K \right); \quad Q_f = A_i \left( 2^{2/3} / 3 \right) \left( 2 \hat{\rho}^{-1} K - \hat{\rho}^2 L \right) \]

Now, some discussion. Notice that if \( A_i \) increase, given productivity in C and given output prices, then \( \hat{\rho} \) decreases. This means an increase in productivity in sector F, given output prices, will cause output of good C to fall and output of good F to increase by more than the productivity increase. In an autarky equilibrium, the increased productivity in sector F must lower the autarky price of F. Thus, technological differences predict the US exports F, as would factor abundance differences. Hence, in this case, the US would unambiguously export F – the “Ricardian technology” differences prediction and the factor abundance predictions coincide. Of course, if the US were capital abundant but had higher productivity in C (not F), then the two forces oppose and the outcome (which good the US exports) cannot be predicted without quantitative knowledge (on a Ricardian basis we expect to export C, but on a factor abundance basis we expect to export F). Finally, assume \( A_c = A_f > 1 \), so the US has an absolute advantage in both goods but (by Ricardian standards) no comparative advantage. Given prices, this joint productivity increase increases outputs by the same proportion and, if demands have unitary income elasticity, the productivity increase will not affect relative prices. Hence, the predictions of the H-O model – which uses factor abundance to predict the trade pattern - would hold. Finally, note that if technology differs across countries, factor price equalization cannot hold. Given output prices, an increase in \( A_i \) has the same effect as an increase in \( P_f \) - it increases the return to the factor used intensively in F (capital) and lowers the real return to the other factor (labor). If, for example, \( A_f > A_c > 1 \), so the US was more productive in both goods, but had the comparative advantage in F, the return on capital would be higher in the US and the return on labor would depend on the magnitudes of \( A_f, A_c \). If \( A_f = A_c > 1 \), then the return to both factors would be higher in the US, but the relative returns \( (W/R) \) would be equalized by free trade.