1. Consider a firm with total costs:

\[ TC(y) = 2y + \left(\frac{y^2}{6}\right); \quad y \geq 0 \]

a) The firm’s profits, as a function of output and price \( \pi(y) = p_y y - TC(y) = \left( p_y y - 2y - \left(\frac{y^2}{6}\right)\right) \).

i. **Find the output level, \( y^* (p) \), that maximizes profits.**

To find the output level, maximize profits with respect to output \( y \):

\[
\frac{d\pi(y)}{dy} = \left( p_y - \frac{y}{3} \right) = 0 \quad \Rightarrow \quad y^* = 0, \quad p \geq 2; \quad y^* = 3\left( p_y - 2 \right), \quad p_y \geq 2.
\]

ii. **Show that this output level (supply curve) is the firm’s marginal cost curve.**

The firm’s marginal cost is:

\[ MC = \frac{dTC}{dy} = 2 + \left(\frac{y}{3}\right) \]  

so:

\[ p_y = MC = 2 + \left(\frac{y}{3}\right) \rightarrow y^* = 3\left( p_y - 2 \right), \quad p \geq 2; \quad y^* < 0 \text{ does not make economic sense so for } p_y < 2, \text{ the firm does not produce any output } \left( p_y < MC \right). \]

iii. Use the firm’s supply curve, \( MC \), show graphically the increase in production costs when the firm’s output increases from \( y = 60 \) to \( y = 78 \).

The firm’s supply curve is shown as MC in figure on next page

So, from class, the change in cost is the area **under** the MC curve between the two output levels:

\[
TC(78) - TC(60) = \text{Area} \left\{60, A, B, 78\right\} = \int_{y=60}^{y=78} \left(2 + \left(\frac{y}{3}\right)\right) dy = \left(2y + \left(\frac{y^2}{6}\right)\right) \bigg|_{y=60}^{y=78} = 450
\]

\{You can ignore the integral, or terms in bracket, if it is not familiar\}. The area under the MC is a trapezoid, with area equal to base times average height: 18*25=450 (or break area up into a rectangle and triangle if that is easier).

**If you use the cost curve:** \( TC(60) = 2*60 + 60^2 \times \frac{1}{6} = 720 \)

\[
TC(78) = 2*78 + \frac{78^2}{6} = 1170 \quad \text{so } TC(78) - TC(60) = 1170 - 720 = 450
\]

iv. Use your answer to part (i) to calculate maximized profits:

\[
\pi^*(p) = p_y y^*(p) - 2y^* - \left(y^*\right)^2 / 6 = \left( p_y - 2 \right) \left( 3\left( p_y - 2 \right) \right) / 6 = (3/2)\left( p_y - 2 \right)^2, \quad p_y \geq 2;
\]

Profits are zero for \( p_y \leq 2 \) since \( y \) is zero.
b) Show how much the firm's profits increase when price increases from 22 to 28.

As discussed in class, this is the (change in) producer surplus, given by area $\{22,A,B,28\}$ in the figure below. Hence, the change in profits is 414.

You can check the answer by using the maximized profit function. From earlier:

$$\pi^* = (3/2)\left(p' - 2\right)^2; \quad \pi^*(22) = (3/2)(20)^2 = 600; \quad \pi^*(28) = (3/2)(26)^2 = 1014$$

Thus:

$$\pi^*(28) - \pi^*(22) = 414$$

![Graph showing profit changes](image)

c) The firm can either sell its output at a price of 22, or can sign an agreement with Walmart whereby it produces 120 shirts for Walmart and sells them at a price of 32. **Note that, at a price of 32, the profit-maximizing output is 90, so if it signs the agreement with Walmart it has to produce more than it would want to.**

To see the impact on profits (as compared to selling at a price of 22) you could just calculate the change in revenue and change in costs:

$$\Delta \pi = \{32 * 120 - C(120)\} - \pi^*(22) = \{3840 - 2640\} - 600 = 600$$

Alternatively (and a bit more complicated) you can use the supply curve above. From earlier results, we know if the firm could sell as much as it wants to Walmart at a price of 32, then the increase in profits would be the change in producer surplus, which is the area $\{22,A,C,32\}$ in the figure.

$$\pi^*(32) - \pi^*(22) = 750$$

That is the area of the trapezoid $\{22,A,C,32\}$

However, the agreement with Walmart requires the firm to produce 120 units, not 90 units. Thus, as the firm expands output from 90 to 120, costs go up by the area under the marginal cost curve $\{90,C,J,120\}$ but revenue goes up by only rectangle $\{90,C,E,120\}$. Therefore, by being forced to produce more than it wants to, the firm incurs a loss equal to area CEJ
Loss due to requirement to produce 120 units = Area \{C,J,E\}=150
Hence, the additional profits the firm earns by signing the agreement with Walmart is 750-150=600, the same as calculate above.

2. A consumer has the (quasi-linear) utility function: \( U(f,y) = f + 52\hat{y} - \left(\frac{y^2}{4}\right) \). \( I \) is income, and \((p_f, p_y)\) denote prices the individual pays for goods \( f \) and \( y \), respectively.

a) Set up the utility maximization problem and derive the individual’s demand functions (maximize utility subject to the budget constraint).

The budget constraint is: \( I - p_f f - p_y \hat{y} = 0 \rightarrow f = \left(\frac{I}{p_f}\right) - \left(\frac{p_y}{p_f}\right) \hat{y} \)

The simplest way to solve is by substituting the budget constraint into the utility function:

\[
U(f, \hat{y}) = \left(\frac{I - p_y \hat{y}}{p_f}\right) + 52 \hat{y} - \left(\frac{y^2}{4}\right) \text{ thus: } \frac{dU}{d\hat{y}} = -\frac{p_y}{p_f} + \left(52 - \frac{y}{2}\right) = 0 \rightarrow \hat{y}^d = 104 - 2\rho \text{ where }
\]

\( \rho \equiv \left(\frac{p_y}{p_f}\right) \), is the relative price of \( y \) (in terms of \( f \)). To guarantee both \( f \) and \( \hat{y} \) are non-negative we requires \( \rho \leq 52 \), and \( \frac{I}{p_f} \geq \rho \hat{y} = 2\rho \left(52 - \rho\right) \). We assume both conditions hold.

An alternative way to get the same demand curves is to set the marginal rate of substitution equal to the price ratio:

\[
\left(\frac{MU_y}{MU_f}\right) = \left(\frac{p_y}{p_f}\right) \rightarrow \left(\frac{52 - \left(\frac{y}{2}\right)}{1}\right) \left(\frac{p_y}{p_f}\right) = \rho , \text{ and use the budget constraint to proceed as above.}
\]

i. Find the individual’s maximized utility by substituting the demand solutions back into the utility function \{this is called the person’s indirect utility function\}.

Note that, with this solution, maximized utility is:

\[
U^* \left( I, P_x, P_y \right) = \frac{I}{p_f} - \rho \hat{y^*} + 52 y^* - \left[\left( y^* \right)^2 / 4 \right] = \frac{I}{p_f} + \left(52 - \rho\right)^2 \text{ since } \hat{y^*} = 2\left(52 - \rho\right).
\]

The function above is called the “indirect utility function” and it shows the individual’s maximized utility as a function of prices and income.

ii. How does an increase in the price of good \( y \) affect maximized utility? Take the (partial) derivative of the indirect utility function with respect to \( p_y \); what does this equal?

An increase in \( p_y \) will make the person worse off, of course; for a finite change, we can measure this loss by the area next to the demand curve between the two prices. For a small price change, the loss is approximated by \( \hat{y} \cdot \Delta p_y \): that is, by the amount consumed times the (small) price change. Taking the (partial) derivative of the indirect utility function with respect to \( p_y \):
\[ U^* (I, P_x, P_y) = \frac{I}{p_f} + (52 - \rho)^2; \quad \frac{\partial U^*}{\partial p_y} = -\frac{2(52 - \rho)}{p_f}. \]

If we let \( p_f = 1 \) then this is the (negative) of the demand for good \( y \). Formally: \( y^* = -\left(\frac{\partial U^*/\partial p_y}{\partial U^*/\partial I}\right) \) - this is called Roy’s identity (you do not need to know this).

b) Currently a consumer with income \( I = 5000 \) can buy goods at prices: \( p_f = 1, p_y = 22 \). A new mall allows the person to buy good \( y \) at a price \( p_y = 13 \).

i. Find the consumer’s purchases at this price. Will the consumer be better or worse off as a result of the opening of this store?

Using the demand curve \( y^* = 2(52 - \rho) \), at \( p_y = 22, y^* = 60 \); at \( p_y = 13, y^* = 78 \); Clearly the consumer is better off if s/he can buy at a lower price. If s/he continued to purchase the same quantity (60) at a price of 13 instead of 22 s/he would save \( 180 \). Thus, the gain in consumer surplus must be at least \( 180 \) – i.e., the consumer would be willing to pay at least \( 180 \) to shop in the store. Of course, the actual amount would be higher.

ii. If the store charges individuals \( SF \) to shop in the store, what is the maximum amount this individual would be willing to pay to shop in the store? Give a numerical answer.

Using the indirect utility function, if the person pays a fee \( F \) to join the store, and buys good \( Y \) at the price \( p_y = 13 \) (his) her utility will be (since \( p_f = 1 \)):

\[ U^* (I - F, p_f, p_y) = \frac{I - F}{p_f} + (52 - \rho)^2 = 5000 - F + (52 - 13)^2 = 6521 - F \]

Before the discount store, she paid a price of 22 and had utility of:

\[ U^* (I, p_f, p_y) = \frac{I}{p_f} + (52 - \rho)^2 = 5000 + 900 = 5900 \]

Thus, the maximum the person would pay is: \( 6521 - F = 5900 \rightarrow F_{\text{max}} = 621 \).

iii. Using the demand curve, show graphically how to calculate your answer for part ii. What is this area called?
There are two ways to calculate this gain. The simple way is to “remember” that the area next to the
demand curve, between two prices, represents the change in “consumer surplus” which – assuming no
income effect – represents the individual’s maximum willingness to pay for a price decrease. *Hence, for
this example, the consumer surplus is the area of the trapezoid \{22, A, B, 13\} or

\[
\text{Consumer surplus} = 9 \cdot \frac{(60 + 78)}{2} = 621, \text{ the same amount as calculated above.}
\]

*The other way to calculate this gain* (which is the way to prove that the area next to the demand curve
actually measures consumer surplus) is to note the following:

The consumer gains from the price decrease in two ways:

1. The original quantity purchased (60) costs less, for a saving of \((22 - 13) \cdot 60 = 540\). *This is the
windfall gain to the consumer, the amount she gains if she does not change the amount purchased.*

2. The other gain is due to the increased purchases. The consumer increases purchases from 60 to
78 units; *since the height of the demand curve (the marginal rate of substitution) gives the value to the
consumer of each additional unit,* the area under the demand curve, corresponding to the quantity
change, represents the maximum amount the consumer would pay for that additional quantity. Hence:

\[
\text{Gross value additional consumption} = \text{Area of } \{60, A, B, 78\} = 18 \cdot \left(\frac{22 + 13}{2}\right) = 315
\]

\[
\text{Cost of additional consumption} = 13 \cdot 18 = 234
\]

\[
\text{Net gain from additional consumption} = 315 - 234 = 81
\]

Thus, overall the consumer gains two ways: (1) by purchasing the original quantity at a lower price, and
(2) by adjusting her purchases to reflect this new price. The total gain is: \(540 + 81 = 621\) which is, of
course, the same answer reached above.
3. {Efficiency of markets}. Use the supply curve from question 1 and the demand curve from question 2 (with \( p_f = 1 \)) and find the equilibrium market price and output level for good \( y \). {In essence, you are assuming there are a lot of identical producers and consumers, but it is simpler to just work with one of each}.

a) Calculate equilibrium price and quantity, and producer and consumer surplus at this equilibrium.

The supply curve from question 1 is \( y^* = 3(p^f_y - 2) \) where \( p^f_y \) is price seller receives

The demand curve from question 2 is \( y^* = 2(52 - p_y) \) since \( p_f = 1 \)

If there are no taxes or subsidies, \( p_y = p^f_y \) and equilibrium requires supply = demand.

Thus: \( y^* = y^* 
\Rightarrow 3(p_y - 2) = 2(52 - p_y) \rightarrow p_y^e = 22; \ y^e = 60 \)

The producer surplus at this point can be calculated using the profit function from problem 1, or the area \( \{2,A,22\} \) in the first figure. Thus:

Producer Surplus = 600

Consumer Surplus: In the original situation, consumer surplus is the area (figure, previous page) of triangle \( \{52,A,22\} = 900 \).

You can get the same answer using the indirect utility function

b) Show how a government production subsidy of 15 per unit sold affects: (i) equilibrium consumer price (\( p_y \)) and producer price (\( p^f_y = (p_y + 15) \)); (ii) equilibrium output; (iii) consumer surplus and (iv) producer surplus. Calculate the change in producer and consumer surplus and the cost to the government (taxpayers) of the subsidy. Discuss how the subsidy affects overall efficiency. Show graphically this change in efficiency.

The production subsidy effectively raises the price producers receive and hence shifts the supply curve down by the amount of the subsidy. Hence: \( p^f_y = p_y + 15 \). The supply curve, written in terms of the consumer price, is: \( y^* = 3((p_y + 15) - 2) = 3(p_y + 13) \) since, when the consumer price is \( p_y \), producers receive \( (p_y + 15) \). In the graph on the next page the new supply curve is the dotted positively sloped line. The new equilibrium is:

\( S = D \rightarrow \left(3p_y + 39\right) = 2(52 - p_y) \rightarrow 5p_y = 65 \rightarrow p_y^e = 13; \ (p_y^e + 15) = 28; \ y^e = 78 \)

Thus, as a result of the subsidy:

(i) The consumer price falls from 22 to 13; BUT the price producers receive, including the subsidy, increases from 22 to 28.

(ii) Equilibrium output increases from 60 to 78

The changes in producer and consumer surplus are the same as calculated in the previous problems, and can be found from the figure below:
(iii) Consumer surplus increases, of course, because of the lower price. The increase in consumer surplus is area \( \triangledown \{22,E,H,13\} = 621 \)

(iv) Producer surplus increases because the price (including the subsidy) increases from 22 to 28; hence, the increase in producer surplus is area \( \triangledown \{22,E,K,28\} = 414 \)

**Does anybody lose?** Sure, the government – or taxpayers – who have to pay for this subsidy. The cost of the subsidy is 15 times the output level or: **Subsidy Cost** = \( 15 \times 78 = 1170 \)

Overall, then the net gain is negative: \( \text{Increase Consumer Surplus} + \text{Increase Producer Surplus} - \text{Cost of Subsidy} = -135 \).

This area is equal to the area of triangle \( \{E,K,H\} \). Why? Because the area represents the amount by which the real costs associated with producing the extra 18 units of output exceed the value to consumers of that additional output – i.e., the area between the marginal cost curve and the marginal value curve.

c) Suppose there is no subsidy but the government allows trade with the rest of the world. The world price is 28, and the country – since it is small – can export as much of the good as it wants at this world price. Show how international trade at \( p^w = 28 \) affects consumer surplus, producer surplus, and overall efficiency. Show the net gain – or loss – from this trade graphically and give an economic explanation of why this area represents the overall welfare change from trade.

*(see next page for continuation of answer)*
If trade takes place at a world price of 28, and there are no import or export barriers the domestic price (for both producers and consumers) rises to 28. Thus, consumer and producer price both increase to 28 (from 22), consumption decreases to 48, production increases to 78, and exports are 30.

Producers gain area \( \{22,E,K,28\} = 414 \)
Consumers lose area \( \{22,E,L,28\} = 324 \)
Increase in sum of producer and consumer surplus = area \( \{L,K,E\} = 90 \)
This increase in welfare is associated with the **increase in production and the decrease in consumption due to trade.**

Output increases from 60 to 78; the increased cost from the increase in output is the area under the MC curve between \( y = 60 \) and \( y = 78 \), which is area \( \{60,E,K,78\} \); while the revenue from selling this output on the world market = area \( \{60,M,K,78\} \). Hence, the area of \( \{M,K,E\} \) represents the gain from increasing output and selling it on the world market.

Similarly, consumption decreases from 60 to 48; the value lost to consumers by that decreased consumption is given by the area under the demand curve, the area of \( \{60,E,L,48\} \). The cost saving to consumers by reducing consumption at the world price is (28) times the amount by which consumption decreased (12), or the area \( \{60,M,L,48\} \). Overall, then, **given the higher price, consumers gain area \( \{L,M,E\} \) due to decreasing consumption (as compared to leaving consumption unchanged. Of course, consumers lose because of the price rise but they would lose even more if they did not reduce consumption from 60 to 48.**

Hence, the overall gain from trade is area \( \{L,E,K\} \). From an economic perspective, it is crucial to understand that the overall gain comes not because price changes (per se), but because **quantity changes in** response to this price change. Finally, note that even though trade is “beneficial”, there are losers as well as winners from free trade.