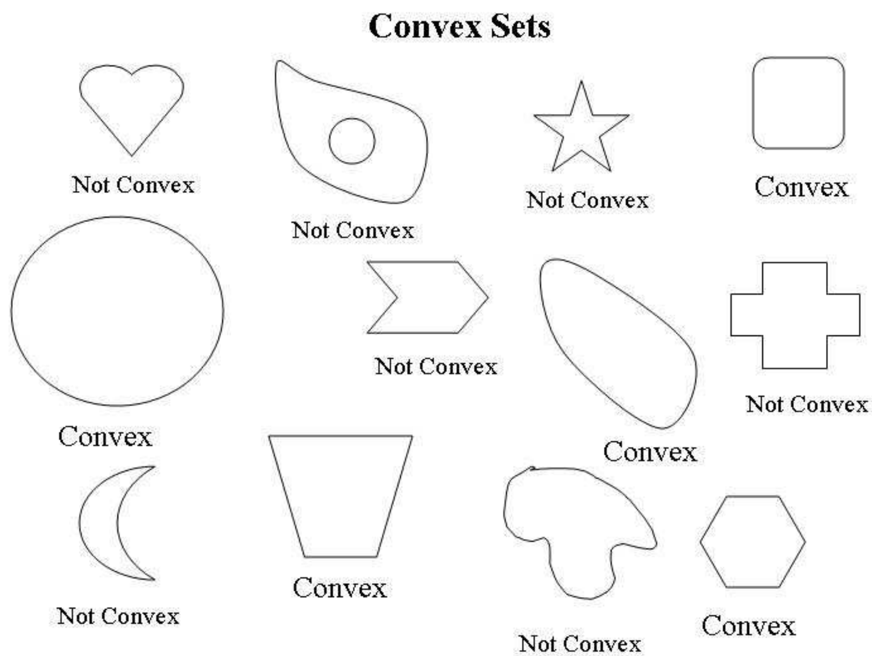
CONVEXITY AND OPTIMIZATION

1. Convex sets

1.1. **Definition of a convex set.** A set S in R^n is said to be convex if for each x_1, $x_2 \in$ S, the line segment $\lambda x_1 + (1-\lambda)x_2$ for $\lambda \in (0,1)$ belongs to S. This says that all points on a line connecting two points in the set are in the set.

FIGURE 1. Examples of Convex Sets



1.2. **Intersections of convex sets.** The intersection of a finite or infinite number of convex sets is convex. Figure 2 contains some examples of convex set intersections.

1.3. **Hyperplanes.** A hyperplane is defined by H = {x: $p'x = \alpha$} where p is a nonzero vector in R^n and α is a scalar. This is a line in two dimensional space, a plane in three dimensional space, etc. For the two dimensional case this gives $p_1 x_1 + p_2 x_2 = \alpha$ which can be rearranged to yield

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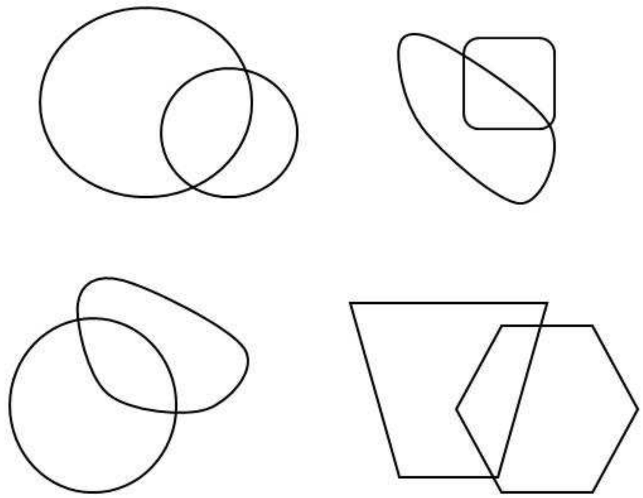
FIGURE 2. Intersections of Convex Sets



$$\begin{aligned} p_1\, x_1 \,+\, p_2\, x_2 &= \alpha \\ \Rightarrow\; p_2\, x_2 &= \alpha \,-\, p_1\, x_1 \\ \Rightarrow\; x_2 &= \frac{\alpha \,-\, p_1\, x_1}{p_2} \\ &= \frac{\alpha}{p_2} \,-\, \frac{p_1}{p_2}\, x_1 \end{aligned}$$

This is just a line with slope ($-p_1/p_2$) and intercept α/p_2.

If $\bar{x}$ is any vector vector in a hyperplane H = {x: $p'x = \alpha$}, then we must have $p'\bar{x}$=α, so that the hyperplane can be equivalently described as

$$\begin{aligned} H &= \{x : p'x = p'\bar{x}\} \\ &= \{x : p'(x \,-\, \bar{x}) \,=\, 0\} \\ &= \bar{x} \,+\, \{x : p'x \,=\, 0\} \end{aligned}$$

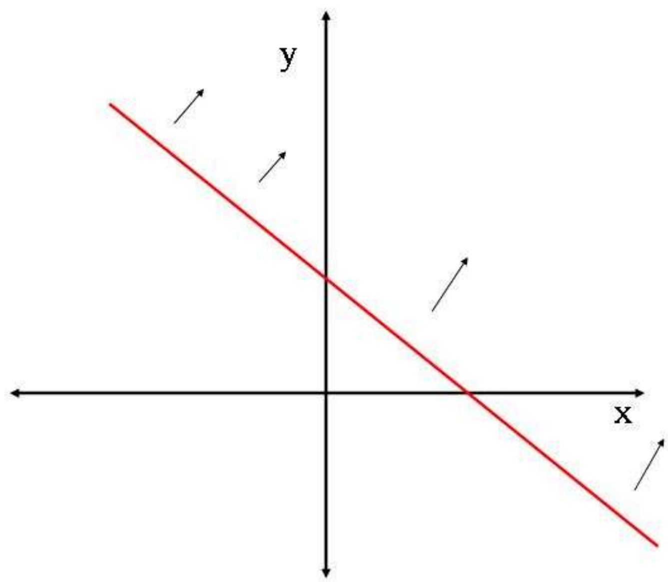
The third expression says thay H is an affine set that is parallel to the subspace {x: $p'x = 0$}. This subspace is orthogonal to the vector p, and consequently, p is called the *normal* vector of the hyperplane H. A hyperplane divides the space into two half-spaces.

1.4. **Half-spaces.** A half-space is defined by S = {x: $p'x \leq \alpha$} or S = {x: $p'x \geq \alpha$} where p is a nonzero vector in R^n and α is a scalar. This is all the points in 2 dimensional space on one side of a straight line or one side of a plane in three dimensional space, etc. The sets above are closed half-spaces. The sets

$$\{x : a'x < \beta\} \text{ and } \{x : a'x > \beta\}$$

are called the open half-spaces associated with the hyperplane {x: $a'x = \beta\}$. A 2-dimensional illustration is presented in figure 3.

FIGURE 3. A Half-space



1.5. **Supporting hyperplane.** Let S be a nonempty convex set in R^n and let $\bar{x}$ be a boundary point. Then there exists a hyperplane that supports S at $\bar{x}$, that is, there exists a nonzero vector p such that $p'(x - \bar{x}) \leq 0$ for each x which is an element of the closure of S. This can also be written as $p'x \leq p'\bar{x}$ for each x which is an element of the closure of S.

1.6. **Separating hyperplanes.** Given two non-empty convex sets S_1 and S_2 in R^n such that $S_1 \cap S_2 = \emptyset$, then there exists as hyperplane H = {x: $p'x = \alpha\}$ that separates them, that is

$$p'x \leq \alpha \, \forall \, x \in S_1$$
$$p'x \geq \alpha \, \forall \, x \in S_2$$

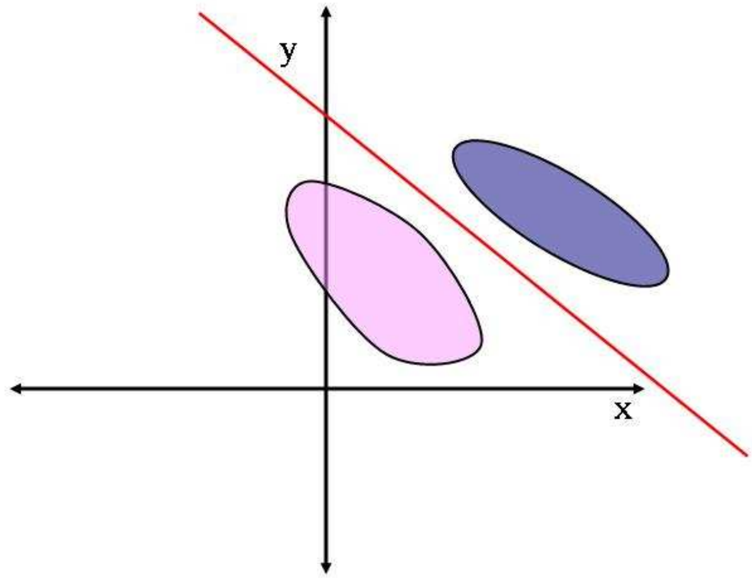
We can also write this as

$$p'x_1 \leq p'x_2, \quad \forall \, x_1 \in S_1 \text{ and } \forall \, x_2 \in S_2$$

A separating hyperplane is illustrated in figure 4.

1.7. **Minkowski's Theorem.** A closed, convex set is the intersection of the half spaces that support it. This is illustrated in figure 5. We can then find a convex set by finding the infinite intersection of half-spaces which support it.

FIGURE 4. A Separating Hyperplane



2. CONVEXITY AND CONCAVITY FOR FUNCTIONS OF A REAL VARIABLE

2.1. **Definitions of convex and concave functions.** If f is continuous in the interval I and twice differentiable in the interior of I (denoted I^0) then we say

1: f is convex on I $\Leftrightarrow$ $f''(x) \geq 0$ for all x in I^0.
2: f is concave on I $\Leftrightarrow$ $f''(x) \leq 0$ for all x in I^0.

We also say that a function is convex on an interval if f´ is increasing on the interval and concave on the interval where f´ is decreasing. Figure 6 shows a concave function. A concave function with a positive first derivative will rise but at a declining rate. If we draw a line between any two points on the graph, the graph will lie above the line. The graph of a concave function will always lie below the tangent line at a given point as shown in figure 7.

Figure 8 shows a convex function. A convex function with a positive first derivative will rise at an increasing rate. If we draw a line between any two points on the graph, the graph will lie below the line. The graph of a convex function will always lie above the tangent line at a given point as shown in figure 9.

FIGURE 5. Minkowski's Theorem

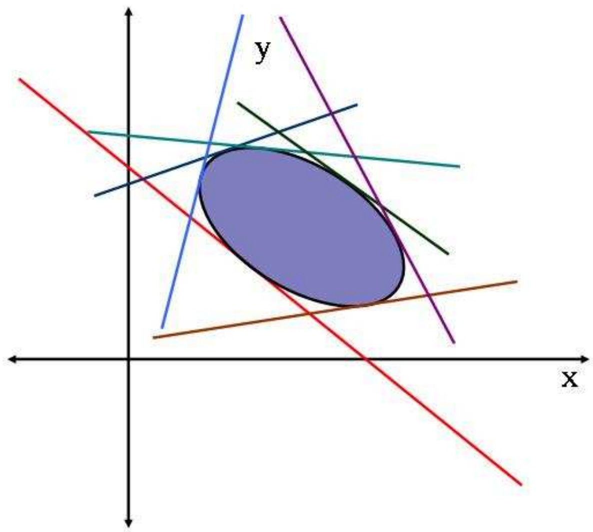
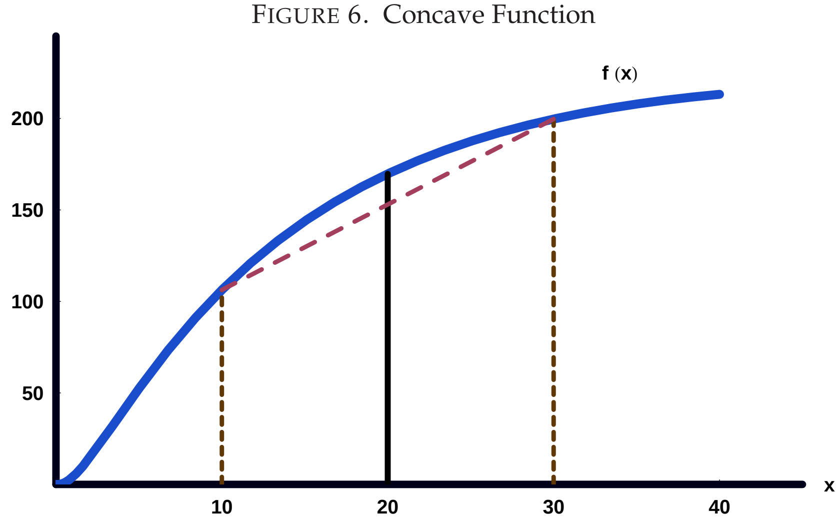


FIGURE 6. Concave Function



2.2. **Inflection points of a function.**

2.2.1. *Definition of an inflection point.* Point c is an **inflection point** for a twice differentiable function f if there is an interval (a, b) containing c such that either of the following two conditions holds:

1: $f''(x) \geq 0 \;\; a < x < c$ and $f''(x) \leq 0 \; if \; c < x < b$
2: $f''(x) \leq 0 \;\; a < x < c$ and $f''(x) \geq 0 \; if \; c < x < b$

FIGURE 7. Tangent Line above Graph for Concave Function

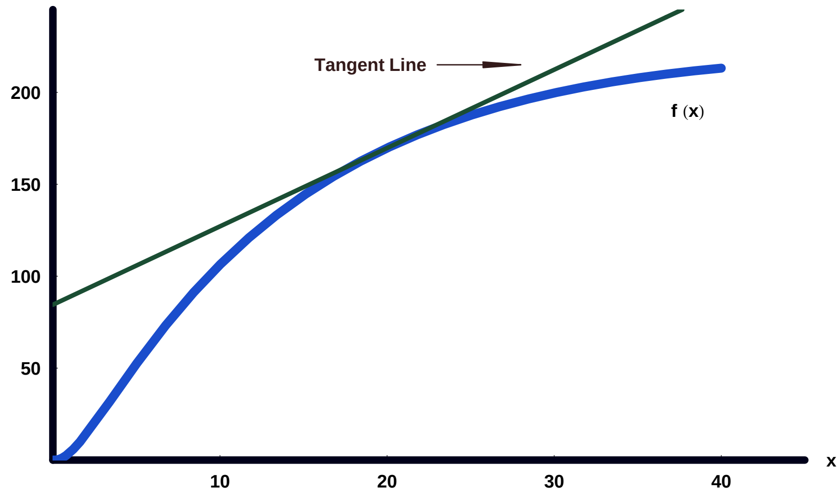
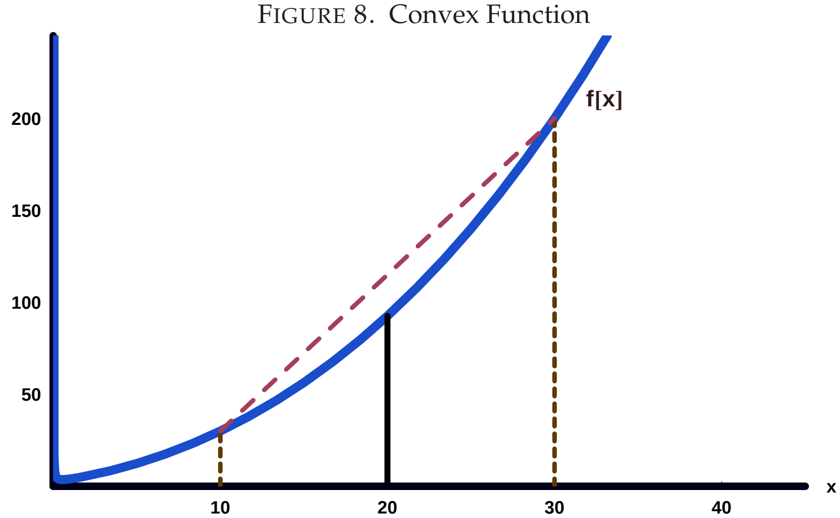


FIGURE 8. Convex Function



Intuitively this says that x = c is an inflection point if f″(x) changes sign at c. Alternatively points at which a function changes from being convex to concave, or vice versa, are called inflection points. Consider the function $f(x) = x^3 - 6x^2 + 9x + 1$. The first derivative is $f' = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$. The second derivative is $f'' = 6x - 12 = 6(x - 2)$. The second derivative is zero at x = 2. When $x < 2, f'' < 0$ and when $x > 2, f'' > 0$. In figure 10 the function has a local maximum at one and a local minimum at three. The function is concave around x = 1 and convex around x = 3. Given that the graph changes from concave to convex, there must be an inflection point. In this case the inflection point is at the point (2,3).

2.2.2. *Test for inflection points.* Let f be a function with a continuous second derivative in an interval I, and suppose c is an interior point of I. Then

1: If c is an inflection point for f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

FIGURE 9. Tangent Line below Graph for Convex Function

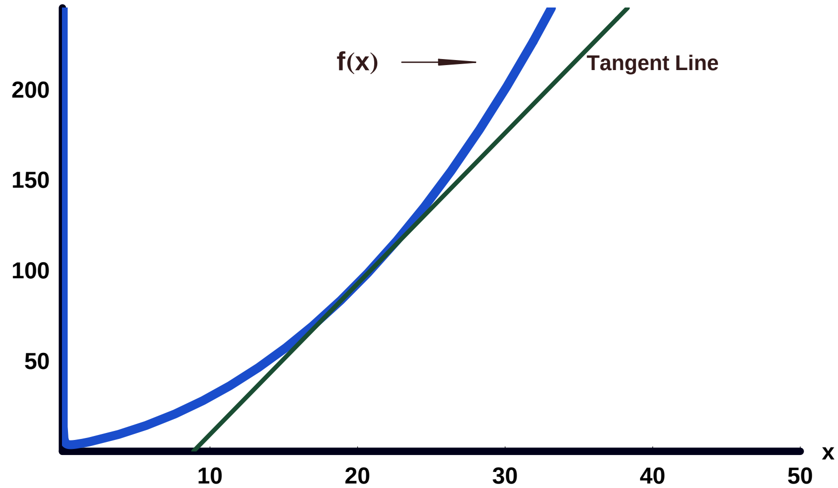
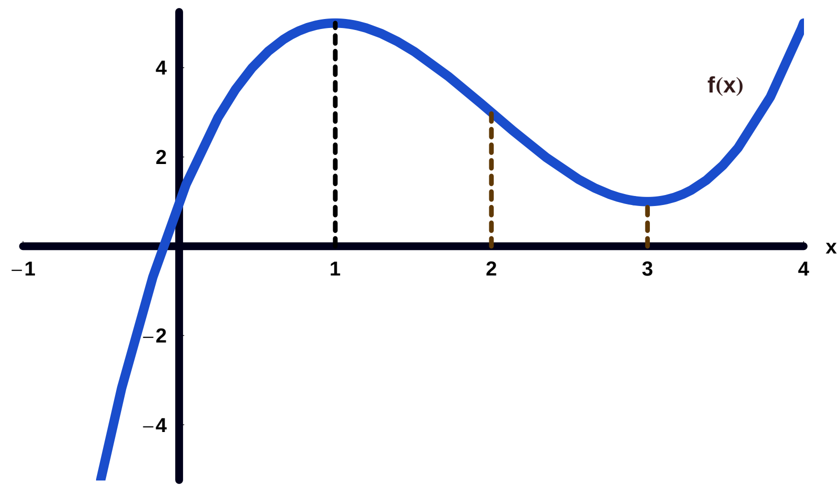


FIGURE 10. Function with Inflection Point

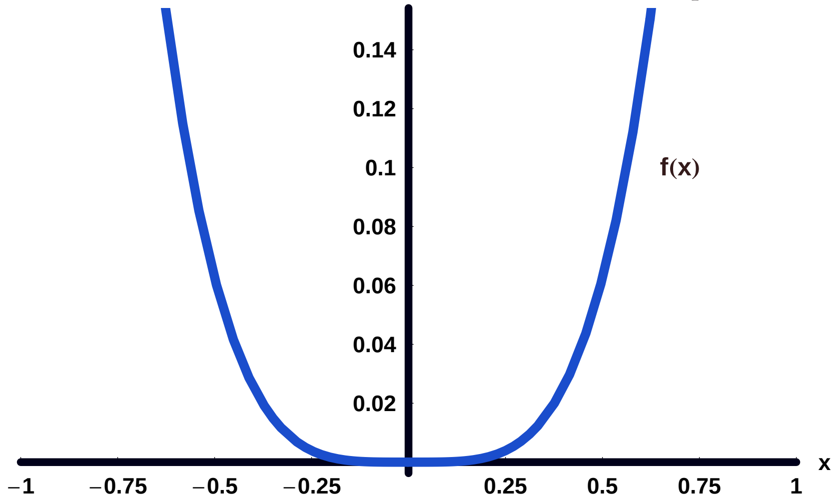


2: If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f.

The condition $f''(c) = 0$ is a necessary condition for c to be an inflection point. It is not a sufficient condition, however, because $f''(c) = 0$ does not imply the f'' changes sign at x = c.

2.2.3. *Example.* Let f(x) = x^4. Then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. At $x = 0$, $f''(x) = 0$. But for this function $f''(x) \geq 0$ for all $x \neq 0$ so f'' does not change sign at x = 0. Thus, x = 0 is not an inflection point for f. We can see this in figure 11.

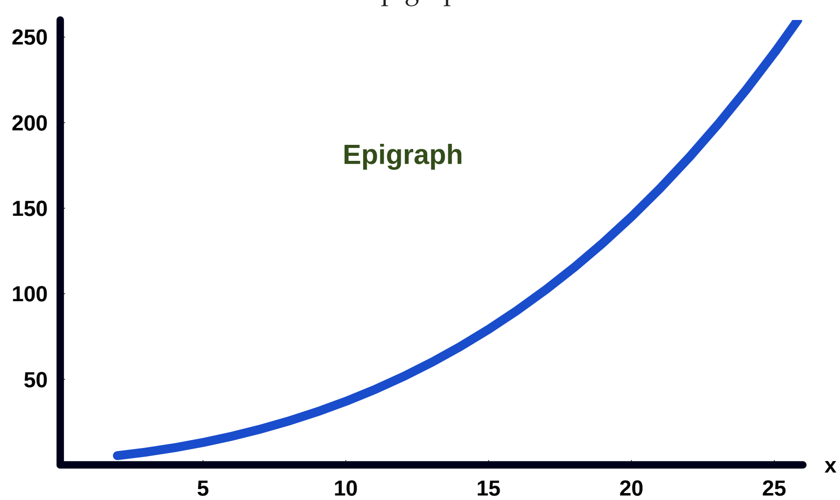
FIGURE 11. Function with $f'' = 0$, but no inflection point



3. EPIGRAPHS AND HYPOGRAPHS

3.1. **Epigraph.** Let f: S → R^1. The epigraph of f is the set $\{(x, y)$: $x \in S$, $y \in R^1$, $y \geq f(x)\}$. The area above the curve in figure 12 is the epigraph of the function.

FIGURE 12. Epigraph of a function



3.2. **Hypograph.** Let f: S → R^1. The hypograph of f is the set $\{(x, y)$, $x \in S$, $y \in R^1$, $y \leq f(x)\}$. The area below the curve in figure 13 is the hypograph of the function.

With a more general function the epigraph and hypograph may have boundaries that move up and down as x increases. We can see this in figure 14.

FIGURE 13. Hypograph of a function

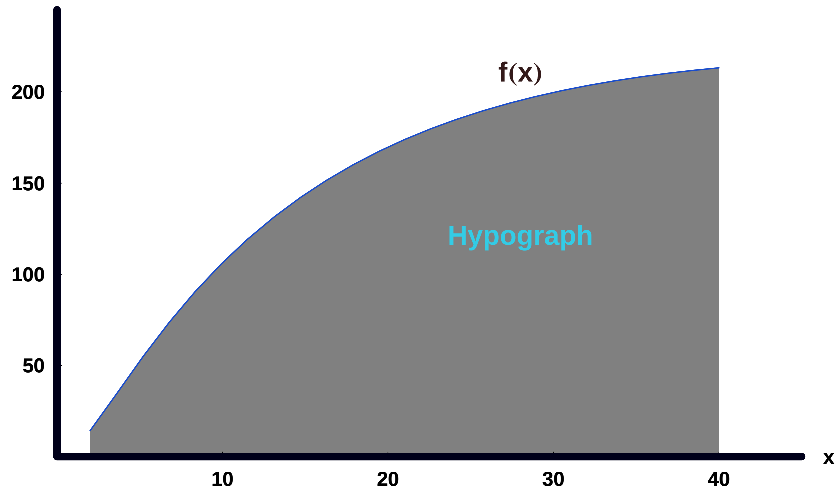
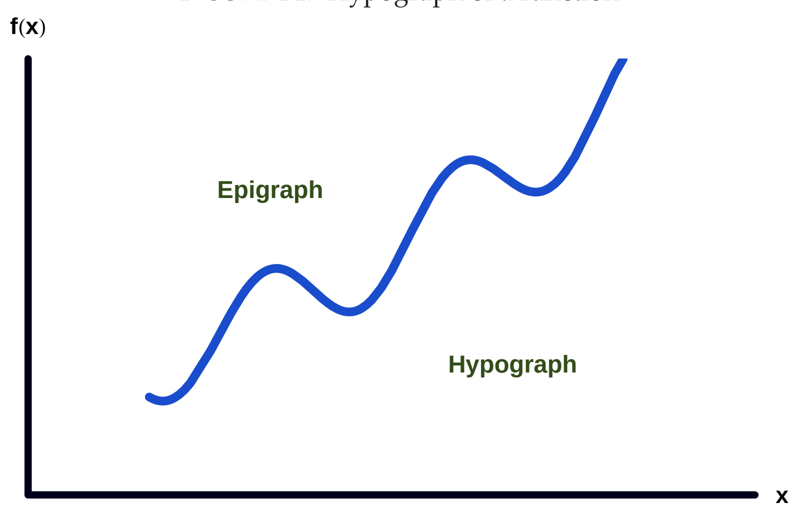


FIGURE 14. Hypograph of a function



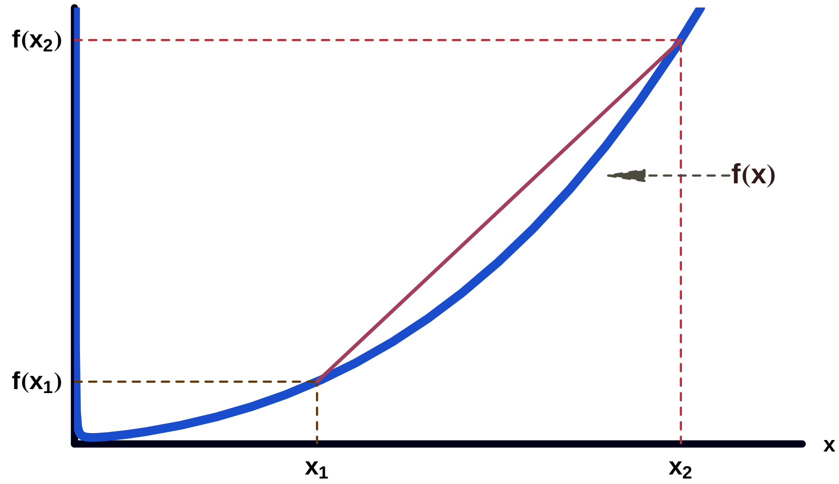
4. GENERAL CONVEX FUNCTIONS

4.1. **Definition of convexity.** Let S be a nonempty convex set in R^n. The function f: S $\rightarrow$ R^1 is said to be convex on S if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ for each x_1, $x_2 \in$ S and for each $\lambda \in [0, 1]$. The function f is said to be strictly convex if the above inequality holds as a strict inequality for each distinct x_1, x_2, $\in$ S and for each $\lambda \in (0, 1)$. This basically says that the function evaluated at at a linear combination of x_1 and x_2 is less than the same linear combination of $f(x_1)$ and $f(x_2)$. Figure 15 shows a convex function.

4.2. **Characteristics of convex functions.**

a: The function f is continuous on the interior of S.

FIGURE 15. A convex function



b: The function f is convex on S if and only if the set $\{(x, y)\colon x \in S, y \geq f(x)\}$ is convex. This set is the epigraph of f. Thus convexity of f is equivalent to convexity of its epigraph.

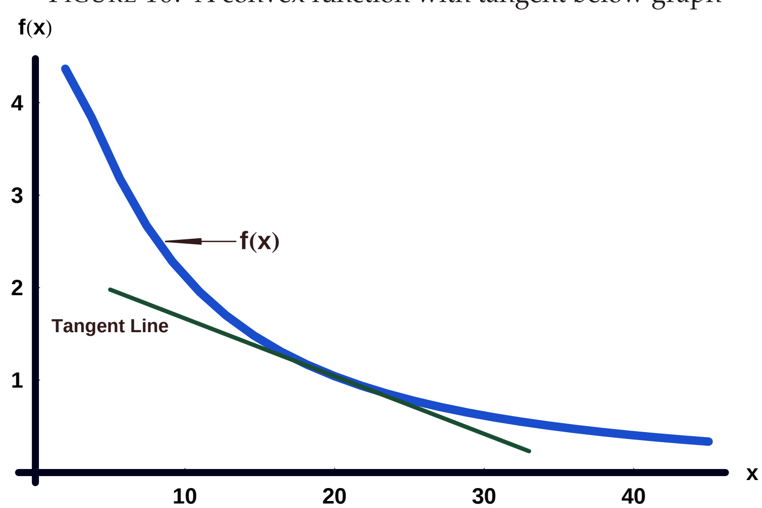
c: The set $\{x \in S, f(x) \leq \alpha\}$ is convex for every real α. This is the lower contour set, so convexity of a function implies convesity of the lower contour set.

d: A differentiable function f is convex on S if and only if

$$f\,(\,x)\;\geq\;f\,(\,\bar{x}\,)\;+\;f'(\bar{x})\,(x\;-\bar{x})\;for\;each\;distinct\;x, \bar{x}\;\in\;S.$$

This implies that tangent line is below the graph as we see in figure 16.

FIGURE 16. A convex function with tangent below graph



e: A function of a single variable f is convex on an interval if for a, x, and b in the interval with a < x < b we have

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

This basically says that the chord between two points lies above the function as in the initial definition.

f: A twice differentiable function f is convex iff the Hessian H(x) is positive semidefinite for each x $\in$ S. For the case of function of two variables, the implication is as follows

$$f \text{ is } convex \Leftrightarrow \frac{\partial^2 f}{\partial x_1^2} \geq 0, \quad \frac{\partial^2 f}{\partial x_2^2} \geq 0, \text{ and } \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 \geq 0$$

g: Let f be twice differentiable. Then if the Hessian H(x) is positive definite for each x $\in$ S, f is strictly concave. Further if f is strictly concave, then the Hessian H(x) is positive semidefinite for each x $\in$ S.

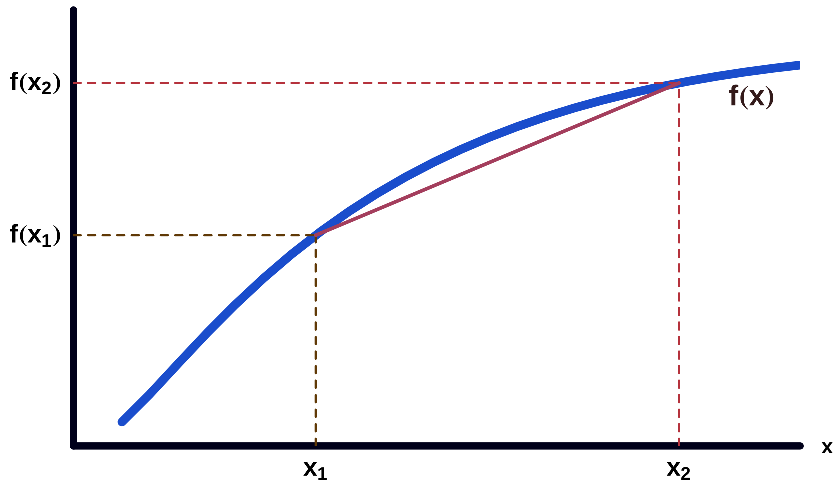
h: Every local minimum of f over a convex set W $\subseteq$ S is a global minimum.

i: If $f'(\bar{x}) = 0$ for a convex function then, $\bar{x}$ is the global minimum of f over S.

5. General concave functions

5.1. **Definition of concavity.** Let S be a nonempty convex set in R^n. The function f: S $\rightarrow$ R^1 is said to be convex on S if $f(\lambda\, x_1 \,+\, (1-\lambda)\, x_2) \,\geq\, \lambda\, f(x_1) \,+\, (1-\lambda) f(x_2)$ for each x_1, $x_2 \in$ S and for each $\lambda \in$ [0, 1]. The function f is said to be strictly concave if the above inequality holds as a strict inequality for each distinct x_1, x_2, $\in$ S and for each $\lambda \in$ (0, 1). This basically says that the function evaluated at a linear combination of x_1 and x_2 is greater than the same linear combination of $f(x_1)$ and $f(x_2)$. Figure 17 shows a concave function.

FIGURE 17. A concave function



5.2. **Characteristics of concave functions.**

a: The function f is continuous on the interior of S.

b: The function f is concave on S if and only if the set {(x, y): x $\in$ S, y $\leq$ f(x)} is convex. This set is the hypograph of f. Thus concavity of f is equivalent to convexity of its hypograph.

c: The set {x $\in$ S, f(x) $\geq \alpha$ } is convex for every real α. This is convexity of the upper contour or level set.

d: A differentiable function f is convex on S if and only if

$$f(x) \;\leq\; f\,(\bar{x}) \;+\; f'(\bar{x})\,(x \,-\, \bar{x}) \; for \; each \; distinct \; x, \bar{x} \;\in\; S.$$

This implies that tangent line is above the graph as we see in figure 18.

e: A function of a single variable f is concave on an interval if for a, x, and b in the interval with a < x < b we have

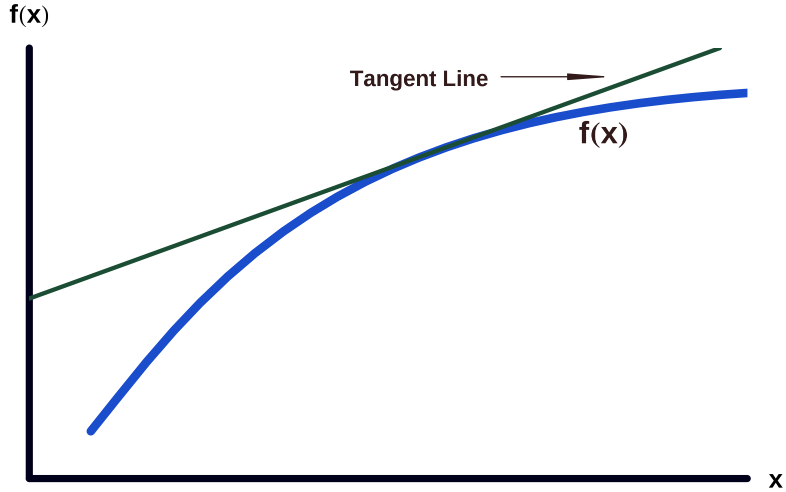
$$\frac{f(x) \;-\; f(a)}{x \;-\; a} \;>\; \frac{f(b) \;-\; f(a)}{b \;-\; a}$$

This basically says that the chord between two points lies below the function as in the initial definition.

f: A twice differentiable function f is concave iff the Hessian H(x) is negative semidefinite for each x $\in$ S.

g: Let f be twice differentiable. Then if the Hessian H(x) is negative definite for each x $\in$ S, f is strictly concave. Further if f is strictly concave, then the Hessian H(x) is negative semidefinite for each x $\in$ S. For the case of function of two variables, the implication is as follows

FIGURE 18. A concave function with tangent above graph



$$f \text{ is } concave \Leftrightarrow \frac{\partial^2 f}{\partial x_1^2} \leq 0, \quad \frac{\partial^2 f}{\partial x_2^2} \leq 0, \text{ and } \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 \geq 0$$

h: Every local maximum of f over a convex set W $\subseteq$ S is a global maximum.
i: If $f'(\bar{x}) = 0$ for a concave function then, $\bar{x}$ is the global maximum of f over S.

6. QUASICONCAVITY

6.1. Definitions of Quasiconcavity.

Definition 1. A real valued function f, defined on a convex set X $\subset$ R^n, a said to be **quasiconcave** if

$$f(\lambda x^1 + +(1-\lambda)x^2) \geq min[f(x^1), f(x^2)] \tag{1}$$

A function f is said to be quasiconvex if - f is quasiconcave.

The following expression also defines a quasi-concave function and is equivalent to equation 1.

$$f(x) \geq f(x^0) \Rightarrow f(\lambda x + +(1-\lambda)x^0) \geq f(x^0) \tag{2}$$

Theorem 1. *Let f be a real valued function defined on a convex set X $\subset$ R^n. The upper contour sets $\{$(x, y): x $\in$ S, α $\leq$ f(x)$\}$ of f are convex for every $\alpha \in$ R if and only if f is a quasiconcave function.*

Proof. Suppose that S(f,α) is a convex set for every $\alpha \in$ R and let x^1 $\in$ X, x^2 $\in$ X, $\bar{\alpha}$ = min[f(x^1), f(x^2)]. Then x^1 $\in$ S(f,$\bar{\alpha}$) and x^2 $\in$ S(f,$\bar{\alpha}$), and because S(f,$\bar{\alpha}$) is convex, (λx^1 + (1-λ)x^2) $\in$ S(f,$\bar{\alpha}$) for arbitrary λ. Hence

$$f(\lambda x^1 + (1-\lambda)x^2) \geq \bar{\alpha} = min[f(x^1), f(x^2)] \tag{3}$$

Conversely, let S(f,α) be any level set of f. Let x^1 $\in$ S(f,α) and x^2 $\in$ S(f,α). Then

$$f(x^1) \geq \alpha, \quad f(x^2) \geq \alpha \tag{4}$$

and because f is quasiconcave, we have

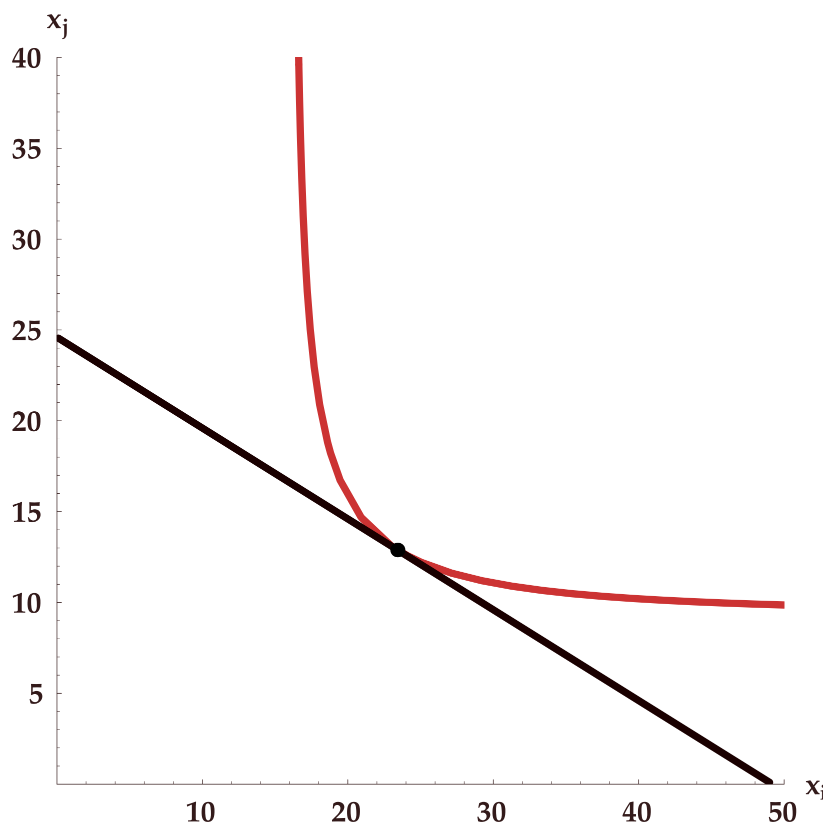
$$f(\lambda x^1 + (1-\lambda) x^2) \geq \alpha \tag{5}$$

and $(\lambda x^1 + (1\text{-}\lambda) x^2) \in S(f,\alpha)$.

□

We can see this is figure 19

FIGURE 19. A level set for a quasi-concave function



Theorem 2. *Let f be differentiable on an open convex set $X \subset R^n$. Then f is quasiconcave if and only if for any $x^1 \in X$, $x^2 \in X$ such that*

$$f(x^1) \geq f(x^2) \tag{6}$$

we have

$$(x^1 - x^2)' \nabla f(x^2) \geq 0 \tag{7}$$

6.2. **Quasi-concavity and bordered Hessians.**

6.2.1. *A set of determinants of a bordered* **Hessian** *matrix.*

Definition 2. The kth-ordered bordered determinant D_k(f,x) of a twice differentiable function f at at point x $\in R^n$ is defined as

$$D_k(f,x) \;=\; det \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_k} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_k \partial x_1} & \frac{\partial^2 f}{\partial x_k \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_k^2} & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_k} & 0 \end{bmatrix} \quad k \;=\; 1,\, 2,\, \ldots,\, n \tag{8}$$

Definition 3. Some authors define the kth-ordered bordered determinant D_k(f,x) of a twice differentiable function f at at point x $\in R^n$ in a different fashion where the first derivatives of the function f border the Hessian of the function on the top and left as compared to in the bottom and right as in equation 8.

$$D_k(f,x) \;=\; det \begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_k} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \frac{\partial^2 f}{\partial x_k \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{bmatrix} \quad k \;=\; 1,\, 2,\, \ldots,\, n \tag{9}$$

The determinant in equation 8 and the determinant in equation 9 will be the same. If we interchange any two rows or any two columns of a determinant, the determinant will change sign but keep its absolute value. A certain number of row exchanges will be necessary to move the bottom row to the top. A like number of column exchanges will be necessary to move the rightmost column to the left. Given that equations 8and 9 are the same except for this even number of row and column exchanges, the determinants will be the same. You can illustrate this to yourself using the following three variable example.

$$\tilde{H}_B \;=\; \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial f}{\partial x_2} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & 0 \end{bmatrix} \qquad \hat{H}_B \;=\; \begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial f}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

6.2.2. *Defining quasi-concavity in terms of determinants of a bordered* **Hessian** *matrix.*

a.: If f(x) is quasi-concave on a solid (non-empty interior) convex set X $\subset R^n$, then

$$(-1)^k \, D_k(f,x) \;\geq\; 0, \quad k \;=\; 1,\, 2,\, \ldots,\, n \tag{10}$$

for every x $\in$ X.

b.: If

$$(-1)^k \, D_k(f,x) \;>\; 0, \quad k \;=\; 1,\, 2,\, \ldots,\, n \tag{11}$$

for every x $\in$ X,then f(x) is quasi-concave on X (Avriel [3, p.149], Arrow and Enthoven [2, p. 781-782]

If f is quasiconcave, then when k is odd, D_k(f,x) will be negative and when k is even, D_k(f,x) will be positive. Thus D_k(f,x)will alternate in sign beginning with positive in the case of two variables.

6.2.3. *Relationship of quasi-concavity to signs of minors (cofactors) of a matrix.* Let

$$F = \begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix} = det\, H_B \tag{12}$$

where det H_B is the determinant of the bordered Hessian of the function f. Now let F_{ij} be the cofactor of $\frac{\partial^2 f}{\partial x_i \partial x_j}$ in the matrix H_B. It is clear that F_{nn} and F have opposite signs because F includes the last row and column of H_B and F_{nn} does not. If the $(-1)^n$ in front of the cofactors is positive then F_{nn} must be positive with F negative and vice versa. Since the ordering of rows since is arbitrary it is also clear that F_{ii} and F have opposite signs. Thus when a function is quasi-concave $\frac{F_{ii}}{F}$ will have a negative sign.

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