Optimization Basics for Electric Power Markets

Presenter:
Leigh Tesfatsion
Professor of Econ, Math, and ECpE
Iowa State University
Ames, Iowa 50011-1070
http://www.econ.iastate.edu/tesfatsi/
tesfatsi@iastate.edu

Last Revised: 20 September 2020

Important Acknowledgement: These lecture materials incorporate edited slides and figures originally developed by R. Baldick, S. Bradley et al., A. Hallam, T. Overbye, and M. Wachowiak
Topic Outline

- ISO Market Optimization on a Typical Operating Day
- Alternative modeling formulations
- Optimization illustration: Real-time economic dispatch
- Classic Nonlinear Programming Problem (NPP): Minimization subject to equality constraints
  - NPP via the Lagrange multiplier approach
  - NPP Lagrange multipliers as shadow prices
- Real-time economic dispatch: Numerical example
- General Nonlinear Programming Problem (GNPP): Minimization subject to equality and inequality constraints
  - GNPP via the Lagrange multiplier approach
  - GNPP Lagrange multipliers as shadow prices
- Necessary versus sufficient conditions for optimization
- Technical references
Key Objective of EE/Econ 458

- Understand the optimization processes undertaken by participants in restructured wholesale power markets

- For ISOs, these processes include:
  - **Security-Constrained Unit Commitment (SCUC)** for determining which Generating Companies (GenCos) will be asked to commit to the production of energy the following day
  - **Security-Constrained Economic Dispatch (SCED)** for determining energy dispatch and locational marginal price (LMP) levels both for the day ahead and in real time
ISO Market Optimization on a Typical Operating Day D: Timing from Midwest ISO (MISO)

- **00:00**: Day-ahead market for day D+1
  - ISO collects demand bids from LSEs and supply offers from GenCos

- **11:00**: ISO evaluates LSE demand bids and GenCo supply offers

- **16:00**: ISO solves D+1 Security Constrained Unit Commitment (SCUC) & Security Constrained Economic Dispatch (SCED) & posts D+1 commitment, dispatch, and LMP schedule

- **23:00**: Day-ahead settlement

- **Real-time spot market (balancing mechanism) for day D**

- **Real-time settlement**
Types of Model Representations
(from Bradley et al. [2])

Main focus of 458 (LT)

Fig. 1.1 Types of model representation.
Classification of Modeling Tools  
(from Bradley et al. [2])

Table 1.1 Classification of Analytical and Simulation Models

<table>
<thead>
<tr>
<th>Certainty</th>
<th>Strategy evaluation</th>
<th>Strategy generation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic simulation</td>
<td>Linear programming</td>
<td></td>
</tr>
<tr>
<td>Econometric models</td>
<td>Network models</td>
<td></td>
</tr>
<tr>
<td>Systems of simultaneous</td>
<td>Integer and mixed-integer programming</td>
<td></td>
</tr>
<tr>
<td>equations</td>
<td>Nonlinear programming</td>
<td></td>
</tr>
<tr>
<td>Input-output models</td>
<td>Control theory</td>
<td></td>
</tr>
<tr>
<td>Monte Carlo simulation</td>
<td>Decision theory</td>
<td></td>
</tr>
<tr>
<td>Econometric models</td>
<td>Dynamic programming</td>
<td></td>
</tr>
<tr>
<td>Stochastic processes</td>
<td>Inventory theory</td>
<td></td>
</tr>
<tr>
<td>Queueing theory</td>
<td>Stochastic programming</td>
<td></td>
</tr>
<tr>
<td>Reliability theory</td>
<td>Stochastic control theory</td>
<td></td>
</tr>
</tbody>
</table>

Statistics and subjective assessment are used in all models to determine values for parameters of the models and limits on the alternatives.
# Optimization in Practice
*(from Bradley et al. [2])*

## Table 5.1 Distinct Characteristics of Strategic, Tactical, and Operational Decisions

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Strategic planning</th>
<th>Tactical planning</th>
<th>Operations control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td>Resource acquisition</td>
<td>Resource utilization</td>
<td>Execution</td>
</tr>
<tr>
<td>Time horizon</td>
<td>Long</td>
<td>Middle</td>
<td>Short</td>
</tr>
<tr>
<td>Level of management involvement</td>
<td>Top</td>
<td>Medium</td>
<td>Low</td>
</tr>
<tr>
<td>Scope</td>
<td>Broad</td>
<td>Medium</td>
<td>Narrow</td>
</tr>
<tr>
<td>Source of information</td>
<td>(External &amp; Internal)</td>
<td>Internal</td>
<td></td>
</tr>
<tr>
<td>Level of detail of information</td>
<td>Highly aggregate</td>
<td>Moderately aggregate</td>
<td>Low</td>
</tr>
<tr>
<td>Degree of uncertainty</td>
<td>High</td>
<td>Moderate</td>
<td>Low</td>
</tr>
<tr>
<td>Degree of risk</td>
<td>High</td>
<td>Moderate</td>
<td>Low</td>
</tr>
</tbody>
</table>
Optimization Techniques

Main focus of 458 (LT)
Optimization Illustration: Hourly Economic Dispatch

- **Initial Problem Simplification:** Ignore Generation Company (GenCo) capacity limits, transmission constraints, line limit losses, and all costs except variable costs.

- **Problem Formulation for Hour H**
  Determine the real-power dispatch levels $P_{Gi} (MW)$ for GenCos $i = 1, 2, \ldots, I$ that minimize total variable cost $TVC$, subject to the constraint that total dispatch equals total real-power demand (load) $P_D$. 
Hourly Economic Dispatch: Mathematical Formulation

\[
\text{minimize} \quad \text{TVC}(P_G) = \sum_{i=1}^{I} VC_i(P_{Gi}) \quad ($/h)
\]

\text{with respect to} \quad P_G = (P_{G1}, \ldots, P_{GI})^T \quad (MW)

\text{subject to} \quad \sum_{i=1}^{I} P_{Gi} = P_D \quad (MW)

Load as a constraint constant for power dispatch

Variable cost of GenCo \(i\)
5-Bus Transmission Grid Example

5 GenCos (I=5), Load $P_D = 500\text{MW}$
Illustration of TVC Determination

\[ \text{MC}(P_G) \]

Optimal Dispatch

\[ P_{G1}^* = 200 \text{MW} \]
\[ P_{G2}^* = 100 \text{MW} \]
\[ P_{G3}^* = 200 \text{MW} \]
\[ P_{G4}^* = P_{G5}^* = 0 \]

P_D (Optimal Dispatch)

G1
G2
G3
G4
G5

P_G (MW)

($/\text{MWh}$)

0 10 20 30 40 50 60 70 80 90

200 400 500 600 800
Illustration of TVC Determination

- **$/MWh**
- **MC($P_G$)**
- **Optimal Dispatch**
  - $P_{G1}^* = 200$ MW
  - $P_{G2}^* = 100$ MW
  - $P_{G3}^* = 200$ MW
  - $P_{G4}^* = P_{G5}^* = 0$

- **$P_D$**
- **$P_G$ (MW)**
Day-Ahead Supply Offers in MISO

Resource Offers
Energy Offer Curves

- An Offer Curve is an offer to sell generation by a Resource
  - Slope (“true”) vs. block (“false”) offer
  - Monotonically increasing in price and non-decreasing in MW
  - Can vary hourly by location (CPNode)
  - Can submit up to 10 MW/price pairs
  - Previous DA offer carries over to DA and the previous day’s RT offer carries over to RT if no supply offer is submitted for the next day

Minimum acceptable price (sale reservation price) for each ΔMW

<table>
<thead>
<tr>
<th>ΔMW</th>
<th>$/MWh</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>35</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>10</td>
<td>45</td>
</tr>
</tbody>
</table>

Copyright © 2006 Midwest Independent Transmission System Operator, Inc. All rights reserved.
Supply Offers in the MISO...Cont’d

Resource Offers
Offer Curves Exercise Key

- Diagram this data set as a Block Offer and Slope Offer

<table>
<thead>
<tr>
<th>Segment</th>
<th>MW</th>
<th>$/MWh</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>$24.50</td>
</tr>
<tr>
<td>2</td>
<td>85</td>
<td>$24.75</td>
</tr>
<tr>
<td>3</td>
<td>95</td>
<td>$25.00</td>
</tr>
<tr>
<td>4</td>
<td>105</td>
<td>$25.50</td>
</tr>
<tr>
<td>5</td>
<td>125</td>
<td>$25.75</td>
</tr>
<tr>
<td>6</td>
<td>135</td>
<td>$26.00</td>
</tr>
</tbody>
</table>

Copyright © 2008 Midwest Independent Transmission System Operator, Inc. All rights reserved.
Important Remark on the Form of GenCo Supply Offers

• **Linear programming (LP)** is used to handle economic dispatch when supply offers have *block* (step-function) form.

• **Nonlinear Programming (NP)** techniques are used to handle economic dispatch when supply offers take a *slope* (piecewise differentiable or differentiable) form. *(K/S assumption)*

• The remainder of these notes use NP techniques for economic dispatch, assuming supply offers take a differentiable form.

• When we later treat GenCo capacity constraints, we will need to relax this to permit supply offers to take a *piecewise* differentiable form.
Hourly Economic Dispatch Once Again

\[
\text{minimize} \quad \text{TVC}(P_G) = \sum_{i=1}^{I} VC_i(P_{Gi}) \quad ($/h) \\
\text{with respect to} \quad P_G = (P_{G1}, \ldots, P_{GI})^T \quad (MW) \\
\text{subject to} \quad \sum_{i=1}^{I} P_{Gi} = P_D \quad (MW)
\]
Minimization with Equality Constraints: General Solution Method?

- For a differentiable function $f(x)$ of an $n$-dimensional vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, a necessary (but not sufficient) condition for $x^*$ to minimize $f(x)$ is that $\nabla_x f(x^*) = 0$.

- This multi-variable gradient condition generalizes the first derivative condition for 1-variable min problems:

$$\nabla \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$
Minimization with Equality Constraints...Cont'd

- When a minimization problem involves equality constraints, we can solve the problem using the **method of Lagrange Multipliers**

- The key idea is to transform the constrained minimization problem into an unconstrained problem
Minimization with Equality Constraints...Cont'd

- n-dimensional vector \( \mathbf{x} = (x_1, \ldots, x_n)^T \) \( \Rightarrow \) \textit{n choice variables}
- m-dimensional vector \( \mathbf{c} = (c_1, \ldots, c_m)^T \) \( \Rightarrow \) \textit{m constraint constants}
- \( \mathbb{R}^n = \text{all real } n\text{-dimensional vectors} \), \( \mathbb{R}^m = \text{all real } m\text{-dimensional vectors} \)
- \( f : \mathbb{R}^n \to \mathbb{R} \) \( \Rightarrow \) \textit{objective function} mapping \( \mathbf{x} \to f(\mathbf{x}) \) on real line
- \( h : \mathbb{R}^n \to \mathbb{R}^m \) \( \Rightarrow \) \textit{constraint function} mapping \( \mathbf{x} \to h(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_m(\mathbf{x}))^T \)

**Nonlinear Programming Problem:**

\((\text{NPP})\) Minimize \( f(\mathbf{x}) \) with respect to \( \mathbf{x} \)

subject to \( h(\mathbf{x}) = \mathbf{c} \)
The **Lagrangian Function** for this problem can be expressed in parameterized form as

\[
L(x, \lambda, c) = f(x) - \lambda^T [h(x) - c]
\]

or equivalently,

\[
L(x, \lambda, c) = f(x) + \lambda^T [c - h(x)]
\]

where \( \lambda^T = (\lambda_1, \ldots, \lambda_m) \)
Regularity Conditions: Suppose f, h are differentiable and either
\[ \text{rank}(\nabla_x h(x^*)) = m \text{ or } h(x) \text{ has form } A_{mxn} x_{nx1} + b_{mx1}. \]

\[ \nabla_x h(x) = \begin{bmatrix}
\frac{\partial h_1(x)}{\partial x_1} & \frac{\partial h_1(x)}{\partial x_2} & \ldots & \frac{\partial h_1(x)}{\partial x_n} \\
\frac{\partial h_2(x)}{\partial x_1} & \frac{\partial h_2(x)}{\partial x_2} & \ldots & \frac{\partial h_2(x)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial h_m(x)}{\partial x_1} & \frac{\partial h_m(x)}{\partial x_2} & \ldots & \frac{\partial h_m(x)}{\partial x_n}
\end{bmatrix}_{(m \times n)} \]
Minimization with Equality Constraints...Cont’d (cf. Fletcher [3])

First-Order Necessary Conditions (FONC) for \( x^* \) to solve Problem (NPP), given regularity conditions:

There exists a value \( \lambda^* \) for the multiplier vector \( \lambda \) such that \( (x^*, \lambda^*) \) satisfies:

\[
0 = \nabla_x L(x^*, \lambda^*, c)_{1 \times n} = \nabla_x f(x^*) - \lambda^T \cdot \nabla_x h(x^*)
\]

\[
0 = \nabla_{\lambda} L(x^*, \lambda^*, c)_{1 \times m} = [c - h(x^*)]^T
\]
Lagrange Multipliers as “Shadow Prices”?

- By construction, the solution \((x^*, \lambda^*)\) is a function of the given vector \(c\) of constraint constants:
  \[(x^*, \lambda^*) = (x(c), \lambda(c))\]

- From FONC condition (2) on previous page:
  \[f(x(c)) = L(x(c), \lambda(c), c)\]
Lagrange Multipliers as “Shadow Prices”...

Then from implicit function theorem, the chain rule, and FONC (1), (2), for each constraint $k$,

\[
\frac{df}{dc_k}(x(c)) = \frac{dL}{dc_k}(x(c), \lambda(c), c) = \nabla_x L \cdot \frac{\partial x}{\partial c_k} + \nabla \lambda L \cdot \frac{\partial \lambda}{\partial c_k} + \frac{\partial L}{\partial c_k}
\]

= 0 + 0 + \lambda_k(c)

This “+” sign follows from the form of Lagrangean function.
Lagrange Multipliers as “Shadow Prices”...

In summary, for each constraint \( k \):

\[
\frac{df}{d c_k}(x(c)) = \lambda_k(c)
\]

Thus the Lagrange multiplier solution \( \lambda_k(c) \) for the \( k \)th constraint gives the change in the *optimized* objective function \( f(x^*) = f(x(c)) \) with respect to a change in the constraint constant \( c_k \) for the \( k \)th constraint.
Example: Economic Dispatch Again

For the economic dispatch we have a minimization constrained with a single equality constraint

\[ L(P_G, \lambda) = \sum_{i=1}^{\mathbb{I}} \text{VC}_i(P_{Gi}) + \lambda(P_D - \sum_{i=1}^{\mathbb{I}} P_{Gi}) \] (no losses)

The necessary conditions for a minimum are

\[ \frac{\partial L(P_G, \lambda)}{\partial P_{Gi}} = \frac{d\text{VC}_i(P_{Gi})}{dP_{Gi}} - \lambda = 0 \] (for \( i = 1 \) to \( \mathbb{I} \))

\[ P_D - \sum_{i=1}^{\mathbb{I}} P_{Gi} = 0 \]
What is economic dispatch for a two generator system \( P_D = P_{G1} + P_{G2} = 500 \text{ MW} \) and

\[
\nabla C_1(P_{G1}) = 1000 + 20P_{G1} + 0.01P_{G1}^2 \text{ $/h}$
\]
\[
\nabla C_2(P_{G2}) = 400 + 15P_{G2} + 0.03P_{G2}^2 \text{ $/h}$
\]

Using the Lagrange multiplier method we know

\[
\frac{dV}{dP_{G1}}\frac{\nabla C_1(P_{G1})}{dP_{G1}} - \lambda = 20 + 0.02P_{G1} - \lambda = 0
\]

\[
\frac{dV}{dP_{G2}}\frac{\nabla C_2(P_{G2})}{dP_{G2}} - \lambda = 15 + 0.06P_{G2} - \lambda = 0
\]

\[
500 - P_{G1} - P_{G2} = 0
\]
Economic Dispatch Example...Continued

We therefore need to solve three linear equations

\[
20 + 0.02P_{G1} - \lambda = 0 \\
15 + 0.06P_{G2} - \lambda = 0 \\
500 - P_{G1} - P_{G2} = 0
\]

\[
\begin{bmatrix}
0.02 & 0 & -1 \\
0 & 0.06 & -1 \\
-1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
P_{G1} \\
P_{G2} \\
\lambda
\end{bmatrix}
=
\begin{bmatrix}
-20 \\
-15 \\
-500
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_{G1}^* \\
P_{G2}^* \\
\lambda^*
\end{bmatrix}
=
\begin{bmatrix}
312.5 \text{ MW} \\
187.5 \text{ MW} \\
26.2 \$/\text{MWh}
\end{bmatrix}
\]

\{ \text{Solution Values} \}
Economic Dispatch Example...Continued

- The solution values for the Economic Dispatch problem are:
  
  \[ P_G^* = (P_{G1}^*, P_{G2}^*)^T = (312.5 \text{ MW}, 187.5 \text{ MW})^T \]
  
  \[ \lambda^* = $26.2/\text{MWh} \]

- By construction, the solution values \( P_G^* \) and \( \lambda^* \) are functions of the constraint constant \( P_D \).

- That is, a change in \( P_D \) would result in a change in these solution values.
Applying previous developments for Lagrange multipliers as shadow prices,

\[ \lambda^* = \frac{\delta TVC(P_G^*)}{\delta P_D} \]  (in $/MWh)

= Change in minimized total variable cost TVC (in $/h) with respect to a change in the total demand \( P_D \) (in MW), where \( P_D \) is the constraint constant for the balance constraint

\[ \text{[Total Dispatch} = \text{Total Demand]} \]
By definition, the *locational marginal price (LMP)* at any bus $k$ in any hour $h$ is the least cost of servicing one more megawatt (MW) of load (fixed demand) at bus $k$.

Consequently,

$$\lambda^* = \frac{\delta \text{TVC}(P_G^*)}{\delta P_D} \quad \text{(in $/\text{MWh}$)}$$

$$\lambda^* = \text*{Locational marginal price (LMP)} \text{ for electric power (same at each bus since we assume there are no line losses and no transmission constraints)}$$
Extension to Inequality Constraints
(cf. Fletcher [3])

General Nonlinear Programming Problem (GNPP):
- $\mathbf{x} = n \times 1$ choice vector;
- $\mathbf{c} = m \times 1$ vector & $\mathbf{d} = s \times 1$ vector (constraint constants)
- $f(\mathbf{x})$ maps $\mathbf{x}$ into $\mathbb{R}$ (all real numbers)
- $h(\mathbf{x})$ maps $\mathbf{x}$ into $\mathbb{R}^m$ (all m-dimensional vectors)
- $z(\mathbf{x})$ maps $\mathbf{x}$ into $\mathbb{R}^s$ (all s-dimensional vectors)

**GNPP**: Minimize $f(\mathbf{x})$ with respect to $\mathbf{x}$ subject to

\[
\begin{align*}
    h(\mathbf{x}) & = \mathbf{c} \\
    z(\mathbf{x}) & \geq \mathbf{d}
\end{align*}
\]
Important Remark on the Representation of Inequality Constraints

**Note:** The inequality constraint for (GNPP) can equivalently be expressed in many different ways:

1. \( z(x) \geq d \);
2. \( z(x) - d \geq 0 \);
3. \( -z(x) - [-d] \leq 0 \);
4. \( r(x) - e \leq 0 \) \((r(x) = -[z(x)], e = -[d])\)
5. \( r(x) \leq e \)
Why Our Form of Inequality?

Our GNPP Form:

Minimize $f(x)$ with respect to $x$ subject to

$h(x) = c$

(*) $z(x) \geq d$

Given this form, we know that an **INCREASE** in $d$ has to result in a new value for the minimized objective function $f(x^*)$ that is **AT LEAST AS GREAT** as before.

Why? When $d$ increases the feasible choice set for $x$ **SHRINKS**, hence $[\text{min } f]$ either $\uparrow$ or stays same.

$\Rightarrow 0 \leq \frac{\delta f(x^*)}{\delta d} = \mu^*^T = \text{Shadow price vector for (*)}$
Extension to Inequality Constraints...Continued

• Define the *Lagrangean Function* as

\[ L(x, \lambda, \mu, c, d) = f(x) + \lambda^T[c - h(x)] + \mu^T[d - z(x)] \]

• Assume *Kuhn-Tucker Constraint Qualification (KTCQ)* holds at \( x^* \), roughly stated as follows:

The true set of feasible directions at \( x^* \)

\[ = \text{Set of feasible directions at } x^* \text{ assuming a linearized set of constraints in place of the original set of constraints.} \]
Extension to Inequality Constraints...Continued

- Given KTCQ, the **First-Order Necessary Conditions (FONC)** for \(x^*\) to solve the (GNPP) are as follows: There exist \(\lambda^*\) and \(\mu^*\) such that \((x^*, \lambda^*, \mu^*)\) satisfy:

\[
0 = \nabla_x L(x^*, \lambda^*, \mu^*, c, d) \\
= [ \nabla_x f(x^*) - \lambda^{**T} \cdot \nabla_x h(x^*) - \mu^{**T} \cdot \nabla x z(x^*) ] ; \\
h(x^*) = c ; \\
z(x^*) \geq d ; \mu^{**T} [d - z(x^*)] = 0 ; \mu^* \geq 0
\]

- These FONC are often referred to as the **Karush-Kuhn-Tucker (KKT) conditions**.
Solution as a Function of \((c, d)\)

By construction, the components of the solution vector \((x^*, \lambda^*, \mu^*)\) are functions of the constraint constant vectors \(c\) and \(d\)

- \(x^* = x(c, d)\)
- \(\lambda^* = \lambda(c, d)\)
- \(\mu^* = \mu(c, d)\)
Given certain additional regularity conditions...

- The solution \( \lambda^* \) for the \( m \times 1 \) multiplier vector \( \lambda \) is the derivative of the minimized value \( f(x^*) \) of the objective function \( f(x) \) with respect to the constraint vector \( c \), all other problem data remaining the same.

\[
\frac{\partial f(x^*)}{\partial c} = \frac{\partial f(x(c,d))}{\partial c} = \lambda^*^T
\]
Given certain additional regularity conditions...

- The solution $\mu^*$ for the $s \times 1$ multiplier vector $\mu$ is the derivative of the minimized value $f(x^*)$ of the objective function $f(x)$ with respect to the constraint vector $d$, all other problem data remaining the same.

$$0 \leq \delta f(x^*)/\delta d = \delta f(x(c,d))/\delta d = \mu^*$$
Consequently...

- The solution $\mathbf{\lambda}^*$ for the multiplier vector $\mathbf{\lambda}$ thus essentially gives the *prices (values)* associated with unit changes in the components of the constraint vector $\mathbf{c}$, all other problem data remaining the same.

- The solution $\mathbf{\mu}^*$ for the multiplier vector $\mathbf{\mu}$ thus essentially gives the *prices (values)* associated with unit changes in the components of the constraint vector $\mathbf{d}$, all other problem data remaining the same.

- Each component of $\mathbf{\lambda}^*$ can take on *any sign*

- Each component of $\mathbf{\mu}^*$ must be *nonnegative*
Sufficient Conditions for Minimization?

First-order **necessary** conditions for $x^*$ to solve NPP/GNPP are **not sufficient in general** to ensure $x^*$ solves NPP/GNPP, or to ensure $x^*$ solves NPP/GNPP uniquely.

- What can go wrong:
  - (1) Local maximum rather than local minimization
  - (2) Inflection point rather than minimum point
  - (3) Local minimum rather than global minimum
  - (4) Multiple minimizing solution points

- Need conditions on second derivatives to rule out 1 & 2, “global” methods/conditions to rule out 3 & 4
Local Max Rather Than Local Min

From A. Hallam, “Simple Multivariate Optimization” (on-line)
Inflection Rather than Min Point

Figure 2. Saddle point of the function $f(x_1, x_2) = x_1^2 - x_2^2$

From A. Hallam, “Simple Multivariate Optimization” (on-line)
Local Min Rather than Global Min: Both Satisfy the FONC $df(x)/dx = 0$
Multiple Minimization Points

**Figure 18.** Graph of the function $f(x_1, x_2) = -x_1 x_2 e^{-\frac{(x_1^2 + x_2^2)}{2}}$

*From A. Hallam, “Simple Multivariate Optimization” (on-line)*
Technical References


