Game Theory: Basic Concepts and Terminology

A **GAME** consists of:

- a collection of decision-makers, called *players*;

- the possible information states of each player at each decision time;

- the collection of possible moves (decisions, actions, plays,...) that each player can choose to make in each of his possible information states;

- a procedure for determining how the move choices of all the players collectively determine the possible outcomes of the game;

- preferences of the individual players over these possible outcomes, typically measured by a *utility* or *payoff* function.
Illustrative Modeling of a Work-Site Interaction as a “Prisoner’s Dilemma Game”

D = Defect (Shirk)  C = Cooperate (Work Hard),

\[(P_1,P_2) = (\text{Worker Payoff, Employer Payoff})\]
A **PURE STRATEGY** for a player in a particular game is a complete contingency plan, i.e., a plan describing what move that player should take in each of his possible information states.

A **MIXED STRATEGY** for a player \( i \) in a particular game is a probability distribution defined over the collection \( S_i \) of player \( i \)'s possible pure strategy choices. That is, a mixed strategy assigns a nonnegative probability \( \text{Prob}(s) \) to each pure strategy \( s \) in \( S_i \), with

\[
\sum_{s \in S_i} \text{Prob}(s) = 1. \tag{1}
\]

**EXPOSITIONAL NOTE:**
For simplicity, the remainder of these brief notes will develop definitions in terms of pure strategies; the unqualified use of “strategy” will always refer to pure strategy. Extension to mixed strategies is conceptually straightforward.
ONE-STAGE SIMULTANEOUS-MOVE N-PLAYER GAME:

- The game is played just once among N players.
- Each of the N players *simultaneously* chooses a strategy (move) based on his current information state, where this information state does *not* include knowledge of the strategy choices of any other player.
- A payoff (reward, return, utility outcome,...) for each player is then determined as a function of the N simultaneously-chosen strategies of the N players.

**Note:** For ONE-stage games, there is only one decision time. Consequently, a choice of a strategy based on a current information state is the same as the choice of a move based on this current information state.
ITERATED SIMULTANEOUS-MOVE N-PLAYER GAME:

- The game is played among N players over successive iterations \( T = 1, 2, \ldots, T_{\text{Max}} \).
- In each iteration \( T \), each of the N players *simultaneously* makes a move (action, play, decision, ...) conditional on his current information state, where this information state does *not* include the iteration-\( T \) move of any other player.
- An iteration-\( T \) payoff (reward, return, utility outcome, ...) is then determined for each player as a function of the N simultaneously-made moves of the N players in iteration \( T \).
- If \( T < T_{\text{Max}} \), the next iteration \( T+1 \) then commences.
- The information states of the players at the beginning of iteration \( T+1 \) are typically updated to include at least some information regarding the moves, payoffs, and/or outcomes from the previous iteration \( T \).

**Note:** For ITERATED games there are multiple decision times. Consequently, a choice of a move based on a current information state does not constitute a strategy (complete contingency plan). Rather, a strategy is the choice of a move for the current iteration, given the current information state, together with a designation of what move to choose in each future iteration conditional on every possible future information state.
"PAYOFF MATRIX" FOR A ONE-STAGE SIMULTANEOUS-MOVE 2-PLAYER GAME:

Consider a one-stage simultaneous-move 2-player game in which each player must choose to play one of $M$ feasible strategies $S_1, \ldots, S_M$. The Payoff Matrix for this 2-player game then consists of an $M \times M$ table that gives the payoff received by each of the two players under each feasible combination of moves the two players can choose to make.

More precisely, each of the $M$ rows of the table corresponds to a feasible strategy choice by Player 1, and each of the $M$ columns of the table corresponds to a feasible strategy choice by Player 2. The entry in the $i$th row and $j$th column of this $M \times M$ table then consists of a pair of values $(P_1(i, j), P_2(i, j))$.

The first value $P_1(i, j)$ denotes the payoff received by Player 1 when Player 1 chooses strategy $S_i$ and Player 2 chooses strategy $S_j$, and the second value $P_2(i, j)$ denotes the payoff received by Player 2 when Player 1 chooses strategy $S_i$ and Player 2 chooses strategy $S_j$. See the 2-player example depicted on the next page.

This definition is easily generalized to the case in which each player has a different collection of feasible strategies to choose from (different by type and/or number).
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NASH EQUILIBRIUM FOR AN N-PLAYER GAME:

A specific combination \((S_1^*, \ldots, S_N^*)\) of feasible strategy choices for an \(N\)-player game, one strategy choice \(S_i^*\) for each player \(i\), is called a \((Pure\ Strategy)\ Nash\ equilibrium\) if no player \(i\) perceives any feasible way of achieving a higher payoff by switching unilaterally to another strategy \(S_i'\).

DOMINANT STRATEGY FOR AN N-PLAYER GAME:

A feasible strategy for a player in an \(N\)-player game is said to be a \(dominant\ strategy\) for this player if it is this player’s \(best\ response\) to \(any\) feasible choice of strategies for the other players.

For example, suppose \(S_1^*\) is a dominant strategy for player 1 in an \(N\)-player game. This means that, no matter what feasible combination of strategies \((S_2, \ldots, S_N)\) players 2 through \(N\) might choose to play, player 1 attains the highest feasible (expected) payoff if he chooses to play strategy \(S_1^*\).

QUESTIONS:

(1) Does the previously depicted worker-employer game have a Nash equilibrium?
(2) Does either player in this game have a dominant strategy?
(3) What is the key distinction between a dominant strategy and a strategy constituting part of a Nash equilibrium?
PARETO EFFICIENCY:

Intuitive Definition:
A feasible combination of decisions for a collection of agents is said to be Pareto efficient if there does not exist another feasible combination of decisions under which each agent is at least as well off and some agent is strictly better off.

More Rigorous Definition: N-Player Game Context
For each \( i = 1, \ldots, N \), let \( P_i \) denote the payoff attained by player \( i \) under a feasible strategy combination \( S = (S_1, \ldots, S_N) \) for the \( N \) players. The strategy combination \( S \) is said to be Pareto efficient if there does not exist another feasible strategy combination \( S' \) under which each player \( i \) achieves at least as high a payoff as \( P_i \) and some player \( j \) achieves a strictly higher payoff than \( P_j \). The payoff outcome \( (P_1, \ldots, P_N) \) is then said to be a Pareto efficient payoff outcome.

QUESTION:
Does the previously depicted worker-employer game have a Pareto efficient strategy combination?
PARETO DOMINATION:

**Intuitive Definition:** A feasible combination of decisions for a collection of agents is said to be *Pareto dominated* if there does exist another feasible combination of decisions under which each agent is at least as well off and some agent is strictly better off.

**More Rigorous Definition:** *N*-Player Game Context For each $i = 1, \ldots, N$, let $P_i$ denote the payoff attained by player $i$ under a strategy combination $S = (S_1, \ldots, S_N)$ for the $N$ players. The strategy combination $S$ is said to be *Pareto dominated* if there does exist another feasible strategy combination $S'$ under which each player $i$ achieves at least as high a payoff as $P_i$ and some player $j$ achieves a strictly higher payoff than $P_j$.

**QUESTION:**

Does the previously depicted worker-employer game have strategy combinations that are Pareto dominated?
COORDINATION FAILURE:

**Intuitive Definition:** A combination of decisions for a collection of agents is said to exhibit *coordination failure* if mutual gains, attainable by a collective switch to a different feasible combination of decisions, are not realized because no individual agent perceives any feasible way to increase their own gain by a unilateral deviation from their current decision.

**More Rigorous Definition:** *N*-Player Game Context A strategy combination \( S = (S_1, \ldots, S_N) \) is said to exhibit *coordination failure* if it is a Pareto-dominated Nash equilibrium.

**QUESTIONS:**

Does the previously depicted worker-employer game have a move combination that exhibits coordination failure?

Might the *iterative* play of this worker-employer game help alleviate coordination failure problems?