Assumptions or Axioms of Arrow

1. **Unanimity** (The Pareto Postulate) - If an individual preference is unopposed by any contrary preference of any other individual, this preference is preserved in the social ordering.

2. **Nondictatorship** - No individual enjoys a position such that whenever she expresses a preference between any two alternatives and all other individuals express the opposite preference, her preference is always preserved in the social ordering.

3. **Transitivity** - The social welfare function gives a consistent ordering of all feasible alternatives. That is, \((aPbPc) \Rightarrow (aPc)\) and \((aIbIc) \Rightarrow aIc\)

4. **Unrestricted Domain** - There is some universal alternative \(u\) such that for every pair of other alternatives \(x\) and \(y\) and for every individual, each of the six possible strict orderings of \(u, x,\) and \(y\) is contained in some admissible ranking of all alternatives for the individual.

5. **Independence of Irrelevant Alternatives** - The social choice between any two alternatives must depend on the orderings of individuals over only these alternatives, and not on their orderings over other alternatives.

**ARROW PARADOX:**

No process (voting, the market, or otherwise) exists that satisfies these axioms.

Any social choice process abiding by these axioms will lead to a dictatorship or inconsistent social orderings.

The theorem states that no social welfare function satisfies these five postulates. To understand the significance of the theorem, it is useful to run through the proof, again following Vickrey. We first define a decisive set \(D\).

**Definition of decisive set:** A set of individuals \(D\) is decisive, for alternatives \(x\) and \(y\) in a given social welfare function, if the function yields a social preference for \(x\) over \(y\), whenever all individuals in \(D\) prefer \(x\) to \(y\), and all others prefer \(y\) to \(x\).
PROOF:

<table>
<thead>
<tr>
<th>Step</th>
<th>Justification</th>
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<tbody>
<tr>
<td>1.</td>
<td>Let D be a set of individuals decisive for x and y</td>
</tr>
<tr>
<td>2.</td>
<td>Assume for all members of D xPyPu, and for all others (those in C) yPuPx</td>
</tr>
<tr>
<td>3.</td>
<td>For society xPy</td>
</tr>
<tr>
<td>4.</td>
<td>For society yPu</td>
</tr>
<tr>
<td>5.</td>
<td>For society xPu</td>
</tr>
<tr>
<td>6.</td>
<td>But for only members of D is xPu</td>
</tr>
<tr>
<td>7.</td>
<td>Society must prefer x to u regardless of changes in rankings of y or any other alternatives</td>
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<tr>
<td>8.</td>
<td>D is decisive for x and u</td>
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<tr>
<td>9.</td>
<td>D is decisive for all pairs of alternatives</td>
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<tr>
<td>10.</td>
<td>D must contain two or more persons</td>
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<tr>
<td>11.</td>
<td>Divide D into two nonempty subsets A and B</td>
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<tr>
<td>12.</td>
<td>Assume for A xPyPu for B yPuPx for C uPxPy</td>
</tr>
<tr>
<td>13.</td>
<td>Since for members of A and B, yPu, for society yPu</td>
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<tr>
<td>14.</td>
<td>If for society yPx, B is decisive for y and x</td>
</tr>
<tr>
<td>15.</td>
<td>If for society xPy, then for society xPu</td>
</tr>
<tr>
<td>16.</td>
<td>But then A is decisive for x and u</td>
</tr>
</tbody>
</table>

In either case, one of the proper subsets of D is decisive for a pair of issues, and therefore by step 9 for all issues. Steps 10-16 can be repeated for this new decisive set, and then continued until the decisive set contains but one member, thus contradicting the nondictatorship postulate.

Q.E.D.

The intuition underlying the proof runs as follows: The unrestricted domain assumption allows any possible constellation of ordinal preferences. When a unanimously preferred alternative does not emerge, some method for choosing among the Pareto preferred alternatives must be found. The independence assumption restricts attention to the ordinal preferences of individuals for any two issues, when deciding those issues. But,
as we have seen in our discussions of majority rule, it is all too easy to construct rules that yield choices between two alternatives but produce a cycle when three successive pairwise choices are made. The transitivity postulate forces a choice among the three, however. The social choice process is not to be left indecisive (Arrow, 1963, p. 120). But with the information at hand—that is, individual ordinal rankings of issue pairs—there is no method for making such a choice that is not imposed or dictatorial.

**Escapes from the Paradox**

Requires a relaxing of one or more of the axioms.

1. **Relaxing Pareto principle and nondictatorship** - Seems hardly worth discussing, if the ideals of individualism and citizen sovereignty are to be maintained.

2. **Transitivity** - Arrow's reason for social choice process to produce a consistent social ordering appear to be
   a. Some social choice be made from any environment.
   b. This choice be independent of the path (or Agenda).

These are different requirements and neither requires full forces of transitivity.

To achieve the 1st goal, one does not have to assume the existence of a social preference ordering defined on the basis of all individual pref. orderings. To make a choice, one needs only a **choice function** that allows one to select a best alternative from any set of feasible alternatives. Transitivity is not required.

Can use

A. **Quasi-Transitivity** - transitivity of preference but not indifference.

or

B. **Acyclicity** allows \( x_1 \) to be only at least as good as \( x_n \), even though \( x_1 P x_2 \ldots x_{n-1} P x_n \).

Possibility theorems have been proven by replacing transitivity by either of these and retaining the other Arrow axioms.

A quasi-transitive ordering of the social choice function produces an oligarchy, which can impose its unanimous preference on the rest of the community. Also, it gives veto power to every member of a subset of the oligarchy—collegium.

1) Hence, dictatorial powers become spread, but don't disappear.
2) Introduce a degree of arbitrariness

  e.g., aPb bPc => aPc or aIc

  Thus, if the property of having a choice made from every environment is to be maintained, there appears to be little lost by sticking to the full transitivity requirement.

3. **Unrestricted Domain** - Freedom of choice or expression.

   **Escapes**

   A. Replace unrestricted domain with other axioms limiting the types of preference orderings the collective choice process is capable of reflecting. This implies placing constitutional constraints on the types of issues that can come up before the collective. E.g., single dimensional issues, issues without distribution effects.

   B.* Restrict community entry to those having preference orderings that do make collective choice possible. E.g., single peakedness along with other four axioms produces a social welfare function--decided by median voter.

4. **Independence of Irrelevant Alternative** - Knowing the social choice made in pairwise comparison in turn determines the entire social ordering and therefore the social choice function for all possible environments.

   This assumption eliminates Borda Counts. The outcomes under the Borda procedure and similar schemes are dependent upon the specific and full set of issues to be decided. Thus, the abandonment of the independence axiom raises the importance of the process which selects the issues to be decided, in a way its acceptance does not. Independence axiom has an appealing economy to it, but it is this property that causes endless cycling.

   This axiom excludes interpersonal comparisons and cardinalizaiton.
Reasoning for excluding cardinality

A. Measurement problems--difficult and arbitrary
B. Procedures are all vulnerable to strategic misrepresentation of preferences.

Independence axiom eliminates all strategic prone procedures.

Duncan Blacks' Unanimity Voting and Escape

Unanimity satisfies the other Arrow axioms and gives rise to a fraction of intransitivities so trivial that for every practical purpose it can be disregarded. For instance, the fraction of intransitivities for a committee of 40 is less than 1 millionth that of a committee of 20.

The larger the committee => less intransitivities

Unanimity

\[(x, y) \ (n, o, o) => x > y\]
\[(x, y) \ (o, o, n) => x < y\]
\[(x, y) \ (h, k, h) => x = y\]
\[(x, y) \ (p, q, r) => x = y\]
\[(x, y) \ (r, q, p) => x = y\]

Unanimity satisfies nondictatorship since if only a single member in collective \(x > y\) while all others \(y > x\), the committee cannot say \(x > y\).

1) Unanimity (Pareto Principle)
2) Nondictatorship
3) Independence of Irrelevant Alternatives
4) Unrestricted Domain
5) Transitivity

Theorem: With Procedure of complete unanimity, the fraction of intransitivities for a committee of \(n\) members (3 motions, strong preference orderings) is

\[
\frac{3^n - 2^{n+1} + 1}{6^{n-1}}
\]

\(n = \) number of people

Unanimity performs worse as number of motions to be considered increases.