Homework #1 - Solution Outline

1. a. 
   \[ f_X(x) = \int_0^2 0.25(2x+y)\,dy = x + 0.5 \quad \text{for } 0 \leq x \leq 1 \quad (0 \text{ otherwise}) \]
   \[ f_Y(y) = \int_0^1 0.25(2x+y)\,dx = 0.25(1 + y) \quad \text{for } 0 \leq y \leq 2 \quad (0 \text{ otherwise}) \]
   \[ f(x,y) \neq f_x(x) \cdot f_y(y) \quad \text{X and Y not independent} \]

b. 
   \[ E[X] = \int_0^1 x(x + 0.5)\,dx = 7/12 \]
   \[ E[Y] = \int_0^2 0.25y(1 + y)\,dy = 7/6 \]

c. 
   \[ E[XY] = \int_0^1 \int_0^1 xy(0.25)(2x+y)\,dxdy = 2/3 \]
   \[ \text{Cov}(X,Y) = E[XY] - E[X] \cdot E[Y] = -1/72 \]

d. 
   \[ f(Y \mid X = 0.25) = \frac{f(0.25,y)}{f_X(0.25)} = \frac{1}{6} + \frac{1}{3}y \quad \text{for } 0 \leq y \leq 2 \quad (0 \text{ otherwise}) \]
   \[ E[Y \mid X = 0.25] = \int_0^2 y\left(\frac{1}{6} + \frac{1}{3}y\right)\,dy = \frac{11}{9} \]
   \[ E[Y^2 \mid X = 0.25] = \int_0^2 y^2\left(\frac{1}{6} + \frac{1}{3}y\right)\,dy = \frac{16}{9} \]
   \[ \text{Var}(Y \mid X = 0.25) = E[Y^2 \mid X = 0.25] - (E[Y \mid X = 0.25])^2 = \frac{23}{81} \]

2. 
   \[ E[(X - \mu)^3] = E[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3 \]
3. a. For both \((i)\) and \((ii)\), we have:

\[
\text{mean} = \sum_{j=1}^{6} j \cdot f(j) = 3.5, \quad E[X^2] = \sum_{j=1}^{6} j^2 \cdot f(j) = 13.7
\]

\[
\text{variance} = E[X^2] - \text{mean}^2 = 1.45, \quad \text{standard deviation} = 1.2042.
\]

b. Histograms show that distribution \((i)\) is symmetric; distribution \((ii)\) is not.

c. For distribution \((i)\):

\[
E[X^3] = \sum_{j=1}^{6} j^3 \cdot f(j) = 58.1
\]

\[
E[(X - \mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3 = 58.1 - 3 \cdot 3.5 \cdot 13.7 + 2 \cdot (3.5)^3 = 0.0
\]

Hence \(\alpha_3 = 0.0\) (a property of all symmetric distributions)

For distribution \((ii)\):

\[
E[X^3] = \sum_{j=1}^{6} j^3 \cdot f(j) = 57.8
\]

\[
E[(X - \mu)^3] = 57.8 - 3 \cdot 3.5 \cdot 13.7 + 2 \cdot (3.5)^3 = -0.3
\]

\[
\alpha_3 = \frac{E[(X - \mu)^3]}{\sigma^3} = \frac{-0.3}{(1.2042)^3} = -0.1718
\]

4. \(\phi(x) = 4 - x^2, \quad \phi'(x) = -2x, \quad \phi''(x) = -2 < 0\) strictly concave

a. \(E[\phi(X)] = E[4 - X^2] = 4 - \int_1^2 x^2 (0.5x + 0.25) dx = \frac{37}{24} = 1.5417\)

b. \(\phi(E[X]) = 4 - (E[X])^2 = 4 - \left(\int_1^2 x (0.5x + 0.25) dx\right)^2 = 4 - \left(\frac{37}{24}\right)^2 = 1.6233\)

c. Comparing \((a)\) and \((b)\): \(E[\phi(X)] < \phi(E[X])\) as required by Jensen's inequality.
5. a. $E[X] = \int_{-1}^{0} x(1+x)dx + \int_{0}^{1} x(1-x)dx = 0$

$E[X^2] = \int_{-1}^{0} x^2(1+x)dx + \int_{0}^{1} x^2(1-x)dx = 1/6$

$E[X^3] = \int_{-1}^{0} x^3(1+x)dx + \int_{0}^{1} x^3(1-x)dx = 0$

$Cov(U, V) = E[U \cdot V] = E[U] \cdot E[V] = E[X^2] = E[X] \cdot E[X^2] = 0$

b. $E[V \mid U = 1/2] = E[X^2 \mid X = 1/2] = \frac{1}{4} \neq \frac{1}{6} = E[X^2] = E[V]$