Homework #2

1. Consider the random sample \( X_1, X_2, \ldots, X_n \) i.i.d. \( N(\mu, \sigma^2) \), and the sample mean, 
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.
\]
Find the smallest value of integer \( n \) such that 
\[
P\left(\left| \bar{X} - \mu \right| \leq 0.5\sigma \right) \geq 0.99.
\]

2. Consider the random sample \( X_1, X_2, \ldots, X_n \) i.i.d. \( N(\mu, \sigma^2) \), and the sample mean, 
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]
and the sample variance, 
\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.
\]
Find the smallest value of integer \( n \) such that 
\[
P(S^2 \geq 1.7\sigma^2) \leq 0.05.
\]

3. Consider the random sample \( X_1, X_2, \ldots, X_n \) i.i.d. \( N(\mu, \sigma^2) \), and the sample mean, 
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]
and the sample variance, 
\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.
\]
Find the smallest value of integer \( n \) such that 
\[
P\left(\left| \bar{X} - \mu \right| \leq 0.75S \right) \geq 0.99.
\]

4. Consider two independent random samples of different sizes from normal populations with a common mean but different variances:
\[
X_1, X_2, \ldots, X_n \text{ i.i.d. } N(\mu, \sigma_1^2) \quad \text{and} \quad Y_1, Y_2, \ldots, Y_m \text{ i.i.d. } N(\mu, \sigma_2^2)
\]
Denote the sample means by 
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \bar{Y} = \frac{1}{m} \sum_{i=1}^{m} Y_i.
\]

\( a. \) Show that 
\[
w\bar{X} + (1-w)\bar{Y}
\]
is an unbiased estimator of \( \mu \) for all \( 0 \leq w \leq 1 \).

\( b. \) Find the value of \( w \) in the range \( 0 \leq w \leq 1 \) that minimizes 
\[
Var\left(w\bar{X} + (1-w)\bar{Y}\right)
\]

5. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables with, for each \( i = 1, 2, \ldots, n \):
\[
X_i = 1 \quad \text{with probability } \theta
\]
\[
0 \quad \text{with probability } 1 - \theta.
\]
Define 
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.
\]
Find \( E[\bar{X}] \) and \( Var(\bar{X}) \).