Homework #2

1. The joint probability density function of random variables $X$ and $Y$ is given by:

$$f(x, y) = \begin{cases} \frac{1}{4}(2x + y) & 0 < x < 1, \ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of $Y$ conditional on $X = \frac{1}{4}$.

2. This problem involves an extension, to the bi-variate case, of Jensen's inequality. A real-valued function of two real variables, $\phi(x, y) : \mathbb{R}^2 \to \mathbb{R}$, is strictly concave if

$$\phi(hx_1 + (1 - h)x_2, hy_1 + (1 - h)y_2) > h\phi(x_1, y_1) + (1 - h)\phi(x_2, y_2)$$

for all $(x_1, y_1) \neq (x_2, y_2) \in \mathbb{R}^2$ and for all $h \in (0, 1)$. For the case of differentiable $\phi(\cdot)$, strict concavity is equivalent to the following first-order derivative property:

$$\phi(x_2, y_2) < \phi(x_1, y_1) + \frac{\partial \phi}{\partial x}(x_1, y_1)(x_2 - x_1) + \frac{\partial \phi}{\partial y}(x_1, y_1)(y_2 - y_1)$$

for all $(x_1, y_1) \neq (x_2, y_2) \in \mathbb{R}^2$.

Write out a proof of the following claim. Let $X$ and $Y$ be continuous random variables with means $\mu_X$ and $\mu_Y$ respectively. Let $\phi(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and strictly concave. Then $E[\phi(X, Y)] < \phi(\mu_X, \mu_Y) = \phi(E[X], E[Y])$.

3. The joint probability density function of random variables $X$ and $Y$ is given by:

$$f(x, y) = \begin{cases} \frac{1}{2}\left(4xy + x + \frac{1}{2}\right) & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Define the function $\phi(x, y) = -(x^2 + xy + y^2)$.
a. Show that \( \varphi(\cdot) \) is strictly concave. (Hint: The easiest way to do this is to use a second-order derivative feature of differentiable strictly concave functions. In particular, a differentiable function \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is strictly concave if its Hessian matrix,

\[
\begin{bmatrix}
\frac{\partial^2 \varphi}{\partial x^2} & \frac{\partial^2 \varphi}{\partial x \partial y} \\
\frac{\partial^2 \varphi}{\partial x \partial y} & \frac{\partial^2 \varphi}{\partial y^2}
\end{bmatrix},
\]

is negative definite throughout \( \mathbb{R}^2 \). Moreover, a \( 2 \times 2 \) matrix is negative definite if its diagonal elements are negative and its determinant is positive.)

b. Evaluate \( E[\varphi(X, Y)] \) and \( \varphi(E[X], E[Y]) \) for this case and show that the bi-variate version of Jensen's inequality is satisfied.

4. A random variable \( X \) is said to have the Weibull distribution with parameters \( \alpha > 0 \) and \( m > 0 \) if its density function is given by:

\[
f_X(x) = \begin{cases} 
\frac{1}{\alpha} m x^{m-1} e^{-x^{\alpha}/\alpha} & x > 0 \\
0 & \text{otherwise}
\end{cases}
\]

Show that the random variable \( Y = X^m \) has the exponential distribution with parameter \( \alpha \). (Hint: Use the transformation of variable technique to find the probability density function for random variable \( Y \).)

5. a. For random variable \( X \) with mean \( \mu \), express the third moment about the mean, \( E[(X - \mu)^3] \), in terms of the first three moments about the origin.

b. For random variable \( X \sim N(\mu, \sigma^2) \), the moment generating function is given by:

\[
m(t) = \exp\left( \mu t + \frac{t^2 \sigma^2}{2} \right).
\]

Use the moment generating function to find the first three moments about the origin, then use the result of part a to find the third moment about the mean for \( X \sim N(\mu, \sigma^2) \).