Homework #3

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$ for $i = 1, 2, \ldots, n$. Define $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Find an expression for $\text{Cov}(\bar{X}, X_i - \bar{X})$ in terms of $\mu$ and $\sigma^2$.

2. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. $N(\mu_1, \sigma^2)$ and that $Y_1, Y_2, \ldots, Y_m$ are i.i.d. $N(\mu_2, \sigma^2)$, and that the two samples are independent. (Note that while the two population distributions have a common variance, the means differ. Note also that the two samples are different sizes: $n$ and $m$.) Find the maximum likelihood estimator of $\sigma^2$.

3. Random variable $X$ has a continuous distribution characterized by the following probability density function:

$$f_X(x) = \begin{cases} \frac{2}{\sqrt{\pi}} \exp\left(-\frac{1}{2}x^2\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

Find the probability density function for random variable $Y = \frac{X}{1 + X}$.

4. A discrete random variable, $Y$, is said to have the Poisson distribution with parameter $\lambda > 0$ if its probability mass function is given by:

$$f(y) = \begin{cases} \frac{\lambda^y}{y!} e^{-\lambda} & \text{for } y = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}.$$ 

Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. Poisson with parameter $\lambda$. Find the maximum likelihood estimator of $\lambda$. 
5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Define $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$. Find the smallest integer $n$ such that $\Pr \left( \left| \frac{\bar{X} - \mu}{S} \right| \leq 1.0 \right) \geq 0.95$. 