Tools for Counting Sample Points

Probability with Equiprobable Events

If a sample space contains N equiprobable sample points and an event A contains n_a sample points, then $P(A) = n_a/N$.

mn Rule

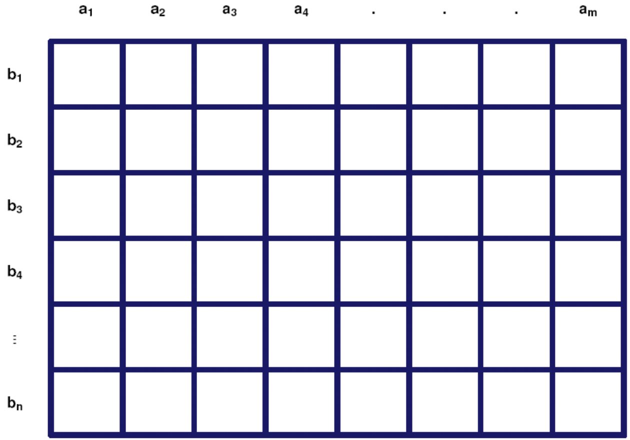
Theorem

With m elements $a_1, a_2, a_3, \ldots, a_m$ and n elements $b_1, b_2, b_3, \ldots, b_n$ it is possible to form $mn = m \times n$ pairs containing one element from each group.

Tools for Counting Sample Points

mn rule

Proof of the theorem can be can be seen by observing the rectangular table below. There is one square in the table for each a_i, b_j pair and hence a total of $m \times n$ squares.



Tools for Counting Sample Points

mn rule

The *mn* rule can be extended to any number of sets. Given three sets of elements, $a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3 \ldots, , b_n$ and $c_1, c_2, c_3, \ldots, c_\ell$, the number of distinct triplets containing one element from each set is equal to $mn\ell$.

Example

Consider an experiment where we toss a red die and a green die and observe what comes up on each die. We can describe the sample space as

$$S_1 = \{(x, y) | x = 1, 2, 3, 4, 5, 6; \; y = 1, 2, 3, 4, 5, 6\}$$

where x denotes the number turned up on the red die and y represents the number turned up on the green die. The first die can result in six numbers and the second die can result in six numbers so the total number of sample points in S is 36.

Example

Consider the event of tossing a coin three times. The sets here are identical and consist of two possible elements, H or T. So we have $2 \times 2 \times 2$ possible sample points. Specifically

$$S_2 = \{(x,\ y,\ z)|x = 1,\ 2;\ \ y = 1,\ 2;\ \ z = 1,\ 2\}$$

mn Rule

Birthday example

- Consider an experiment that consists of recording the birthday for each of 20 randomly selected persons.
- Assume that there are only 365 possible distinct birthdays.
- Find the number of points in the sample space S for this experiment.
- If we assume that each of the possible sets of birthdays is equiprobable, what is the probability that each person in the 20 has a different birthday?
  - Number the days of the year 1, 2, . . . , 365. A sample point is $(x_1, ..., x_{20})$ where x_i is the birthdate of person i.
  - Repeated applications of the *mn* rule tell us there are $(365)^{20}$ such twenty-tuples. Thus S contains $N = (365)^{20}$ sample points.
  - If we assume them to be equiprobable, $P(E_i) = 1/(365)^{20}$ for each simple event.

mn Rule

Birthday example continued

- Denote the event that each person has a different birthday by A.
- To find $P(A)$ we need to determine n_a, the number of sample points in A.
- The set of numbers from which the first element in a 20-tuple in A can be selected contains 365 numbers
- The set from which the second element can be selected contains 364 numbers
- The set from which the third can be selected contains 363 .....
- The set from which the twentieth element can be selected contains 346 elements
- Then:

$$n_a = (365) \times (364) \times \ldots \times (346).$$

- and

$$P(A) = \frac{n_a}{N} = \frac{365 \times 364 \times \ldots \times 346}{(365)^{20}} = .5886$$

Permutations

Definition

A ordered arrangement of r distinct objects is called a permutation. The number of ways of ordering n distinct objects taking r at a time is distinguished by the symbol P_r^n

- Useful notation:

$$\begin{aligned} n! &= n(n-1)(n-2)(n-3)\cdots(2)(1) \\ 0! &= 1 \end{aligned}$$

Permutations

Theorem

$$P_r^n = n(n-1)(n-2)(n-3)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof of Theorem

- Note that the first object can be chosen in one of n ways.
- The second can be chosen in $(n-1)$ ways, the third in $(n-2)$ ways, and so forth. Thus

$$\begin{aligned} P_r^n &= n(n-1)(n-2)(n-3)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)(n-r)\cdots 1}{(n-r)(n-r-1)\cdots 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Permutations

Example

Consider a bowl containing six balls with the letters A, B, C, D, E, F. Now draw one ball from the bowl and write down its letter and then draw a second ball and write down its letter. The outcome is then an ordered pair. According to theorem 17 the number of distinct ways of doing this is given by

$$P_2^6 = \frac{6!}{4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 6 \times 5 = 30$$

We can also see this by enumeration:

AB	AC	AD	AE	AF
BA	BC	BD	BE	BF
CA	CB	CD	CE	CF
DA	DB	DC	DE	DF
EA	EB	EC	ED	EF
FA	FB	FC	FD	FE

Permutations

Example

Suppose that a club consists of 25 members. A president and a secretary are to be chosen from the membership. Since the positions can be filled by first choosing one of the 25 members to be president and then choosing one of the remaining 24 members to be secretary, the possible number of choices is

$$P_2^{25} = \frac{n!}{(n-r)!} = \frac{25 \times 24 \times 23 \times \cdots \times 1}{23 \times 22 \times 21 \times \cdots \times 1} = 25 \times 24 = 600$$

Example

Suppose that six different books are to be arranged on a shelf. The number of possible permutations of the books is

$$P_6^6 = \frac{n!}{(n-r)!} = \frac{6 \times 5 \times 4 \times \cdots \times 1}{0!} = 6! = 720$$

Sampling with replacement

- Sampling with replacement: The balls are put back in the bowl.
- Suppose there are n balls and k selections are to be made.
  - The sample space S will contain all vectors of the form $(x_1, \ldots, x_k)$, where x_i is the outcome of the ith selection $(i = I, \ldots, k)$.
  - There are n possible outcomes for each of the k selections, so the total number of vectors in S is n^k.
  - If each ball has equal probability of being picked, the probability assigned to each vector in S is $1/n^k$.

Sampling with replacement

Example

- Consider a bowl containing six balls with the letters A, B, C, D, E, F on the respective balls.
- Draw one ball from the bowl, write down its letter and **return it to the bowl**.
- Draw a second ball and write down its letter.
  - Note that this is the same experiment as before, except for the fact that we return the first ball to the bowl.
  - By the *mn* rule there are 36 different outcomes.
  - Since they all have identical probabilities, the probability assigned to each outcome is $1/36$.
- Because we allow replacement there are now 36 possible outcomes instead of 30.

Partitioning n Distinct Objects into k Distinct Groups

Theorem

The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, n_3, \ldots, n_k$ *objects, respectively, where each object appears in exactly one group and* $\Sigma_{i=1}^{k} n_i = n$ *, is*

$$N = \binom{n}{n_1 \quad n_2 \quad n_3 \quad \ldots \quad n_k} = \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

Proof of Theorem

The number of distinct arrangements of n objects, assuming all objects are distinct, is $P_n^n = n!$. Then P_n^n equals the number of ways of partitioning the n objects into k groups (N) multiplied by the number of ways of ordering the n_1, n_2, ... and n_k elements within each group.

$$P_n^n = n! = (N)\,(n_1!\,n_2!\,n_3!\,\ldots\,n_k!)$$
$$\Rightarrow N = \frac{n!}{n_1!\,n_2!\,n_3!\,\ldots\,n_k!} \equiv \binom{n}{n_1 \quad n_2 \quad n_3 \quad \ldots \quad n_k}$$

where $n_i!$ is the number of distinct arrangements of n_i objects in group i.

Partitioning
Example 1

Consider the letters A, B, C, D. Now consider all the ways of grouping them into two groups of two. Using the formula we obtain

$$
\begin{aligned}
N &= \frac{n!}{n_1!\, n_2!\, n_3! \cdots n_{k!}} \\
&= \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} \\
&= 6
\end{aligned}
$$

Enumerating we obtain

$$
\begin{array}{cc}
AB & CD \\
AC & BD \\
AD & BC
\end{array}
$$

Partitioning

Example 2

- In how many ways can two paintings by Monet, three paintings by Renoir and two paintings by Degas be hung side by side on a museum wall if we do not distinguish between paintings by the same artists.
- Substituting $n = 7$, $n_1 = 2$, $n_2 = 3$ and $n_3 = 2$ into the formula, we obtain

$$N = \frac{n!}{n_1!\, n_2!\, n_3!} = \frac{7!}{2!\,3!\,2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1} = 210$$

Combinations

Arrangement of Symbols Representing Sample Points does not Matter

- C_r^n : The number of combinations of n objects taken r at a time.
- The number of subsets, each of size r, that can be formed from the n objects.
- This number is denoted by

$$C_r^n \quad \text{or} \quad \binom{n}{r}$$

Combinations

Theorem

The number of unordered subsets of size r chosen (without replacement) from n available objects is

$$\binom{n}{r} = C_r^n = \frac{P_r^n}{r!} = \frac{n!}{r!\,(n-r)!}$$

Proof of Theorem

- The selection of r objects from a total of n is equivalent to partitioning the n objects into $k = 2$ groups: the r selected and the $(n - r)$ remaining.
- This is a special case of the general partitioning problem of theorem 21. In the present case, $k = 2$, $n_1 = r$ and $n_2 = (n - r)$. Therefore we have,

$$\binom{n}{r} = \binom{n}{r \quad n-r} = \frac{n!}{r!(n-r)!}$$

Combinations

Example

- Consider a bowl containing six balls with the letters A, B, C, D, E, F .
- Draw two balls from the bowl and write down the letter on each of them, not paying any attention to the order.
- The number of distinct ways of doing this is given by

$$C_2^6 = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 4 \times 3 \times 2 \times 1} = \frac{6 \times 5}{2} = 15$$

- We can also see this by enumeration.

AB	AC	AD	AE	AF
	BC	BD	BE	BF
		CD	CE	CF
			DE	DF
				FE

Combinations
Example

- A club consists of 25 members
- Two of them are to be chosen to go to a special meeting with the regional president.
- Using the formula:

$$C_2^{25} = \frac{25!}{2!\,23!} = \frac{25 \times 24 \times 23 \times 22 \times \cdots \times 1}{2 \times 1 \times 23 \times 22 \times \cdots \times 1} = \frac{25 \times 24}{2} = 300$$

- Another example:
- Eight politicians meet at a fund raising dinner.
- How many greetings can be exchanged if each politician shakes hands with every other politician exactly once?
- Using the formula for a combination.

$$C_2^{8} = \frac{8!}{2!\,6!} = \frac{8 \times 7 \times 6 \times 5 \times \cdots \times 1}{2 \times 1 \times 6 \times 5 \times \cdots \times 1} = \frac{8 \times 7}{2} = 28$$