Econ 500 Fall 2006
Professor Bunzel
Due date: Wednesday, October 4, beginning of class.

Problemset 4

1. Let X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 3x, & 0 \leq y \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

   (a) Find the marginal density functions of X and Y.
   (b) Find $P\left(X \leq \frac{3}{4} | Y \leq \frac{1}{2}\right)$.
   (c) Find the conditional density function of X given that $Y = y$.
   (d) Find $P\left(X \leq \frac{3}{4} | Y = \frac{1}{2}\right)$.
   (e) Find $E(Y|X = x)$.
   (f) Use the law of iterated expectations to find $E(Y)$.
   (g) Find $E(Y)$ directly from the marginal density of Y.

2. Let Y be a random variable with a density given by:

$$f(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

   (a) Find the density function of $U_1 = 2Y - 1$.
   (b) Find the density function of $U_2 = 1 - 2Y$.
   (c) Find the density function of $U_3 = Y^2$.
   (d) Using the density functions found above, find $E(U_1)$, $E(U_2)$, and $E(U_3)$.
   (e) Using the standard metod to find $E(g(Y))$, find $E(U_1)$, $E(U_2)$, and $E(U_3)$.

3. The Weibull density function is given by

$$f(y) = \begin{cases} \frac{1}{\alpha} m y^{m-1} e^{-\frac{y^m}{\alpha}}, & 0 < y \\ 0 & otherwise \end{cases}$$

   (a) Find the density of $U = Y^m$.
   (b) Find $E\left(Y^k\right)$ for any positive integer k.
   (c) Now suppose X has an exponential distribution with mean β. Prove that $Z = \sqrt{X}$ has a Weibull density with $\alpha = \beta$ and $m = 2$.
   (d) Find $E\left(X^{\frac{k}{2}}\right)$ for any positive integer k.

4. Suppose that Y_1 and Y_2 are idenpendent and that both are uniformly distributed on the interval $(0, 1)$, and let $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$.

(a) Show that the joint density of U_1 and U_2 are given by

$$f_{U_1,U_2}(u_1, u_2) = \begin{cases} \frac{1}{2} & \begin{array}{c} -u_1 < u_2 < u_1,\ 0 < u_1, \\ u_1 - 2 < u_2 < 2 - u_1,\ 1 \le u_1 < 2 \end{array} \\ 0 & otherwise \end{cases}$$

(b) Sketch the region where $f_{U_1,U_2}(u_1, u_2) > 0$.

(c) Show that the marginal density of U_1 is

$$f_{U_1}(u) = \begin{cases} u_1 & 0 < u_1 < 1 \\ 2 - u_1 & 1 \le u_1 < 2 \\ 0 & otherwise \end{cases}$$

(d) Show that the marginal density of U_2 is

$$f_{U_1}(u) = \begin{cases} 1 + u_2 & -1 < u_2 < 0 \\ 1 - u_2 & 0 \le u_2 < 1 \\ 0 & otherwise \end{cases}$$

(e) Are U_1 and U_2 independent? Why or why not?