8.2 Suppose that $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$, $V(\hat{\theta}_1) = \sigma_1^2$, and $V(\hat{\theta}_2) = \sigma_2^2$. Consider the estimator $\hat{\theta}_3 = a\hat{\theta}_1 + (1 - a)\hat{\theta}_2$.

a Show that $\hat{\theta}_3$ is an unbiased estimator for $\theta$.

b If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, how should the constant $a$ be chosen in order to minimize the variance of $\hat{\theta}_3$?

8.3 Consider the situation described in Exercise 8.2. How should the constant $a$ be chosen to minimize the variance of $\hat{\theta}_3$ if $\hat{\theta}_1$ and $\hat{\theta}_2$ are not independent but are such that $\text{Cov}(\hat{\theta}_1, \hat{\theta}_2) = c \neq 0$?

8.4 Suppose that $Y_1, Y_2, Y_3$ denote a random sample from an exponential distribution with density function

$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-y/\theta}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the following five estimators of $\theta$:

$$\hat{\theta}_1 = Y_1, \qquad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \qquad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \qquad \hat{\theta}_4 = \min(Y_1, Y_2, Y_3), \qquad \hat{\theta}_5 = \overline{Y}.$$

a Which of these estimators are unbiased?

b Among the unbiased estimators, which has the smallest variance?

8.7 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size $n$ from a population with mean 3. Assume that $\hat{\theta}_2$ is an unbiased estimator of $E(Y^2)$ and that $\hat{\theta}_3$ is an unbiased estimator of $E(Y^3)$. Give an unbiased estimator for the third central moment of the underlying distribution.

8.37 Suppose that $Y$ is normally distributed with mean 0 and unknown variance $\sigma^2$. Then $Y^2/\sigma^2$ has a $\chi^2$ distribution with 1 degree of freedom. Use the pivotal quantity $Y^2/\sigma^2$ to find:

a a 95% confidence interval for $\sigma^2$.

b a 95% upper confidence limit for $\sigma^2$.

c a 95% lower confidence limit for $\sigma^2$.

8.81 Recently, the EPA set a maximum noise level for heavy trucks at 83 decibels (dB). The manner in which this limit is applied will greatly affect the trucking industry and the public. One way to apply the limit is to require all trucks to conform to the noise limit. A second, but less satisfactory, method is to require the truck fleet's mean noise level to be less than the limit. If the latter rule is adopted, variation in the noise level from truck to truck becomes important, because a large value of $\sigma^2$ would imply that many trucks exceed the limit, even if the mean fleet level were 83 dB. A random sample of six heavy trucks produced the following noise levels (in decibels):

| | | | | | |
|---|---|---|---|---|---|
| 85.4 | 86.8 | 86.1 | 85.3 | 84.8 | 86.0. |

Use these data to construct a 90% confidence interval for $\sigma^2$, the variance of the truck noise emission readings. Interpret your results.

8.82 In Exercise 8.69 we gave the carapace lengths of ten mature *Thenus orientalis* lobsters caught in the seas in the vicinity of Singapore. For your convenience, the data are reproduced here. Suppose that you wished to describe the variability of the carapace lengths of this population of lobsters. Find a 90% confidence interval for the population variance $\sigma^2$.

| Lobster field number | *A061* | *A062* | *A066* | *A070* | *A067* | *A069* | *A064* | *A068* | *A065* | *A063* |
|---|---|---|---|---|---|---|---|---|---|---|
| Carapace length (mm) | 78 | 66 | 65 | 63 | 60 | 60 | 58 | 56 | 52 | 50 |

8.83 Suppose that $S^2$ is the sample variance based on a sample of size $n$ from a normal population with unknown mean and variance.

a Derive a $100(1-\alpha)\%$ upper confidence bound for $\sigma^2$.

b Derive a $100(1-\alpha)\%$ lower confidence bound for $\sigma^2$.