Chi-square random variable

Probability Density Function

- X is said to be a chi-square random variable with ν degrees of freedom $(\chi^2(\nu))$ if

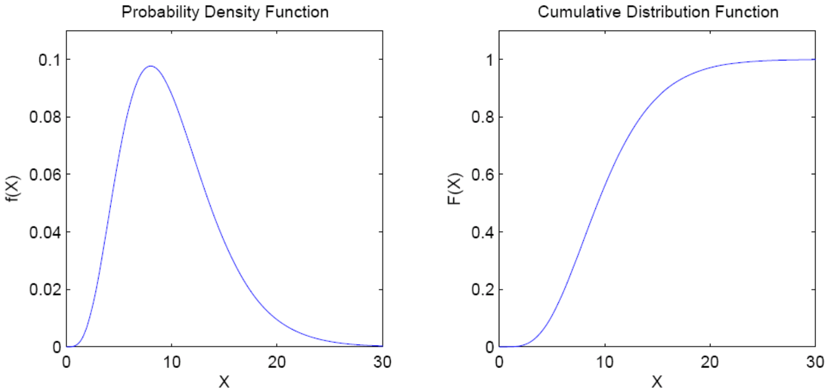
$$f(x;\nu) = \begin{cases} \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{\frac{-x}{2}} & 0 < x \\ 0 & \mathit{otherwise} \end{cases}$$

- where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(r) = \int_0^\infty u^{r-1} e^{-u}\, du, \quad r > 0$$

- Note that for positive integer values of r, $\Gamma(r) = (r-1)!$

- The following diagram shows the pdf and cdf for the chi-square distribution with parameters $\nu = 10$.



Properties of the chi-square random variable

- χ^2 and $N(0,1)$
  - Consider n independent random variables.

$$X_i \sim N(0,1), \;\; i = 1, 2, ..., n$$
$$\textit{then} \; \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$$

  - It can also be shown that

$$\textit{If} \;\; X_i \sim N(0,1), \;\; i = 1, 2, ..., n$$
$$\textit{and} \; \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$
$$\textit{then} \; \sum_{i=1}^{n-1} (X_i - \bar{X})^2 \sim \chi^2(n)$$

  - because this is the sum of $(n-1)$ independent random variables given that $\bar{X}$ and $(n-1)$ of the x's are independent.

Properties of the chi-square random variable

- χ^{2} and $N(\mu, \sigma^{2})$

$$
\begin{aligned}
If \;\; X_i &\sim N(\mu,\sigma^2), \;\; i = 1,2,...,n \\
and \;\; \bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
then \;\; \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 &\sim \chi^{2}(n) \\
and \;\; \sum_{i=1}^{n-1}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 &\sim \chi^{2}(n)
\end{aligned}
$$

- Sums of chi-square random variables: If

$$
\begin{aligned}
y_1 &\sim \chi^2(\nu_1), \text{ and} \\
y_2 &\sim \chi^2(\nu_2)
\end{aligned}
$$

and y_1 and y_2 are independent, then

$$
y_1 + y_2 \sim \chi^2(\nu_1+\nu_2).
$$

Moments of chi-square random variables

- The most commonly used moments are:

$$\begin{aligned} Mean\ \left(\chi^2(\nu)\right) &= \nu = degrees\ of\ freedom \\ Var\ \left(\chi^2(\nu)\right) &= 2\nu \\ Mode\ \left(\chi^2(\nu)\right) &= \nu - 2 \end{aligned}$$

- The distribution function of $\chi^2(\nu)$ is defined as

$$F(x;\nu) = \int_0^x f(s;\nu)ds$$

- It is tabulated in most statistics and econometrics texts.

The Student's t random variable

- The ratio

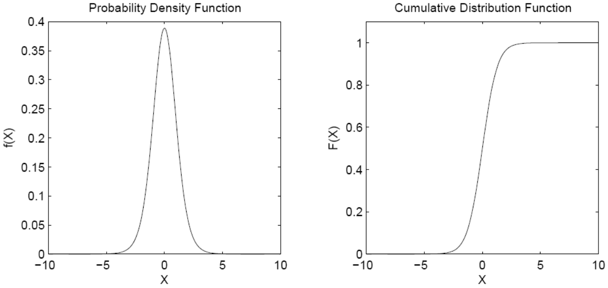
$$t = \frac{N(0,1)}{\sqrt{\frac{\chi^2(\nu)}{\nu}}}$$

  has the Student's t density function with ν degrees of freedom when the $N(0,1)$ is independent of the $\chi^2(\nu)$ variable.
- Tables of the distribution are in most statistics and econometrics books.
- The density of Student's t distribution is given by:

$$f(t;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{\frac{-(\nu+1)}{2}}, \quad -\infty < t < \infty$$

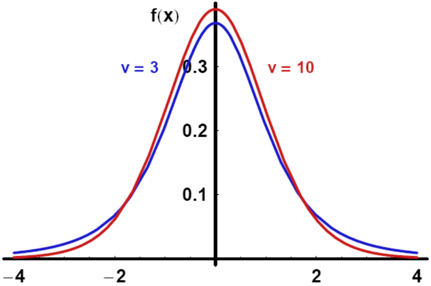
The Student's t random variable

- This is the pdf and cdf for the Student's t-distribution with parameter $\nu = 10$.



- Note that it is symmetric about origin.

- This is the pdf for the Student's t-distribution with parameters $\nu = 10$ and $\nu = 3$.



The Student's t random variable

- The mean and variance of the $t(\nu)$ distribution are:

$$\begin{aligned} Mean\ (t(\nu)) &= 0 \\ Var\ (t(\nu)) &= \frac{\nu}{\nu - 2} \end{aligned}$$

The F distribution

- The distributon function of the F distribution is constructed in the following manner:
  - If $\chi_1^2(\nu_1)$ and $\chi_2^2(\nu_2)$ are independently distributed chi-square variates, then

$$F(\nu_1,\nu_2) = \frac{\frac{\chi_1^2(\nu_1)}{\nu_1}}{\frac{\chi_2^2(\nu_2)}{\nu_2}} = \frac{\nu_2}{\nu_1}\cdot\frac{\chi_1^2(\nu_1)}{\chi_2^2(\nu_2)}$$

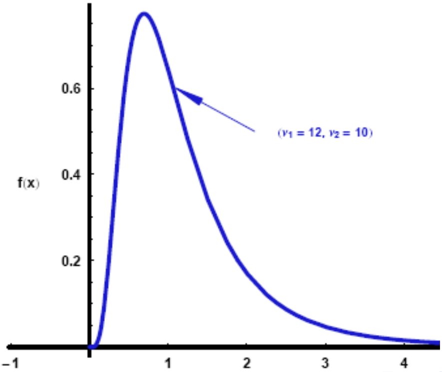
    has the F density with ν_1 and ν_2 degrees of freedom.

- The density of the F distribution is

$$f(F;\nu_1,\,\nu_2) = \begin{cases} \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\,\Gamma\left(\frac{\nu_2}{2}\right)}\cdot\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}\cdot F^{\frac{\nu_1}{2}-1}\cdot\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{-(\nu_1+\nu_2)}{2}} & F > 0 \\ 0 & \textit{otherwise} \end{cases}$$

The F distribution

- Tabulations of the distribution of $F(\nu_1, \nu_2)$ are widely available.
  - Note that $F_{\nu_1, \nu_2} \sim \left(\frac{1}{F_{\nu_2, \nu_1}}\right)$
- The figure below shows the pdf for the F distribution with parameters $\nu_1 = 12$ and $\nu_2 = 20$.



The F distribution

- The relevant moments of the F distribution are:

$$E(F) = \frac{\nu_2}{\nu_2 - 2}$$

$$Var(F) = \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}$$

Exponential Distribution

- A continuous random variable X has the exponential distribution with parameter $\theta > 0$ if

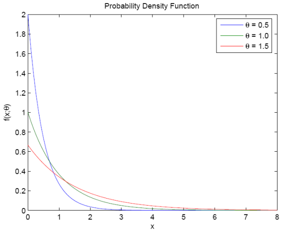
$$\begin{aligned} f(x;\theta) &= \frac{1}{\theta} \cdot e^{-x/\theta} \quad x > 0 \\ &= 0 \quad otherwise \end{aligned}$$

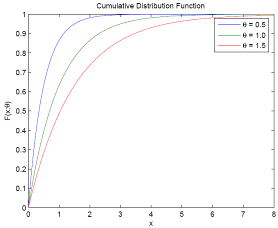
- The CDF of X is

$$F(x;\theta) = 1 - e^{-x/\theta} \qquad x > 0$$

Exponential Distribution

- These are the pdf and cdf for various values of θ:





Properties of the exponential distribution

- For a continuous random variable X, $X \sim Exp(\theta)$ if and only if
$$P[X > a + t \mid X > a] = P[X > t] \tag{1}$$
for all $a > 0$ and $t > 0$.
- Proof that if $X \sim Exp(\theta)$ then (1) holds:
$$\begin{aligned} P[X &> a + t \mid X > a] = \frac{P[X > a + t \text{ and } X > a]}{P[X > a]} \\ &= \frac{P[X > a + t]}{P[X > a]} = \frac{e^{-(a+t)/\theta}}{e^{-a/\theta}} \\ &= P[X > t] \end{aligned}$$
- Because of this property the exponential distribution is often used to model lifetimes.
- Note that in this model, an old component which is still working is just as reliable as a new component. Failure of such a component is not due to fatigue or wear.

Properties of the exponential distribution

- The moments of the exponential distribution are

$$
\begin{aligned}
E\left(X\right) &= \theta \\
E\left(X^2\right) &= 2\theta^2 \\
Var[X] &= \theta^2
\end{aligned}
$$

The Gamma Distribution

- The general formula for the probability density function of the gamma distribution is

$$f(x;\gamma,\beta) = \frac{(\frac{x-\mu}{\beta})^{\gamma-1} e^{(-\frac{x-\mu}{\beta})}}{\beta\Gamma(\gamma)} \qquad x \geqslant \mu, \gamma > 0, \beta > 0$$

  where γ is the shape parameter, μ is the location parameter and β is the scale parameter.
- Γ is the gamma function which is defined as

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

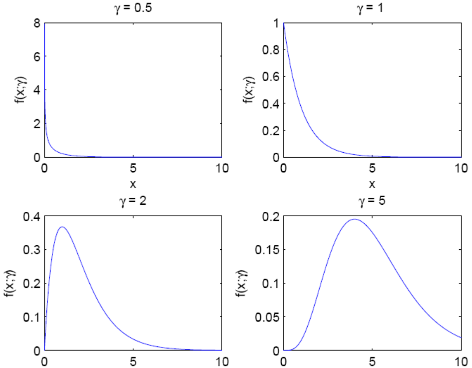
- The "standard" case is $\mu = 0$. Therefore, usually we denote the random variable X which has the pdf form above as $X \sim Gamma(\gamma, \beta)$.

The Gamma Distribution

- When $\mu = 0$ and $\beta = 1$ it is called the standard gamma distribution and has pdf:

$$f(x;\gamma) = \frac{(x)^{\gamma-1} e^{(-x)}}{\Gamma(\gamma)}$$

- The pdf looks like:



The Gamma Distribution

- The cumulative distribution function (Cdf) of standard gamma distribution is

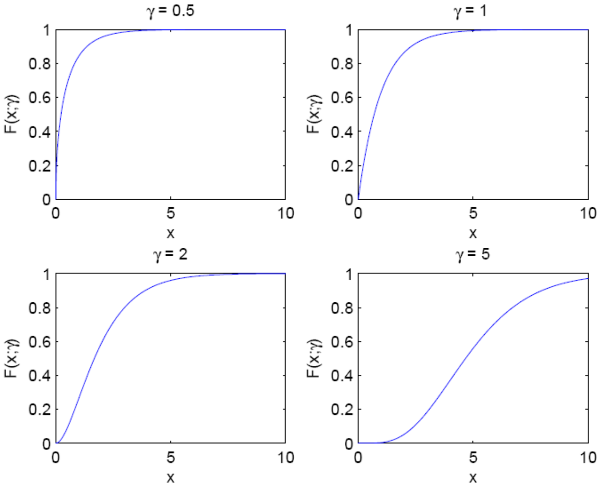
$$F(x;\gamma) = \frac{\Gamma_x(\gamma)}{\Gamma(\gamma)} \qquad x \geqslant 0, \gamma > 0.$$

where $\Gamma_x(\gamma)$ is the **incomplete** gamma function, defined as

$$\Gamma_x(\gamma) = \int_0^x t^{\gamma-1} e^{-t} dt$$

The Gamma Distribution

- The pdf looks like:



The Gamma Distribution

Properties of gamma distribution

When $\mu = 0$, the gamma distribution has the following properties.

- If $X_i \sim Gamma(\gamma, \beta)$ for $i = 1, 2, ..., N$, and $\bar{\gamma} = \sum_{i=1}^{N} \gamma_i$, then

$$Y = \sum_{i=1}^{N} X_i \quad \sim Gamma(\bar{\gamma}, \beta)$$

  provided that X_i is independently distributed.
- If $X \sim Gamma(\gamma, \beta)$, then $\frac{X}{\beta} \sim Gamma(\gamma, 1)$.
- If $X \sim Gamma(1, \beta)$, then $X \sim Exp(\beta)$, i.e. X is exponentially distributed.
- If $X \sim Gamma(\gamma = \delta/2, \beta = 2)$, then $X \sim \chi^2(\delta)$, X is chi-square distributed.

- The first raw moment is

$$
\begin{aligned}
E(X) &= \int_0^\infty x \frac{(x)^{\gamma-1} e^{\left(-\frac{x}{\beta}\right)}}{\beta^\gamma \Gamma(\gamma)} dx \\
&= \frac{1}{\beta^\gamma \Gamma(\gamma)} \int_0^\infty (x)^\gamma e^{\left(-\frac{x}{\beta}\right)} dx \quad \left(let \frac{x}{\beta} = t, x = \beta t,\ dx = \beta dt\right) \\
&= \frac{1}{\beta^\gamma \Gamma(\gamma)} \int_0^\infty (\beta t)^\gamma e^{(-t)} \beta dt \\
&= \frac{\beta^{\gamma+1}}{\beta^\gamma \Gamma(\gamma)} \int_0^\infty (t)^{(\gamma+1)-1} e^{(-t)} dt
\end{aligned}
$$

- Recall that $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$, so

$$\begin{aligned} E\left(X\right) &= \frac{\beta^{\gamma+1}}{\beta^{\gamma}\Gamma(\gamma)} \int_0^\infty (t)^{(\gamma+1)-1} e^{(-t)} dt \\ &= \frac{\beta^{\gamma+1}\Gamma(\gamma+1)}{\beta^{\gamma}\Gamma(\gamma)} \\ &= \frac{\gamma\beta^{\gamma+1}\Gamma(\gamma)}{\beta^{\gamma}\Gamma(\gamma)} \\ &= \gamma\beta \end{aligned}$$

- The second raw moment is

$$E\left(X^2\right) = (\gamma+1)\beta^2\gamma$$

- It follows that

$$Var[X] = E\left(X^2\right) - E^2\left(X\right) = (\gamma+1)\beta^2\gamma - (\beta\gamma)^2 = \gamma\beta^2$$