SOME THEOREMS ON QUADRATIC FORMS AND NORMAL VARIABLES

1. The Multivariate Normal Distribution

The $n \times 1$ vector of random variables, y, is said to be distributed as a multivariate normal with mean vector μ and variance covariance matrix Σ (denoted $y \sim N(\mu,\, \Sigma)$) if the density of y is given by

$$f(y;\, \mu,\, \Sigma) = \frac{e^{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)}}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}\,. \tag{1}$$

Consider the special case where $n = 1$: $y = y_1$, $\mu = \mu_1$, $\Sigma = \sigma^2$.

$$\begin{aligned} f(y_1;\, \mu_1, \sigma) &= \frac{e^{-\frac{1}{2}(y_1-\mu_1)\left(\frac{1}{\sigma^2}\right)(y_1-\mu_1)}}{(2\pi)^{\frac{1}{2}}(\sigma^2)^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y_1-\mu_1)^2}{2\sigma^2}} \end{aligned} \tag{2}$$

is just the normal density for a single random variable.

2. Theorems on Quadratic Forms in Normal Variables

2.1. Quadratic Form Theorem 1.

Theorem 1. *If $y \sim N(\mu_y,\, \Sigma_y)$, then*

$$z = Ay \sim N(\mu_z = A\mu_y;\; \Sigma_z = A\Sigma_y A')$$

where A is a matrix of constants.

2.1.1. *Proof.* We prove the form of the mean and variance, but not that z is normally distributed

$$\begin{aligned} E(z) &= E(Ay) = AE(y) = A\mu_y \\ \text{var}(z) &= E[(z - E(z))\,(z - E(z))'] \\ &= E[(Ay - A\mu_y)(Ay - A\mu_y)'] \\ &= E[A(y - \mu_y)(y - \mu_y)'A'] \\ &= AE(y - \mu_y)(y - \mu_y)'A' \\ &= A\Sigma_y A' \end{aligned} \tag{3}$$

Date: October 6, 2006.

2.1.2. *Example.* Let $Y_1, \dots, Y_n$ denote a random sample drawn from $N(\mu,\, \sigma^2)$. Then

$$Y = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ Y_n \end{pmatrix} \sim N \left[\begin{pmatrix} \mu \\ \cdot \\ \cdot \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & \dots & 0 \\ \cdot & \sigma^2 & \cdot \\ \cdot & & \\ 0 & & \sigma^2 \end{pmatrix} \right] . \tag{4}$$

Now theorem 1 implies that:

$$\begin{aligned} \bar{Y} &= \frac{1}{n} Y_1 + \cdots + \frac{1}{n} Y_n \\ &= \left(\frac{1}{n}, \dots, \frac{1}{n} \right) Y = AY \\ &\sim N(\mu,\, \sigma^2/n) \quad \text{since} \\ &\left(\frac{1}{n}, \dots, \frac{1}{n} \right) \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} = \mu \quad \text{and} \\ &\left(\frac{1}{n}, \dots, \frac{1}{n} \right) \sigma^2 I \begin{pmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} . \end{aligned} \tag{5}$$

2.2. **Quadratic Form Theorem 2.**

Theorem 2. *Let the $n \times 1$ vector $y \sim N(0,\, I)$. Then $y'y \sim \chi^2(n)$.*

Proof: Consider that each y_i is an independent standard normal variable. Write out $y'y$ in summation notation as

$$y'y = \Sigma_{i=1}^n y_i^2 \tag{6}$$

which is the sum of squares of n standard normal variables.

2.3. **Quadratic Form Theorem 3.**

Theorem 3. *If $y \sim N(0, \sigma^2 I)$ and M is a symmetric idempotent matrix of rank m then*

$$\frac{y'My}{\sigma^2} \sim \chi^2(\text{tr } M) \tag{7}$$

Corollary: If the $n \times 1$ vector $y \sim N(0, I)$ and the $n \times n$ matrix A is idempotent and of rank m. Then

$$y'Ay \sim \chi^2(m)$$

2.4. Quadratic Form Theorem 4.

Theorem 4. *If* $y \sim N(0,\, \sigma^2 I)$, M *is a symmetric idempotent matrix of order* n, *and* L *is a* $k \times n$ *matrix, then* Ly *and* $y'My$ *are independently distributed if* $LM = 0$.

2.5. Quadratic Form Theorem 5.

Theorem 5. *Let the* $n \times 1$ *vector* $y \sim N(0, I)$, *let* A *be an* $n \times n$ *idempotent matrix of rank* m, *let* B *be an* $n \times n$ *idempotent matrix of rank* s, *and suppose* $BA = 0$. *Then* $y'Ay$ *and* $y'By$ *are independently distributed* χ^2 *variables.*

2.6. Quadratic Form Theorem 6 (Craig's Theorem).

Theorem 6. *If* $y \sim N(\mu,\, \Omega)$ *where* Ω *is positive definite, then* $q_1 = y'Ay$ *and* $q_2 = y'By$ *are independently distributed if* $A\Omega B = 0$.

2.7. Quadratic Form Theorem 7.

Theorem 7. *If* y *is a* $n \times 1$ *random variable and* $y \sim N(\mu,\, \Sigma)$ *then*

$$(y - \mu)'\Sigma^{-1}(y - \mu) \sim \chi^2(n)\,.$$

2.8. **Quadratic Form Theorem 8.** Let $y \sim N(0,\, I)$. Let M be a non-random idempotent matrix of dimension $n \times n$ (rank $(M) = r \leq n$). Let A be a non-random matrix such that $AM = 0$. Let $t_1 = My$ and let $t_2 = Ay$. Then t_1 and t_2 are independent random vectors.

REFERENCES

[1] Cramer, H. *Mathematical Methods of Statistics.* Princeton: Princeton University Press, 1946.
[2] Driscoll, M. F. and W. R. Grundberg. "A History of the Development of Craig's Theorem." *The American Statistician* 40(1986):65-69.
[3] Driscoll, M. F. and B. Krasnicka. "An accessible proof of Craig's theorem in the general case." *The American Statistician* 49(1995):59-61.
[4] Goldberger, A. S. *Econometric Theory.* New York: Wiley, 1964.
[5] Goldberger, A. S. *A Course in Econometrics.* Cambridge: Harvard University Press, 1991.
[6] Hocking, R. R. *The Analysis of Linear Models*. Monterey: Brooks/Cole, 1985.
[7] Hocking, R. R. *Methods and Applications of Linear Models*. New York: Wiley, 1996.
[8] Rao, C. R. *Linear Statistical Inference and its Applications.* 2nd edition. New York: Wiley, 1973.
[9] Schott, J. R. *Matrix Analysis for Statistics*. New York: Wiley, 1997.