Quadratic forms in normal variables, Econ 500

Helle Bunzel

Fall 2006

The Multivariate Normal Distribution

Reminder

- The $n \times 1$ vector of random variables, y, is distributed as a multivariate normal with mean μ and variance covariance matrix Σ if

$$f(y;\, \mu,\, \Sigma) = \frac{e^{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)}}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}.$$

- We say

$$y \sim N(\mu,\, \Sigma)$$

- Consider the special case where $n = 1$: $y = y_1$, $\mu = \mu_1$, $\Sigma = \sigma^2$.

$$f(y_1;\, \mu_1, \sigma) = \frac{e^{-\frac{1}{2}(y_1-\mu_1)\left(\frac{1}{\sigma^2}\right)(y_1-\mu_1)}}{(2\pi)^{\frac{1}{2}}(\sigma^2)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y_1-\mu_1)^2}{2\sigma^2}}$$

- This is the normal density for a single random variable.

Theorems on Quadratic Forms in Normal Variables

Theorem

If $y \sim N(\mu_y, \Sigma_y)$, *then*

$$z = Ay \sim N(\mu_z = A\mu_y;\ \Sigma_z = A\Sigma_y A')$$

where A *is a matrix of constants.*

Proof.

We prove the form of the mean and variance, but not that z is normally distributed

$$\begin{aligned} E(z) &= E(Ay) = AE(y) = A\mu_y \\ Var(z) &= E[(z - E(z))(z - E(z))'] \\ &= E[(Ay - A\mu_y)(Ay - A\mu_y)'] \\ &= E[A(y - \mu_y)(y - \mu_y)'A'] \\ &= AE(y - \mu_y)(y - \mu_y)'A' \\ &= A\Sigma_y A' \end{aligned}$$

□

- Let $Y_1, \ldots, Y_n$ denote a random sample drawn from $N(\mu, \sigma^2)$. Then

$$Y = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ Y_n \end{pmatrix} \sim N\left[\begin{pmatrix} \mu \\ \cdot \\ \cdot \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & \ldots & 0 \\ \cdot & \sigma^2 & \cdot \\ \cdot & & \\ 0 & & \sigma^2 \end{pmatrix}\right]. \tag{1}$$

- Now theorem 1 implies that:

$$\begin{aligned} \bar{Y} &= \frac{1}{n}Y_1 + \cdots + \frac{1}{n}Y_n \\ &= \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) Y = AY \end{aligned}$$

- Now,

$$A\mu = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} = \mu$$

- and

$$AV[Y]A' = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \sigma^2 I \begin{pmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

- So

$$\bar{Y} = AY \sim N(\mu, \sigma^2/n)$$

Theorems on Quadratic Forms in Normal Variables

Theorem

Let the $n \times 1$ *vector* $y \sim N(0, I)$. *Then* $y'y \sim \chi^2(n)$.

Proof.

Consider that each y_i is an independent standard normal variable. Write out $y'y$ in summation notation as

$$y'y = \Sigma_{i=1}^{n} y_i^2$$

which is the sum of squares of n standard normal variables. □

Theorems on Quadratic Forms in Normal Variables

Theorem

If $y \sim N(0, \sigma^2 I)$ *and* M *is a symmetric idempotent matrix of rank* m *then*

$$\frac{y'My}{\sigma^2} \sim \chi^2(\mathrm{tr}\ M) \tag{2}$$

Corollary

If the $n \times 1$ *vector* $y \sim N(0, I)$ *and the* $n \times n$ *matrix* A *is idempotent and of rank* m. *Then*

$$y'Ay \sim \chi^2(m)$$

Theorems on Quadratic Forms in Normal Variables

Theorem

If $y \sim N(0, \sigma^2 I)$*, M is a symmetric idempotent matrix of order n, and L is a $k \times n$ matrix, then Ly and $y'My$ are independently distributed if* $LM = 0$.

Theorem

Let the $n \times 1$ vector $y \sim N(0, I)$, let A be an $n \times n$ idempotent matrix of rank m, let B be an $n \times n$ idempotent matrix of rank s, and suppose $BA = 0$. Then $y'Ay$ and $y'By$ are independently distributed χ^2 variables.

Theorem

If $y \sim N(\mu, \Omega)$ where Ω is positive definite, then $q_1 = y'Ay$ and $q_2 = y'By$ are independently distributed if $A\Omega B = 0$.

Theorems on Quadratic Forms in Normal Variables

Theorem

If y is a $n \times 1$ random variable and $y \sim N(\mu, \Sigma)$ then

$$(y - \mu)' \Sigma^{-1} (y - \mu) \sim \chi^2(n) \, .$$

Theorem

Let $y \sim N(0, I)$. Let M be a non-random idempotent matrix of dimension $n \times n$ (rank $(M) = r \leq n$). Let A be a non-random matrix such that $AM = 0$. Let $t_1 = My$ and let $t_2 = Ay$. Then t_1 and t_2 are independent random vectors.