Chebyshev's inequality

Corollary

Let X be a random variable with mean μ and variance σ^2. Then for any $k > 0$ or any $\varepsilon > 0$

$$P\left[|X - \mu| \geq k\sigma\right] \leq \frac{1}{k^2}$$
$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \tag{9}$$
$$P\left[|X - \mu| \geq \varepsilon\right] \leq \frac{\sigma^2}{\varepsilon^2}$$

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Proof

Let $g(x) = \frac{(x-\mu)^2}{\sigma^2}$, where $\mu = E(X)$ and $\sigma^2 = Var(X)$. Note that $g(x) \geq 0$. Then let $r = k^2$. Then, by Markov's inequality

$$P\left[\frac{(X-\mu)^2}{\sigma^2} \geq k^2\right] \leq \frac{1}{k^2} E\left(\frac{(X-\mu)^2}{\sigma^2}\right) \tag{10}$$

$$= \frac{1}{k^2} \frac{E(X-\mu)^2}{\sigma^2} = \frac{1}{k^2}$$

because $E(X-\mu)^2 = \sigma^2$. We can then rewrite equation 10 as follows

$$P\left[\frac{(X-\mu)^2}{\sigma^2} \geq k^2\right] \leq \frac{1}{k^2}$$

$$\Rightarrow P\left[(X-\mu)^2 \geq k^2\sigma^2\right] \leq \frac{1}{k^2}$$

$$\Rightarrow P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2}$$

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Example

- The number of accidents that occur during a given month at a particular intersection, X, is found to have a mean of 12 and a standard deviation of 2.
- The underlying distribution is not known.
- What is the probability that, next month, X will be greater than eight but less than sixteen.
- Note

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \Leftrightarrow$$

$$P[(\mu - k\sigma) < X < (\mu + k\mu)] \geq 1 - \frac{1}{k^2}$$

- For this problem $\mu = 12$ and $\sigma = 2$ so $\mu - k\sigma = 12 - 2k$.
- We can solve this equation for the k that gives us the desired bounds on the probability.

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Example

$$\begin{aligned} \mu - k\mu &= 12 - (k)(2) = 8 \\ &\Rightarrow 2k = 4 \Rightarrow k = 2 \end{aligned}$$

and

$$\begin{aligned} 12 + (k)(2) &= 16 \\ \Rightarrow 2k - 4 &\Rightarrow k = 2 \end{aligned}$$

- We then obtain

$$P[(8) < X < (16)] \geq 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

- Therefore the probability that X is between 8 and 16 is at least 3/4.