SAMPLE MOMENTS

1. Population Moments

1.1. Moments about the origin (raw moments). Recall that the rth moment about the origin of a random variable X, denoted by μ'_r, is the expected value of X^r; symbolically,

$$\mu'_r = E(X^r) = \sum_x x^r f(x)$$

for $r = 0, 1, 2, ...$ when X is discrete and

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

when X is continuous. The rth moment about the origin is only defined if $E[X^{\mathrm{r}}]$ exists. A moment about the origin is sometimes called a raw moment. Note that $\mu'_1 = E(X) = \mu_X$, the mean of the distribution of X, or simply the mean of X. The rth moment is sometimes written as function of θ where θ is a vector of parameters that characterize the distribution of X.

If there is a sequence of random variables, $X_1, X_2, ...X_{\mathrm{n}}$, we will call the rth population moment of the ith random variable $\mu'_{i,\,r}$ and define it as

$$\mu'_{i,r} \; = \; E\left(X_i^r\right)$$

1.2. Central moments. The rth moment about the mean of a random variable X, denoted by μ_r, is the expected value of $(\,X - \mu_X)^r$ symbolically,

$$\mu_r = E[\,(\,X \;-\; \mu_X\,)^r\,] = \sum_x \,(\,x - \mu_X)^r\, f(x)$$

for $r = 0, 1, 2, ...$ when X is discrete and

$$\mu_r = E[\,(X \;-\; \mu_X)^r\,] = \int_{-\infty}^{\infty} (x \;-\; \mu_X)^r f(x)\, dx$$

when X is continuous. The r^{th} moment about the mean is only defined if $E[(X - \mu_X)^r]$ exists. The rth moment about the mean of a random variable X is sometimes called the rth central moment of X. The rth central moment of X about a is defined as $E[(X - a)^{\mathrm{r}}]$. If $a = \mu_{\mathrm{X}}$, we have the rth central moment of X. Note that

$$\begin{aligned}
\mu_1 \;&=\; E[X \;-\; \mu_X] \;=\; \int_{-\infty}^{\infty} (x \;-\; \mu_X)\, f(x)\, dx \;=\; 0 \\
\mu_2 \;&=\; E[(X \;-\; \mu_X)^2] \;=\; \int_{-\infty}^{\infty} (x \;-\; \mu_X)^2\, f(x)\, dx \;=\; Var(X) \;=\; \sigma^2
\end{aligned}$$

Also note that all odd moments of X around its mean are zero for symmetrical distributions, provided such moments exist.

If there is a sequence of random variables, $X_1, X_2, \ldots X_n$, we will call the r^{th} central population moment of the ith random variable $\mu_{i,r}$ and define it as

$$\mu_{i,r} \;=\; E\left(X_i^r - \mu'_{i,1}\right)^r \tag{1}$$

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When the variables are identically distributed, we will drop the i subscript and write μ'_r and μ_r .

2. Sample Moments

2.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, ...X_n$. The first sample moment, usually called the average is defined by

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Corresponding to this statistic is its numerical value, $\bar{x}_n$, which is defined by

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

where x_i represents the observed value of X_i. The rth sample moment

$$\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} X_i^r \tag{2}$$

This too has a numerical counterpart given by

$$\bar{x}_n^r = \frac{1}{n} \sum_{i=1}^{n} x_i^r$$

2.2. **Properties of Sample Moments.**

2.2.1. *Expected value of* $\bar{X}_n{}^r$. Taking the expected value of equation 2 we obtain

$$E\left[\bar{X}_n^r\right] = E\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} E\ X_i^r = \frac{1}{n} \sum_{i=1}^{n} \mu'_{i,r} \tag{3}$$

If the X's are identically distributed, then

$$E\left[\bar{X}_n^r\right] = E\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} \mu'_r = \mu'_r \tag{4}$$

2.2.2. *Variance of* $\bar{X}_n^r$. First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

$$Var\left(\bar{X}_n^r\right) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i^r\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i^r\right)$$

If the X's are independent, then

$$Var\left(\bar{X}_n^r\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var\left(X_i^r\right)$$

If the X's are independent and identically distributed, then

$$Var\left(\bar{X}_n^r\right) = \frac{1}{n}\ Var\left(X^r\right)$$

where X denotes any one of the random variables (because they are all identical). In the case where $r = 1$, we obtain

$$Var\left(\bar{X}_n\right) = \frac{1}{n} Var\left(X\right) = \frac{\sigma^2}{n}$$

3. Sample Central Moments

3.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots, X_n$. We define the sample central moments as

$$
\begin{aligned}
C_n^r &= \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \mu'_{i,1} \right)^r, \; r = 1, 2, 3, \ldots, \\
\Rightarrow C_n^1 &= \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \mu'_{i,1} \right) \\
\text{and } C_n^2 &= \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \mu'_{i,1} \right)^2
\end{aligned}
$$

These are only defined if $\mu'_{i,1}$ is known.

3.2. **Properties of Sample Central Moments.**

3.2.1. *Expected value of C_n^r.* The expected value of C_n^r is given by

$$
E \left(C_n^r \right) = \frac{1}{n} \sum_{i=1}^{n} E \left(X_i - \mu'_{i,1} \right)^r = \frac{1}{n} \sum_{i=1}^{n} \mu_{i,r}
$$

The last equality follows from equation 1.

If the X_i are identically distributed, then

$$
\begin{aligned}
E \left(C_n^r \right) &= \mu_r \\
E \left(C_n^1 \right) &= 0
\end{aligned}
$$

3.2.2. *Variance of C_n^r.* First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

$$
Var \left(C_n^r \right) = Var \left(\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \mu'_{i,1} \right)^r \right) = \frac{1}{n^2} Var \left(\sum_{i=1}^{n} \left(X_i - \mu'_{i,1} \right)^r \right)
$$

If the X's are independently distributed, then

$$
Var \left(C_n^r \right) = \frac{1}{n^2} \sum_{i=1}^{n} Var \left[\left(X_i - \mu'_{i,1} \right)^r \right]
$$

If the X's are independent and identically distributed, then

$$
Var \left(C_n^r \right) = \frac{1}{n} Var \left[\left(X - \mu'_1 \right)^r \right]
$$

where X denotes any one of the random variables (because they are all identical). In the case where $r = 1$, we obtain

$$
\begin{aligned}
Var \left(C_n^1 \right) &= \frac{1}{n} Var \left[X - \mu'_1 \right] = \frac{1}{n} Var \left[X - \mu \right] \\
&= \frac{1}{n} \sigma^2 - 2 \, Cov \left[X, \mu \right] + Var \left[\mu \right] = \frac{1}{n} \sigma^2
\end{aligned}
$$

4. Sample About the Average

4.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, ...X_{\rm n}$. Define the rth sample moment about the average as

$$M_n^r \;=\; \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^r , \quad r \;=\; 1, 2, 3, \ldots ,$$

This is clearly a statistic of which we can compute a numerical value. We denote the numerical value by, m_n^r, and define it as

$$m_n^r = \frac{1}{n} \sum_{i=1}^{n} \left(\, x_i - \bar{x}_n\right)^r$$

In the special case where $r = 1$ we have

$$M_n^1 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right) = \frac{1}{n} \sum_{i=1}^{n} X_i - \bar{X}_n = \bar{X}_n - \bar{X}_n \;=\; 0$$

4.2. **Properties of Sample Moments about the Average when r = 2.**

4.2.1. *Alternative ways to write M_n^r.* We can write M_n^2 in an alternative useful way by expanding the squared term and then simplifying as follows

$$\begin{aligned}
M_n^r \;&=\; \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^r \\
\Rightarrow \;\; M_n^2 \;&=\; \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2 = \frac{1}{n} \left(\sum_{i=1}^{n} \left[X_i^2 - 2\, X_i \bar{X}_n + \bar{X}_n^2 \right] \right) \\
&=\; \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \frac{2\bar{X}_n}{n} \sum\nolimits_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} \bar{X}_n^2 \\
&=\; \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\bar{X}_n^2 + \bar{X}_n^2 = \frac{1}{n} \left(\sum_{i=1}^{n} X_i^2 \right) - \bar{X}_n^2
\end{aligned}$$

4.2.2. *Expected value of M_n^2.* The expected value of M_n^2 is then given by

$$\begin{aligned}
E\, \left(M_n^2\right) \;&=\; \frac{1}{n}\, E \left[\sum_{i=1}^{n} X_i^2 \right] - E\left[\bar{X}_n^2\right] \\
&=\; \frac{1}{n} \sum_{i=1}^{n} E\left[X_i^2\right] - \left(E\left[\bar{X}_n\right]\right)^2 - Var(\bar{X}_n) \\
&=\; \frac{1}{n} \sum_{i=1}^{n} \mu'_{i\,,\,2} - \left(\frac{1}{n} \sum_{i=1}^{n} \mu'_{i\,,\,1} \right)^2 - Var(\bar{X}_n\,),
\end{aligned}$$

where the third line follows from equation 3.

If the X_i are independent and identically distributed, then

$$\begin{aligned} E\left(M_n^2\right) &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i^2\right] - E\left[\bar{X}_n^2\right] \qquad (5) \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu'_{i,2} - \left(\frac{1}{n} \sum_{i=1}^{n} \mu'_{i,1}\right)^2 - Var(\bar{X}_n) \\ &= \mu'_2 - \left(\mu'_1\right)^2 - \frac{\sigma^2}{n} = \sigma^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2 \end{aligned}$$

where μ'_1 and μ'_2 are the first and second population moments, and σ^2 is the second central population moment for the identically distributed variables. Note that this obviously implies

$$E\left[\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2\right] = n\,E\left(M_n^2\right) = n\left(\frac{n-1}{n}\right)\sigma^2 = (n-1)\,\sigma^2$$

4.2.3. *Variance of* M_n^2. By definition,

$$Var\left(M_n^2\right) = E\left[\left(M_n^2\right)^2\right] - \left(E\,M_n^2\right)^2 \qquad (6)$$

The second term on the right on equation 6 is easily obtained by squaring the result in equation 5.

$$\begin{aligned} E\left(M_n^2\right) &= \frac{n-1}{n}\sigma^2 \qquad (7) \\ \Rightarrow \left(E\left(M_n^2\right)\right)^2 &= \left(E\,M_n^2\right)^2 = \frac{(n-1)^2}{n^2}\sigma^4 \end{aligned}$$

Now consider the first term on the right hand side of equation 6. Write it as

$$E\left[\left(M_n^2\right)^2\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)^2\right] \qquad (8)$$

Now consider writing $\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$ as follows

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 &= \frac{1}{n}\sum_{i=1}^{n}\left((X_i - \mu) - (\bar{X} - \mu)\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 \\ \text{where } Y_i &= X_i - \mu \text{ and } \bar{Y} = \bar{X} - \mu \end{aligned}$$

Obviously,

$$\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 = \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2, \text{ where } Y_i = X_i - \mu, \bar{Y} = \bar{X} - \mu$$

Now consider the properties of the random variable Y_i which is a transformation of X_i. First the expected value.

$$E\left(Y_i\right) = E\left(X_i\right) - E\left(\mu\right) = \mu - \mu = 0$$

The variance of Y_i is

$$Var\left(Y_i\right) = Var\left(X_i\right) = \sigma^2$$

if X_i are independently and identically distributed. Also consider $E(Y_i^4)$. We can write this as

$$E(Y^4) = \int_{-\infty}^{\infty} y^4 f(x)\, dx = \int_{-\infty}^{\infty} (x-\mu)^4 f(x)\, dx = \mu_4$$

Now write equation 8 as follows

$$E\left[\left(M_n^2\right)^2\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)^2\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right]$$
$$= \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right]$$

Ignoring $\frac{1}{n^2}$ for now, expand equation 9 as follows

$$E\left[\left(\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right] = E\left[\left(\sum_{i=1}^{n}\left(Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2\right)\right)^2\right]$$
$$= E\left[\left(\sum_{i=1}^{n} Y_i^2 - 2\bar{Y}\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n}\bar{Y}^2\right)^2\right]$$
$$= E\left[\left(\left\{\sum_{i=1}^{n} Y_i^2\right\} - 2n\bar{Y}^2 + n\bar{Y}^2\right)^2\right]$$
$$= E\left[\left(\left\{\sum_{i=1}^{n} Y_i^2\right\} - n\bar{Y}^2\right)^2\right]$$
$$= E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2 - 2n\bar{Y}^2\sum_{i=1}^{n} Y_i^2 + n^2\bar{Y}^4\right]$$
$$= E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2\right] - 2nE\left[\bar{Y}^2\sum_{i=1}^{n} Y_i^2\right] + n^2 E\left(\bar{Y}^4\right)$$

Now consider the first term on the right of 10 which we can write as

$$E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2\right] = E\left[\sum_{i=1}^{n} Y_i^2 \sum_{j=1}^{n} Y_j^2\right] = E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2\right]$$
$$= \sum_{i=1}^{n} E\, Y_i^4 + \sum\sum\nolimits_{i \neq j} E\, Y_i^2\; E\, Y_j^2$$
$$= n\,\mu_4 + n\,(n-1)\,\mu_2^2 = n\,\mu_4 + n\,(n-1)\,\sigma^4$$

Now consider the second term on the right of 10 (ignoring the $2n$ for now) which we can write as

$$E\left[\bar{Y}^2 \sum_{i=1}^{n} Y_i^2\right] = \frac{1}{n^2} E\left[\sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{i=1}^{n} Y_i^2\right]$$

$$= \frac{1}{n^2} E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum_{i \neq j} Y_i^2\, Y_j^2 + \sum\sum_{j \neq k} Y_j Y_k \sum_{\substack{i \neq j \\ i \neq k}} Y_i^2\right]$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} E\, Y_i^4 + \sum\sum_{i \neq j} E Y_i^2\, E Y_j^2 + \sum\sum_{j \neq k} E\, Y_j E Y_k \sum_{\substack{i \neq j \\ i \neq k}} E Y_i^2\right]$$

$$= \frac{1}{n^2}\left[n\,\mu_4 + n\,(n - 1)\,\mu_2^2 + 0\right] = \frac{1}{n}\left[\mu_4 + (n - 1)\sigma^4\right]$$

Now consider the third term on the right side of 10 (ignoring n^2 for now) which we can write as

$$E\left[\bar{Y}^4\right] = \frac{1}{n^4} E\left[\sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{\ell=1}^{n} Y_\ell\right]$$

$$= \frac{1}{n^2} E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum_{i \neq k} Y_i^2\, Y_k^2 + \sum\sum_{i \neq j} Y_i^2\, Y_j^2 + \sum_{i \neq j} Y_i^2\, Y_j^2 + \cdots\right]$$

where for the first double sum ($i = j \neq k = \ell$), for the second ($i = k \neq j = \ell$), and for the last ($i = \ell \neq j = k$) and ... indicates that all other terms include Y_{i} in a non-squared form, the expected value of which will be zero. Given that the Y_{i} are independently and identically distributed, the expected value of each of the double sums is the same, which gives

$$E\left[\bar{Y}^4\right] = \frac{1}{n^4} E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum_{i \neq k} Y_i^2\, Y_k^2 + \sum\sum_{i \neq j} Y_i^2\, Y_j^2 + \sum_{i \neq j} Y_i^2\, Y_j^2 + \cdots\right]$$

$$= \frac{1}{n^4}\left[\sum_{i=1}^{n} E\, Y_i^4 + 3\sum\sum_{i \neq j} Y_i^2\, Y_j^2 \;+\; \text{terms containing } EY_i\right]$$

$$= \frac{1}{n^4}\left[\sum_{i=1}^{n} E\, Y_i^4 + 3\sum\sum_{i \neq j} Y_i^2\, Y_j^2\right] = \frac{1}{n^4}\left[n\,\mu_4 + 3\,n\,(n - 1)\,(\mu_2)^2\right]$$

$$= \frac{1}{n^4}\left[n\,\mu_4 + 3\,n\,(n - 1)\,\sigma^4\right] = \frac{1}{n^3}\left[\mu_4 + 3\,(n - 1)\,\sigma^4\right]$$

Now combining the information in equations 12, 13, and 14 we obtain

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right] &= E\left[\left(\sum_{i=1}^{n}\left(Y_i^2-2Y_i\bar{Y}+\bar{Y}^2\right)\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} Y_i^{\,2}\right)^2\right] - 2nE\left[\bar{Y}^2\sum_{i=1}^{n} Y_i^2\right] + n^2E\left(\bar{Y}^4\right) \\
&= n\mu_4 + n(n-1)\mu_2^2 - 2n\left[\frac{1}{n}\left[\mu_4 + (n-1)\mu_2^2\right]\right] + n^2\left[\frac{1}{n^3}\left[\mu_4 + 3(n-1)\mu_2^2\right]\right] \\
&= n\mu_4 + n(n-1)\mu_2^2 - 2\left[\mu_4 + (n-1)\mu_2^2\right] + \left[\frac{1}{n}\left[\mu_4 + 3(n-1)\mu_2^2\right]\right] \\
&= \frac{n^2}{n}\mu_4 - \frac{2n}{n}\mu_4 + \frac{1}{n}\mu_4 + \frac{n^2(n-1)}{n}\mu_2^2 - \frac{2n(n-1)}{n}\mu_2^2 + \frac{3(n-1)}{n}\mu_2^2 \\
&= \frac{n^2-2n+1}{n}\mu_4 + \frac{(n-1)(n^2-2n+3)}{n}\mu_2^2 \\
&= \frac{n^2-2n+1}{n}\mu_4 + \frac{(n-1)(n^2-2n+3)}{n}\sigma^4
\end{aligned}
$$

Now rewrite equation 9 including $\frac{1}{n^2}$ as follows

$$
\begin{aligned}
E\left[\left(M_n^2\right)^2\right] &= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right] \\
&= \frac{1}{n^2}\left(\frac{n^2-2n+1}{n}\mu_4 + \frac{(n-1)(n^2-2n+3)}{n}\sigma_4\right) \\
&= \frac{n^2-2n+1}{n^3}\mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3}\sigma_4 \\
&= \frac{(n-1)^2}{n^3}\mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3}\sigma_4
\end{aligned}
$$

Now substitute equations 7 and 16 into equation 6 to obtain

$$
\begin{aligned}
Var\left(M_n^2\right) &= E\left[\left(M_n^2\right)^2\right] - \left(E\,M_n^2\right)^2 \\
&= \frac{(n-1)^2}{n^3}\mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3}\sigma^4 - \frac{(n-1)^2}{n^2}\sigma^4
\end{aligned}
$$

We can simplify this as

$$
\begin{aligned}
Var\left(M_n^2\right) &= \frac{(n-1)^2}{n^3}\,\mu_4 + \frac{(n-1)\,(n^2-2n+3)}{n^3}\,\sigma^4 - \frac{n\,(n-1)^2}{n^3}\,\sigma^4 \\
&= \frac{\mu_4\,(n-1)^2 + \left[(n-1)\,\sigma^4\right]\,\left(n^2 - 2n + 3 - n\,(n-1)\right)}{n^3} \\
&= \frac{\mu_4\,(n-1)^2 + \left[(n-1)\,\sigma^4\right]\,\left(n^2 - 2n + 3 - n^2 + n\right)}{n^3} \\
&= \frac{\mu_4\,(n-1)^2 + \left[(n-1)\,\sigma^4\right]\,(3-n)}{n^3} \\
&= \frac{\mu_4\,(n-1)^2 - \left[(n-1)\,\sigma^4\right]\,(n-3)}{n^3} \\
&= \frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\sigma^4}{n^3}
\end{aligned}
$$

5. Sample Variance

5.1. Definition of sample variance. The sample variance is defined as

$$
S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2
$$

We can write this in terms of moments about the mean as

$$
\begin{aligned}
S_n^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 \\
&= \frac{n}{n-1}M_n^2 \text{ where } M_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2
\end{aligned}
$$

5.2. Expected value of S^2. We can compute the expected value of S^2 by substituting in from equation 5 as follows

$$
E\left(S_n^2\right) = \frac{n}{n-1}\,E\left(M_n^2\right) = \frac{n}{n-1}\,\frac{n-1}{n}\,\sigma^2 = \sigma^2
$$

5.3. Variance of S^2. We can compute the variance of S^2 by substituting in from equation 17 as follows

$$
\begin{aligned}
Var\left(S_n^2\right) &= \frac{n^2}{(n-1)^2}\,Var\left(M_n^2\right) \qquad (18)\\
&= \frac{n^2}{(n-1)^2}\left(\frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\sigma^4}{n^3}\right) \\
&= \frac{\mu_4}{n} - \frac{(n-3)\sigma^4}{n(n-1)}
\end{aligned}
$$

5.4. **Definition of $\hat{\sigma}^2$.** S^2 is a possible estimate of the population variance. Another is $\hat{\sigma}^2$ which is given by

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n \right)^2 = M_n^2$$

5.5. **Expected value of $\hat{\sigma}^2$.** We can compute the expected value of $\hat{\sigma}^2$ by substituting in from equation 5 as follows

$$E\left(\hat{\sigma}^2\right) = E\left(M_n^2\right) = \frac{n - 1}{n} \sigma^2$$

5.6. **Variance of $\hat{\sigma}^2$.** We can compute the variance of $\hat{\sigma}^2$ by substituting in from equation 17 as follows

$$\begin{aligned} Var\left(\hat{\sigma}^2\right) &= Var\left(M_n^2\right) = \frac{(n-1)^2 \mu_4}{n^3} - \frac{(n-1)(n-3)\sigma^4}{n^3} \\ &= \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \end{aligned}$$

We can also write this in an alternative fashion

$$\begin{aligned} Var\left(\hat{\sigma}^2\right) &= Var\left(M_n^2\right) = \frac{(n-1)^2 \mu_4}{n^3} - \frac{(n-1)(n-3)\sigma^4}{n^3} \\ &= \frac{(n-1)^2 \mu_4}{n^3} - \frac{(n-1)(n-3)\mu_2^2}{n^3} \\ &= \frac{n^2 \mu_4 - 2n\mu_4 + \mu_4}{n^3} - \frac{n^2 \mu_2^2 - 4n \mu_2^2 + 3\mu_2^2}{n^3} \\ &= \frac{n^2 (\mu_4 - \mu_2^2) - 2n(\mu_4 - 2\mu_2^2) + \mu_4 - 3\mu_2^2}{n^3} \\ &= \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \end{aligned}$$

6. Normal populations

6.1. **Central moments of the normal distribution.** The first central moment is

$$E(X - \mu) = E(X) - \mu = 0$$

The second central moment is

$$E(X - \mu)^2 = E\left(X^2\right) - \mu^2 = \sigma^2$$

The third central moment is

$$E(X - \mu)^3 = 0$$

because the density of the normal distribution is symmetric. The fourth central moment is

$$E(X - \mu)^4 = 3\sigma^4 \tag{19}$$

6.2. **Variance of $\mathbf{S}^2$.** Let $X_1, X_2, ...X_n$ be a random sample from a normal population with mean μ and variance $\sigma^2 < \infty$.

We know from equation 18 that

$$\begin{aligned} Var\ (\ S_n^2\ ) &= \frac{n^2}{(n\ -\ 1)^2}\ Var\left(M_n^2\right) \\ &= \frac{n^2}{(n\ -\ 1)^2}\left(\frac{(n\ -\ 1)^2\ \mu_4}{n^3} - \frac{(n\ -\ 1)\ (n-3)\ \sigma^4}{n^3}\right) \\ &= \frac{\mu_4}{n} - \frac{(n-3\ )\ \sigma^4}{n\ (n\ -\ 1)} \end{aligned}$$

If we substitute in for μ_4 from equation 19 we obtain

$$\begin{aligned} Var\left(S_n^2\ \right) &= \tfrac{\mu_4}{n} - \tfrac{(n-3\ )\ \sigma^4}{n\ (n\ -\ 1)} = \frac{3\,\sigma^4}{n} - \frac{(n-3\ )\ \sigma^4}{n\ (n\ -\ 1)} \\ &= \tfrac{(\,3\,(\,n\ -\ 1\,)-(n-3\ ))\ \sigma^4}{n\ (\,n\ -\ 1\,)} = \frac{(\,3\,n - 3 - n + 3\ ))\ \sigma^4}{n\ (\,n\ -\ 1\,)} \\ &= \tfrac{2\,n\,\sigma^4}{n\ (\,n\ -\ 1\,)} = \frac{2\ \sigma^4}{(\,n\ -\ 1\,)} \end{aligned}$$

6.3. **Variance of $\hat{\sigma}^2$.** It can be shown that

$$Var\,(\hat{\sigma}^2\,) = \frac{2\,\sigma^4\ (\,n\ -\ 1\,)}{n^2}$$