

Conditional Expectations

Law of Iterated Expectations

- The following theorem is extremely useful:

Theorem

Law of Iterated Expectations

Let X and Y denote random variables. Then

$$E[X] = E_Y [E_{X|Y}[X | Y]] \quad (54)$$

- The inner expectation is with respect to the conditional distribution of X given Y and the outer expectation is with respect to the distribution of Y .

Conditional Expectations

Law of Iterated Expectations

Proof.

Suppose that X and Y are jointly continuous densities $f_{X,Y}(x,y)$, $f_X(x)$ and $f_Y(y)$. Then

$$\begin{aligned} & E_Y [E_{X|Y}[X | Y]] \\ &= \int_{-\infty}^{\infty} E_{X|Y}[X | Y] f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \right] f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y) dx dy \\ &= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \right] dx = \int_{-\infty}^{\infty} x f_X(x) dx = E[X] \end{aligned}$$

The proof is similar for the discrete case. □

Conditional Expectations

Law of Iterated Expectations, Example

- In the example from before, we can use the law of iterated expectations:

$$\begin{aligned} E[X] &= E_Y[E[X | Y]] = \int_0^1 \frac{1-y}{2} (2(1-y)) dy = \int_0^1 (1-y)^2 dy \\ &= \frac{-(1-y)^3}{3} \Big|_0^1 = -\frac{1}{3} [(1-1)^3 - (1-0)^3] = -\frac{1}{3} [0 - 1] = \frac{1}{3} \end{aligned}$$

- We can also find $E[X]$ directly:

$$\begin{aligned} E[X] &= \int_0^1 \int_0^{1-y} 2x dx dy = \int_0^1 \left(x^2 \Big|_0^{1-y} \right) dy \\ &= \int_0^1 (1-y)^2 dy = \int_0^1 \frac{-(1-y)^3}{3} dy = \frac{-(1-y)^3}{3} \Big|_0^1 \\ &= -\frac{1}{3} [(1-1)^3 - (1-0)^3] = -\frac{1}{3} [0 - 1] = \frac{1}{3} \end{aligned}$$

Conditional Variance

- The variance of the random variable X may be different for different values of Y .

Definition

The conditional variance is

$$\begin{aligned}\text{var}[X \mid Y = y] &= E[(X - E[X \mid Y = y])^2 \mid Y = y] \\ &= E[X^2 \mid Y = y] - [E[X \mid Y = y]]^2\end{aligned}\tag{55}$$

- We can write the variance of X as a function of the expected value of the conditional variance:

$$\text{var}[X] = E[\text{var}[X \mid Y = y]] + \text{var}[E[X \mid Y = y]]\tag{56}$$

- We can show this (different from notes):

$$\begin{aligned} & V_X [E [Y|X]] + E_X [V [Y|X]] \\ = & E_X (E [Y|X] - E_X [E [Y|X]])^2 + E_X (E [Y^2|X] - E [Y|X]^2) \\ = & E_X (E [Y|X]^2 + E (Y)^2 - 2E (Y) E [Y|X] \\ & + E_X E [Y^2|X] - E [Y|X]^2) \\ = & E_X (E (Y)^2 - 2E (Y) E [Y|X] + E [Y^2|X]) \\ = & E (Y)^2 - 2E (Y)^2 + E (Y^2) = E (Y^2) - E (Y)^2 = V (Y) \end{aligned}$$

Conditional Variance

Example

- Let the joint density of two random variables X and Y be given by

$$f(x, y) = \begin{cases} \frac{1}{4}(2x + y) & 0 \leq x \leq 1, \quad 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- We can find the marginal density of X by integrating the joint density with respect to y as follows

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \int_0^2 \frac{1}{4}(2x + y) dy \\ &= \frac{1}{4} \left(2xy + \frac{y^2}{2} \right) \Big|_0^2 = \frac{1}{4} \left(4x + \frac{4}{2} \right) \quad (57) \\ &= \frac{1}{4} (4x + 2), \quad 0 \leq x \leq 1 \end{aligned}$$

Conditional Variance

Example

- We can find the marginal density of Y by integrating the joint density with respect to x as follows:

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \int_0^1 \frac{1}{4}(2x + y) dx \\&= \frac{1}{4} (x^2 + xy) \Big|_0^1 \\&= \frac{1}{4} (1 + y), \quad 0 \leq y \leq 2\end{aligned}\tag{58}$$

Conditional Variance

Example

- We find the expected value of X by multiplying the marginal density by x and then integrating over x .

$$\begin{aligned} E[X] &= \int_0^1 \frac{1}{4} x (4x + 2) dx = \int_0^1 \frac{1}{4} (4x^2 + 2x) dx \\ &= \frac{1}{4} \left(\frac{4}{3} x^3 + x^2 \right) \Big|_0^1 = \frac{1}{4} \left(\frac{4}{3} + 1 \right) = \frac{1}{4} \left(\frac{7}{3} \right) \quad (59) \\ &= \frac{7}{12} \end{aligned}$$

Conditional Variance

Example

- To find the variance of X , we first need to find the $E[X^2]$. We do this as follows

$$\begin{aligned} E[X^2] &= \int_0^1 \frac{1}{4} x^2 (4x + 2) dx = \int_0^1 \frac{1}{4} (4x^3 + 2x^2) dx \\ &= \frac{1}{4} \left(x^4 + \frac{2}{3} x^3 \right) \Big|_0^1 = \frac{1}{4} \left(1 + \frac{2}{3} \right) = \frac{1}{4} \left(\frac{5}{3} \right) \\ &= \frac{5}{12} \end{aligned} \quad (60)$$

Conditional Variance

Example

- The variance of X is then given by

$$\begin{aligned}\text{var}(X) &\equiv E[(X - E(X))^2] \equiv E(x^2) - E^2(x) \\ &= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \frac{49}{144} \\ &= \frac{60}{144} - \frac{49}{144} = \frac{11}{144}\end{aligned}\tag{61}$$

Conditional Variance

Example

- We find the conditional density of X given that $Y = y$ by forming the ratio

$$\begin{aligned} f_{X|Y}(x | y) &= \frac{f(x, y)}{f_Y(y)} = \frac{\frac{1}{4}(2x + y)}{\frac{1}{4}(1 + y)} \\ &= \frac{(2x + y)}{(1 + y)}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 2 \end{aligned} \tag{62}$$

Conditional Variance

Example

- We then form the expected value of X given Y by multiplying the density by x and then integrating over x .

$$\begin{aligned} E[X | Y] &= \int_0^1 x \frac{(2x + y)}{(1 + y)} dx = \frac{1}{1 + y} \int_0^1 (2x^2 + xy) dx \\ &= \frac{1}{1 + y} \left(\frac{2}{3}x^3 + \frac{1}{2}x^2y \right) \Big|_0^1 = \frac{1}{(1 + y)} \left(\frac{2}{3} + \frac{1}{2}y \right) \quad (63) \\ &= \frac{\left(\frac{2}{3} + \frac{y}{2}\right)}{(1 + y)} = \frac{(4 + 3y)}{(6 + 6y)} = \left(\frac{1}{6}\right) \frac{(4 + 3y)}{(1 + y)} \end{aligned}$$

Conditional Variance

Example

- We can find the unconditional expected value of X as follows:

$$\begin{aligned} E[X] &= E_Y [E[X | Y]] = \int_0^2 \frac{(4 + 3y)}{(6 + 6y)} \frac{1}{4} (1 + y) dy \\ &= \frac{1}{4} \int_0^2 \frac{(4 + 3y)(1 + y)}{6(1 + y)} dy = \frac{1}{24} \int_0^2 (4 + 3y) dy \quad (64) \\ &= \frac{1}{24} \left(4y + \frac{3}{2}y^2 \right) \Big|_0^2 = \frac{1}{24} (8 + 6) = \frac{14}{24} = \frac{7}{12} \end{aligned}$$

Conditional Variance

Example

- We find the conditional variance by finding the expected value of X^2 given Y and then subtracting the square of $E[X | Y]$.

$$\begin{aligned} E[X^2 | Y] &= \int_0^1 x^2 \frac{(2x + y)}{(1 + y)} dx = \frac{1}{1 + y} \int_0^1 (2x^3 + x^2 y) dx \\ &= \frac{1}{1 + y} \left(\frac{1}{2} x^4 + \frac{1}{3} x^3 y \right) \Big|_0^1 = \frac{1}{1 + y} \left(\frac{1}{2} + \frac{1}{3} y \right) \quad (65) \\ &= \left(\frac{1}{6} \right) \left(\frac{3 + 2y}{1 + y} \right) \end{aligned}$$

Conditional Variance

Example

- Now square $E[X | Y]$.

$$\begin{aligned} E^2[X | Y] &= \left(\left(\frac{1}{6} \right) \frac{(4 + 3y)}{(1 + y)} \right)^2 \\ &= \frac{1}{36} \frac{(4 + 3y)^2}{(1 + y)^2} \end{aligned} \tag{66}$$

Conditional Variance

Example

- Now subtract equation 66 from equation 65

$$\begin{aligned}\text{var}[X | Y] &= \left(\frac{1}{6}\right) \left(\frac{3 + 2y}{1 + y}\right) - \frac{1}{36} \frac{(4 + 3y)^2}{(1 + y)^2} \\ &= \left(\frac{1}{36}\right) \left(\frac{(18 + 12y)(1 + y) - (4 + 3y)^2}{(1 + y)^2}\right) \\ &= \left(\frac{12y^2 + 30y + 18 - (16 + 24y + 9y^2)}{36(1 + y)^2}\right) \\ &= \frac{3y^2 + 6y + 2}{36(1 + y)^2}\end{aligned}\tag{67}$$

Conditional Variance

Example

- For example, if $y = 1$, we obtain

$$\begin{aligned}\text{var}[X | Y = 1] &= \left. \frac{3y^2 + 6y + 2}{36(1+y)^2} \right|_{y=1} \\ &= \frac{11}{144}\end{aligned}\tag{68}$$

- The expected value of $\text{var}[X | Y]$ is:

$$\begin{aligned}E[\text{var}[X | Y]] &= \int_0^2 \frac{3y^2 + 6y + 2}{36(1+y)^2} \frac{1}{4}(1+y) dy \\ &= \frac{1}{144} \int_0^2 \frac{3y^2 + 6y + 2}{(1+y)} dy\end{aligned}\tag{69}$$

Conditional Variance

Example

- Divide the numerator by the denominator:

$$1 + y \overline{) \begin{array}{r} 3y + 3 \\ 3y^2 + 6y + 2 \\ \underline{3y^2 + 3y} \\ 3y + 2 \\ \underline{3y + 3} \\ -1 \end{array}} \quad (70)$$

$$\Rightarrow \frac{3y^2 + 6y + 2}{1 + y} = (3y + 3) - \frac{1}{1 + y}$$

Conditional Variance

Example

- Then

$$\begin{aligned} E[\text{var}[X|Y]] &= \frac{1}{144} \int_0^2 \frac{3y^2 + 6y + 2}{(1+y)} dy \\ &= \frac{1}{144} \int_0^2 (3y + 3) - \frac{1}{(1+y)} dy \\ &= \frac{1}{144} \left[\frac{3}{2}y^2 + 3y - \log(1+y) \Big|_0^2 \right] \\ &= \frac{1}{144} [6 + 6 - \log(3) + \log 1] \\ &= \frac{1}{144} (12 - \log(3)) \end{aligned}$$

Conditional Variance

Example

- Recall that

$$V[E[X | Y]] = E_Y[(E[X | Y])^2] - (E_Y[E[X | Y]])^2$$

- Second term first:

- From equation 63, the expected value of X given Y was:

$$E[X | Y] = \left(\frac{1}{6}\right) \frac{(4 + 3y)}{(1 + y)} \quad (71)$$

- From equation 64 we got

$$E_Y(E[X | Y]) = \frac{7}{12} \quad (72)$$

Conditional Variance

Example

- Now find the first term

$$\begin{aligned} E_Y\left(\left(E[X | Y]\right)^2\right) &= \int_0^2 \left(\frac{1}{36}\right) \frac{(4+3y)^2}{(1+y)^2} \frac{1}{4}(y+1) dy \\ &= \frac{1}{144} \int_0^2 \frac{(4+3y)^2}{1+y} dy \\ &= \frac{1}{144} \int_0^2 \frac{9y^2 + 24y + 16}{1+y} dy \end{aligned} \tag{73}$$

Conditional Variance

Example

- Again, use long division to simplify the expression:

$$\begin{array}{r} 9y + 15 \\ 1 + y \overline{) 9y^2 + 24y + 16} \\ \underline{9y^2 + 9y} \\ 15y + 16 \\ \underline{15y + 15} \\ 1 \end{array} \quad (74)$$
$$\Rightarrow \frac{9y^2 + 24y + 16}{1 + y} = (9y + 15) + \frac{1}{1 + y}$$

Conditional Variance

Example

- Then we get:

$$\begin{aligned} E_Y \left((E[X | Y])^2 \right) &= \frac{1}{144} \int_0^2 \frac{9y^2 + 24y + 16}{1+y} dy \\ &= \frac{1}{144} \int_0^2 9y + 15 + \frac{1}{1+y} dy \\ &= \frac{1}{144} \left[\frac{9y^2}{2} + 15y + \log[y+1] \right] \Big|_0^2 \\ &= \frac{1}{144} \left[\frac{36}{2} + 30 + \log[3] \right] = \frac{1}{144} [48 + \log[3]] \end{aligned} \tag{75}$$

Conditional Variance

Example

- We can now calculate the variance:

$$\begin{aligned}\text{var}[E[X | Y]] &= E_Y \left((E[X | Y])^2 \right) - \left(E_Y(E[X | Y]) \right)^2 \\ &= \frac{1}{144} [48 + \log[3]] - \left(\frac{7}{12} \right)^2 \\ &= \frac{1}{144} [48 + \log[3]] - \frac{49}{144} \\ &= \frac{1}{144} [\log[3] - 1]\end{aligned}\tag{76}$$

- We can now verify that

$$\text{var}[X] = E[\text{var}[X | Y = y]] + \text{var}[E[X | Y = y]]$$

Conditional Variance

Example

- Recall from equation 61 that $\text{var}[X] = \frac{11}{144}$.
- On the RHS we got:

$$\begin{aligned} E[\text{var}[X | Y = y]] + \text{var}[E[X | Y = y]] \\ &= \frac{1}{144} [\log[3] - 1] + \frac{1}{144} [12 - \log[3]] \quad (77) \\ &= \frac{\log[3] - 1 + 12 - \log[3]}{144} = \frac{11}{144} \end{aligned}$$

which is the same!

Statement of Inequality: For any random variables X and Y :

$$(E(XY))^2 \leq (E(X))^2 (E(Y))^2$$

Corollary:

$$|\text{cov}(X_1, X_2)| < (\text{var}(X_1))^{\frac{1}{2}} (\text{var}(X_2))^{\frac{1}{2}} \quad (78)$$