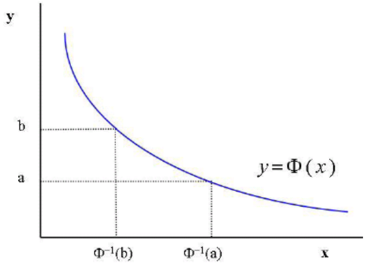
General change of variable or transformation formula

Proof

- Now consider the case where Φ is a decreasing function:



- Note that each point on the y axis maps into a point on the x axis. Therefore

$$P(a < Y < b) = P\left(\Phi^{-1}(b) < X < \Phi^{-1}(a)\right) = \int_{\Phi^{-1}(b)}^{\Phi^{-1}(a)} f_X(x)\, dx$$

- Making a change of variable for $x = \Phi^{-1}(y)$ as before we can write

$$
\begin{aligned}
P(a < Y < b) &= P\left(\Phi^{-1}(b) < X < \Phi^{-1}(a)\right) \\
&= \int_{\Phi^{-1}(b)}^{\Phi^{-1}(a)} f_X(x)\, dx \\
&= \int_b^a f_X\left(\Phi^{-1}(y)\right) \frac{d\,\Phi^{-1}(y)}{dy}\, dy \\
&= -\int_a^b f_X\left(\Phi^{-1}(y)\right) \frac{d\,\Phi^{-1}(y)}{dy}\, dy \\
&= \int_a^b f_X\left(\Phi^{-1}(y)\right) \left(-\frac{d\,\Phi^{-1}(y)}{dy}\right) dy
\end{aligned}
$$

- Because

$$\frac{d\,\Phi^{-1}(y)}{d\,y} = \frac{d\,x}{d\,y} = \frac{1}{\frac{dy}{dx}} > 0$$

when the function $y = \Phi(x)$ is increasing and

$$-\frac{d\,\Phi^{-1}(y)}{d\,y}$$

is positive when $y = \Phi(x)$ is decreasing, we can combine the two cases by writing

$$g(y) = f_Y(y) = f_X\left[\Phi^{-1}(y)\right] \cdot \left|\frac{d\,\Phi^{-1}(y)}{d\,y}\right|$$

- Let X have the probability density function given by

$$f_X(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$$

- Find the density function of $Y = \Phi(X) = 6X - 3$.
- Notice that $f_X(x)$ is positive for all x such that $0 \leq x \leq 2$.
- The function Φ is increasing for all X.
- We can then find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= 6x - 3 \\ &\Rightarrow 6x = y + 3 \\ &\Rightarrow x = \frac{y+3}{6} = \Phi^{-1}(y) \end{aligned}$$

- We can then find $\frac{d\ \Phi^{-1}(y)}{d\ y}$ as

$$\frac{d\Phi^{-1}}{dy} = \frac{d}{dy}\left(\frac{y\ +\ 3}{6}\right) = \frac{1}{6}$$

- The density of Y is then

$$\begin{aligned} g(y) &= f_Y(y) = f_X\left[\Phi^{-1}(y)\right] \cdot \left|\frac{d\Phi^{-1}(y)}{d\ y}\right| \\ &= \left(\frac{1}{2}\right)\left(\frac{3\ +\ y}{6}\right)\left|\frac{1}{6}\right|, \quad 0 \leq \frac{3\ +\ y}{6} \leq 2 \end{aligned}$$

- For all other values of y, $g(y) = 0$.
- Simplifying the density and the bounds we obtain

$$g(y) = f_Y y) = \begin{cases} \frac{3\ +\ y}{72}, & -3 \leq y \leq 9 \\ 0 & elsewhere \end{cases}$$

- Let X have the probability density function given by $f_X(x)$.
- Consider the transformation $Y = \Phi(X) = \sigma X + \mu$, $\sigma \neq 0$.
- The function Φ is increasing for all X.
- We can find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= \sigma x + \mu \\ &\Rightarrow \sigma x = y - \mu \\ &\Rightarrow x = \frac{y - \mu}{\sigma} = \Phi^{-1}(y) \end{aligned}$$

- We can then find $\frac{d\Phi^{-1}}{dy}$ as:

$$\frac{d\Phi^{-1}}{dy} = \frac{d}{dy}\left(\frac{y - \mu}{\sigma}\right) = \frac{1}{\sigma}$$

- The density of Y is then

$$f_Y(y) = f_X\left[\Phi^{-1}(y)\right] \cdot \left|\frac{d\,\Phi^{-1}(y)}{dy}\right| = f_X\left[\frac{y-\mu}{\sigma}\right] \cdot \left|\frac{1}{\sigma}\right|$$

- New example: Let X have the pdf given by

$$f_X(x) = \begin{cases} e^{-x}, & 0 \leq x \leq \infty \\ 0, & elsewhere \end{cases}$$

- Find the density function of $Y = X^{1/2}$.
- Notice that $f_X(x)$ is positive for all x such that $0 \leq x \leq \infty$.
- The function Φ is increasing for all X.

General change of variable or transformation formula

Example

- We can then find the inverse function Φ^{-1} as follows

$$
\begin{aligned}
y &= x^{\frac{1}{2}} \\
&\Rightarrow \quad y^2 = x \\
&\Rightarrow \quad x = \Phi^{-1}(y) = y^2
\end{aligned}
$$

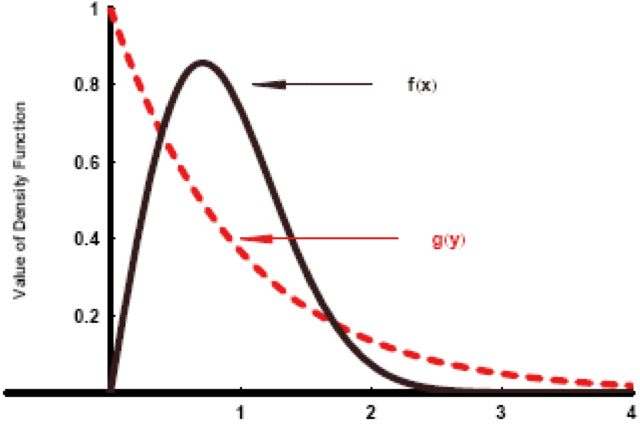
- We can then find the derivative of Φ^{-1} with respect to y as

$$\frac{d\,\Phi^{-1}}{dy} = \frac{d}{dy}\, y^2 = 2y$$

- The density of Y is then

$$f_Y(y) = f_X\left[\Phi^{-1}(y)\right] \cdot \left|\frac{d\Phi^{-1}(y)}{dy}\right| = e^{-y^2}\ |2y|$$

- Here is a graph of the two density functions:



The Two Density Functions.

- General definition of a transformation:

Definition

Let Φ be any function from R^k to R^m, $k, m \geq 1$, such that $\Phi^{-1}(A) = \{x \epsilon R^k : \Phi(x) \epsilon A\}$. If we write $y = \Phi(x)$, the function Φ defines a mapping from the sample space of the variable $X(\Xi)$ to a sample space of the random variable $Y(\Psi)$. Specifically

$$\Phi(x) : \Xi \rightarrow \Psi$$

and

$$\Phi^{-1}(A) = \{x \epsilon \Psi : \Phi(x) \epsilon A\}$$

Transformations involving multiple functions of multiple random variables

Theorem

Let $f_{X_1 X_2}(x_1 \; x_2)$ *be the value of the joint pdf of the continuous random variables* X_1 *and* X_2 *at* (x_1, x_2). *If the functions given by* $y_1 = u_1(x_1, x_2)$ *and* $y_2 = u_2(x_1, x_2)$ *are partially differentiable with respect to* x_1 *and* x_2 *and represent a one-to-one transformation for all values within the range of* X_1 *and* X_2 *for which* $f_{X_1 X_2}(x_1 \; x_2) \neq 0$, *then, the equations* $y_1 = u_1(x_1, x_2)$ *and* $y_2 = u_2(x_1, x_2)$ *can be uniquely solved for* x_1 *and* x_2 *to give* $x_1 = w_1(y_1, \; y_2)$ *and* $x_2 = w_2(y_1, \; y_2)$ *and for corresponding values of* y_1 *and* y_2, *the joint probability density of* $Y_1 = u_1(X_1, X_2)$ *and* $Y_2 = u_2(X_1, X_2)$ *is given by*

$$f_{Y_1 Y_2}(y_1 \; y_2) = f_{X_1 X_2}\left[w_1(y_1, \; y_2), \; w_2(y_1 \; y_2)\right] \cdot |J|$$

Transformations involving multiple functions of multiple random variables

Theorem

where J is the Jacobian of the transformation and is defined as the determinant

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

At all other points $f_{Y_1 Y_2}(y_1 \ y_2) = 0$.

Transformations involving multiple functions of multiple random variables

Example

- Let the pdf of X_1 and X_2 be given by

$$f_{X_1 X_2}(x_1, x_2) = \begin{cases} e^{-(x_1 + x_2)} & x_1 \geq 0, \ x_2 \geq 0 \\ 0 & \textit{elsewhere} \end{cases}$$

- Consider two random variables Y_1 and Y_2, defined in the following manner.

$$Y_1 = X_1 + X_2 \tag{5}$$

$$Y_2 = \frac{X_1}{X_1 + X_2} \tag{6}$$

- To find the joint density of Y_1 and Y_2 we first need to solve the system of equations:

$$X_1 = Y_1 - X_2 \tag{7}$$

Transformations involving multiple functions of multiple random variables

Example

- Plugging this into 6, we get

$$\begin{aligned} Y_2 &= \frac{Y_1 - X_2}{Y_1 - X_2 + X_2} = \frac{Y_1 - X_2}{Y_1} \Leftrightarrow \\ Y_1 Y_2 &= Y_1 - X_2 \Leftrightarrow \\ X_2 &= Y_1 - Y_1 Y_2 = Y_1 (1 - Y_2) \end{aligned}$$

- Plugging this into 7, we obtain

$$X_1 = Y_1 - X_2 = Y_1 - (Y_1 - Y_1 Y_2) = Y_1 Y_2$$

Transformations involving multiple functions of multiple random variables

Example

- The Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix}$$
$$= -y_2 y_1 - y_1 (1 - y_2)$$
$$= -y_2 y_1 - y_1 + y_1 y_2 = -y_1$$

- This transformation is one-to-one and maps $x_1 > 0$ and $x_2 > 0$ in the x_1 x_2-plane into t $y_1 > 0$ and $0 < y_2 < 1$.
- If we apply theorem 4 we obtain

$$f_{Y_1 Y_2}(y_1 \; y_2) = f_{X_1 X_2}\left[w_1 \left(y_1 , \; y_2 \right), \; w_2 \left(y_1 \; y_2 \right) \right] \cdot \left| J \right|$$
$$= e^{-(y_1 y_2 + y_1 - y_1 y_2)} \left| -y_1 \right|$$
$$= e^{-y_1} \left| -y_1 \right| = y_1 e^{-y_1}$$

Transformations involving multiple functions of multiple random variables

Example

- Considering all possible values of values of y_1 and y_2 we obtain

$$f_{Y_1\,Y_2}(y_1, y_2) \;=\; \begin{cases} y_1 e^{-y_1} & \textit{for } y_1 \;\geq\; 0\,,\; 0 \;<\; y_2 \;<\; 1 \\ 0 & \textit{elsewhere} \end{cases}$$

- We can then find the marginal density of Y_2 :

$$f_{Y_2}(y_1,\; y_2) \;=\; \int_0^\infty f_{Y_1\,Y_2}\,(y_1,\; y_2)\,dy_1 = \int_0^\infty y_1\; e^{-\,y_1}\; dy_1$$

- We make a uv substitution to integrate. u, v, du, and dv are defined as

$$\begin{aligned} u \;&=\; y_1 & v \;&=\; -\,e^{-y_1} \\ du \;&=\; dy_1 & dv \;&=\; e^{-y_1}\,dy_1 \end{aligned}$$

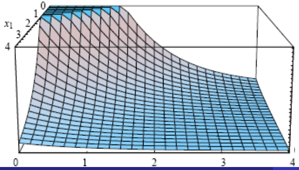
Transformations involving multiple functions of multiple random variables

Example

- This then implies

$$
\begin{aligned}
f_{Y_2}(y_1,\ y_2) &= \int_0^{\infty} y_1\, e^{-y_1} dy_1 = -\,y_1 e^{-\,y_1}\Big|_0^{\infty} - \int_0^{\infty} -\,e^{-\,y_1} d\,y_1 \\
&= (\,0\ -\ 0\,)\ -\ \left(e^{-\,y_1}\right)\Big|_0^{\infty} =\ 0\ -\ \left(e^{-\,\infty}\ -\ e^{0}\right) \\
&= 0\ -\ 0\ +\ 1 =\ 1
\end{aligned}
$$

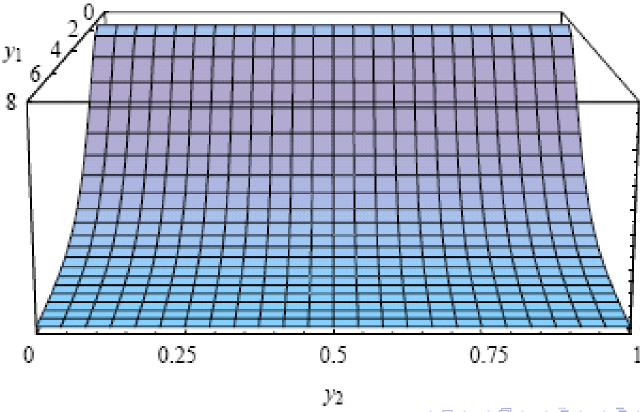
  for all y_2 such that $0 < y_2 < 1$.
- This is the joint density of (X_1, X_2)



Transformations involving multiple functions of multiple random variables

Example

- The joint density of (Y_1, Y_2):



Transformations involving multiple functions of multiple random variables

Example

- The marginal density of Y_2: