1. REAL-VALUED FUNCTIONS OF SEVERAL VARIABLES

1.1. Definition of a real-valued function of several variables. Suppose D is a set of n-tuples of real numbers \((x_1, x_2, x_3, \ldots, x_n)\). A real-valued function \(f\) on D is a rule that assigns a unique (single) real number 

\[ y = f(x_1, x_2, x_3, \ldots, x_n) \]

to each element in D. The set D is the function’s domain. The set of y-values taken on by \(f\) is the range of the function. The symbol \(y\) is the dependent variable of \(f\), and \(f\) is said to be a function of the n independent variables \(x_1\) to \(x_n\). We also call the \(x\)’s the function’s input variables and we call \(y\) the function’s output variable.

A real-valued function of two variables is just a function whose domain is \(\mathbb{R}^2\) and whose range is a subset of \(\mathbb{R}^1\), or the real numbers. If we view the domain D as column vectors in \(\mathbb{R}^n\), we sometimes write the function as

\[
f \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]

1.2. Examples.

a: The volume of a right circular cylinder is a function of the radius and height, \(V = f(r, h)\) or \(V = \pi r^2 h\). A cylinder is represented in figure 1. The volume of the cylinder as a function of its radius and height is shown graphically in figure 2. Notice that the volume increases as both the radius and the height increase.

**Figure 1. A Cylinder**
**Figure 2.** The Volume of a Cylinder as a Function of Radius and Height

**Figure 3.** Cobb-Douglas Production Function $f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2}$

**b:** The level of production from a given technology as a function of two inputs $x_1$ and $x_2$ is represented by $y = f(x_1, x_2)$. For example, a quadratic function might be $f(x_1, x_2) = 20x_1 + 16x_2 - 2x_1^2 - x_2^2$. A Cobb-Douglas function might be $f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2}$. We can represent the Cobb-Douglas function with a graph in three dimensions as in figure 3.

1.3. **Interior and boundary points in the plane or $\mathbb{R}^2$.**

1.3.1. **Interior points of regions in the plane ($\mathbb{R}^2$).** A point $(x_1^0, x_2^0)$ in a region $B$ in the $x_1$-$x_2$ plane is an interior point of $B$ if it is the center of a disk that lies entirely in $B$.

1.3.2. **Boundary points of regions in the plane ($\mathbb{R}^2$).** A point $(x_0, y_0)$ is a boundary point of $B$ if every disk centered at $(x_0, y_0)$ contains points that lie outside of $B$ as well as points that lie in $B$. (The boundary point itself needs to belong to $B$).

The interior points of a region, as a set, make up the interior of the region. The region’s boundary points make up its boundary.
1.3.3. Open and closed regions in the plane ($\mathbb{R}^2$). A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all of its boundary points.

1.3.4. Bounded regions in the plane. A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Consider the region in the plane bounded by line segments in figure 4. Call this region the set $B$. The point $a$ is an interior point because all points in a small disk centered at $a$ lie within $B$. The point $c$ is a boundary point because no matter how small the radius of the disk centered at $c$, it contains points both within and outside of $B$. The boundary of $B$ consists of all points on the line segments defining $B$, and the interior of $B$ consists of all points bounded by the line segments but which do not lie on them. The set $B$ is a closed set.

**Figure 4. Interior and Boundary Points of a Region in the Plane**

Consider the region in the plane bounded by the dotted line in figure 5. Call this region $B$. The set consists all points within the dotted circle but does not include points on the circle. The point $a$ is an interior point because all points in a small disk centered at $a$ lie within $B$. The point $c$ is a boundary point because no matter how small the radius of the disk centered at $c$, it contains points both within and outside of $B$. In this case the boundary point $c$ is not contained in the set $B$. The boundary of $B$ consists of all points on the circle defining $B$, and the interior of $B$ consists of all points bounded by circle which do not lie on it. The set $B$ is an open set.

Consider the region in the plane on and to the northeast of the curved line in figure 6. Call this region $B$. The point $a$ is an interior point because all points in a small disk centered at $a$ lie within $B$. The point $c$ is
a boundary point because no matter how small the radius of the disk centered at \( c \), it contains points both within and outside of \( B \). The interior of \( B \) consists of all points to the northeast of the curved line in figure 6. The set \( B \) is an unbounded set. The set \( B \) is also a closed set.

1.4. Interior and boundary points in space or \( \mathbb{R}^3 \).

1.4.1. Interior points of regions in space (\( \mathbb{R}^3 \)). A point \((x_1^0, x_2^0, x_3^0)\) in a region \( D \) in space is an interior point of \( D \) if it is the center of a ball that lies entirely in \( D \).
1.4.2. **Boundary points of regions in space** \( (R^3) \). A point \((x_0^1, x_0^2, x_0^3)\) is a boundary point of \( D \) if every sphere centered at \((x_0^1, x_0^2, x_0^3)\) encloses points that lie outside of \( D \) and well as points that lie in \( D \).

The **interior** of \( D \) is the set of interior point of \( D \). The boundary of \( D \) is the set of **boundary** points of \( D \).

1.4.3. **Open and closed regions in space** \( (R^3) \). A region \( D \) is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

1.5. **Interior and boundary points in** \( R^n \). We can extend the idea of open and closed sets to \( R^n \). Let \( a = (a_1, a_2, \ldots, a_n) \) be a point in \( R^n \) and let \( r \) be a given positive number. The set of all points \( x \in R^n \) such that

\[
(\|x - a\| \cdot |x - a|)^{1/2} < r
\]

is called an open **ball** of radius \( r \) with center at \( a \). We call \((\|x - a\| \cdot |x - a|)^{1/2}\) the distance between the point \( x \) and the point \( a \). We denote this set by \( B(a;r) \). The ball \( B(a;r) \) consists of all points whose distance from \( a \) is less than \( r \). In \( R^1 \), this is an open interval with center at \( a \). In \( R^2 \), this is a circular disk with radius \( r \) and center at \( a \). In \( R^3 \), this is spherical solid with center at \( a \) and radius \( r \).

1.5.1. **Interior points of sets in** \( R^n \). Let \( S \) be a subset of \( R^n \), and assume that the point \( a \) is an element of \( S \). Then \( a \) is called an **interior** point of \( S \) if there is an open \( n \)-ball with center at \( a \), all of whose points belong to \( S \).

1.5.2. **Open sets in** \( R^n \). A set \( S \) in \( R^n \) is called open if all its points are interior points.

1.5.3. **Closed sets in** \( R^n \). A set \( S \) in \( R^n \) is called closed if its complement \( R^n \setminus S \) is open.

Consider the region in space consisting of the doughnut or torus in figure 7. Call this region B in figure 8. The point \( a \) is an interior point because all points in a ball or sphere centered at \( a \) lie within \( B \) as shown in figure 9. The point \( c \) is not an interior point because a sphere centered at point \( c \) would contain points both inside and outside the doughnut.

**Figure 7.** A Subset of \( R^3 \)

1.6. **Graphs and level curves for functions with domain in** \( R^2 \).

1.6.1. **The graph of a function with domain in** \( R^2 \). The set of all points \( \{(x_1, x_2), f(x_1, x_2)\} \) in space \( (R^3) \), for \((x_1, x_2)\) in the domain of \( f \), is called the **graph** of \( f \). The graph of \( f \) is also called the **surface** \( z = f(x_1, x_2) \). Consider the function

\[
z = f(x_1, x_2) = 100 - x_1^2 - x_2^2
\]

This function has a graph in \( R^3 \) because its domain is \( R^2 \). The graph of \( f(x_1, x_2) = 100 - x_1^2 - x_2^2 \) is shown in figure 10.
1.6.2. The level curve of a function with domain in $\mathbb{R}^2$. The set of points in the plane ($\mathbb{R}^2$) where a function $f(x_1, x_2)$ has a constant value $f(x_1, x_2) = c$ is called a level curve of $f$. One can think of a level curve like contour lines on a map. Points on the same curve or line represent the same height of the function. Level curves are created by intersecting a plane at a given height (or value of the function $f(x_1, x_2)$) with the graph of $f(x_1, x_2)$, noting the values of $(x_1, x_2)$ where these intersections occur, and then plotting them in the $x_1$-$x_2$ plane. As an example, consider the function

$$z = f(x_1, x_2) = 100 - x_1^2 - x_2^2$$

from figure 10. In figure 11, we show both the graph of the function and a plane at function value of 30.

A series of level curves for $f(x_1, x_2) = 100 - x_1^2 - x_2^2$ at various valued of the function are contained in figure 12.
1.6.3. Second example of a function with domain in $\mathbb{R}^2$. Consider the function

$$ f(x_1, x_2) = 100 - x_1^2 - x_2^2 $$
FIGURE 12. Contour lines for the function \( f(x_1, x_2) = 100 - x_1^2 - x_2^2 \)

The graph of the function and a plane at \( z = 2 \) is depicted in figure 13. A level curve for the function when \( z = 2 \) is given in figure 14. A more general set of level curves is depicted in figure 15.

\[
z = f(x_1, x_2) = \frac{x_1^2 x_2^2}{(1 + x_1^2)(1 + x_2^2)}
\]

The graph of the function and a plane at \( z = 2 \) is depicted in figure 13. A level curve for the function when \( z = 2 \) is given in figure 14. A more general set of level curves is depicted in figure 15.

FIGURE 13. Graph of the function \( z = \frac{x_1^2 x_2^2}{(1 + x_1^2)(1 + x_2^2)} \)

1.6.4. Third example of a function with domain in \( \mathbb{R}^2 \). Consider the function

\[
z = f(x_1, x_2) = \frac{-x_1 x_2}{e^{x_1^2 + x_2^2}}
\]

The graph of the function is depicted in figure 16. A set of level curves is depicted in figure 17.
1.6.5. Functions with domain in $\mathbb{R}^3$. The set of points $(x_1, x_2, x_3)$ in space where a function of three independent variables $f(x_1, x_2, x_3)$ has a constant value $f(x_1, x_2, x_3) = c$ is called a level surface of $f$. The set of all points $\{x_1, x_2, x_3, f(x_1, x_2, x_3)\}$ in space, for $(x_1, x_2, x_3)$ in the domain of $f$, is called the graph of $f$. The graph of $f$ is also called the surface $w = f(x_1, x_2, x_3)$. One example of the level set for a function with a domain in $\mathbb{R}^3$ is contained in figure 18 while a second example is shown in figure 19.

2. Limits

2.1. Definition of a limit for a real valued function with a domain in $\mathbb{R}^2$.

Definition 1 (Limit). We say that a function $f(x_1, x_2)$ approaches the limit $L$ as $(x_1, x_2)$ approaches $(x_1^0, x_2^0)$, and write
Figure 16. Graph of the function \( f(x_1, x_2) = \frac{x_1 x_2}{e^{x_1^2} + x_2^2} \)

Figure 17. Levels curves for the function \( f(x_1, x_2) = \frac{x_1 x_2}{e^{x_1^2} + x_2^2} \)

\[
\lim_{(x_1, x_2) \to (x_1^0, x_2^0)} f(x_1, x_2) = L
\]

if, for every number \( \varepsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \((x_1, x_2)\) in the domain of \( f \),

\[0 < \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} < \delta \Rightarrow |f(x_1, x_2) - L| < \varepsilon\]
Notice that \( |f(x_1, x_2) - L| \) is the distance between the numbers \( f(x_1, x_2) \) and \( L \), and \( \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} \) is the distance between the point \( \{x_1, x_2\} \) and the point \( \{x_1^0, x_2^0\} \). Thus definition 1 says that the distance between \( f(x_1, x_2) \) and \( L \) can be made arbitrarily small by making the distance between \( \{x_1, x_2\} \) and \( \{x_1^0, x_2^0\} \) sufficiently small (but not zero).

Figure 20 shows a the a rectangular neighborhood for a point in the \( x_1-x_2 \) plane and the corresponding portion of the surface. The minimum value that the function achieves over the plane in denoted “\( \text{min} \)” on the vertical axis, while the maximum value that the function achieves over the plane in denoted “\( \text{max} \)” on the vertical axis. The idea of a limit is that as we make the rectangle smaller and smaller, the difference between the “\( \text{min} \)” and “\( \text{max} \)” points will get smaller and smaller. Formally, we consider the distance of
points from a given point in the domain of the function (a disc in the case of $\mathbb{R}^2$), as the diameter of this disk approaches zero, the value of the function will approach a fixed number.

**Figure 20.** Finding the Limiting Value for the Function $f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2}$ as $\{x_1, x_2\} \to \{17, 17\}$

The definition of limit applies to boundary points $(x_0^1, x_0^2)$ as well as interior points of the domain of $f$. The only requirement is that the point $(x_1, x_2)$ remain in the domain at all times. It can be shown that

$$\lim_{(x_1, x_2) \to (x_0^1, x_0^2)} x_1 = x_0^1 \quad (1a)$$
$$\lim_{(x_1, x_2) \to (x_0^1, x_0^2)} x_2 = x_0^2 \quad (1b)$$
$$\lim_{(x_1, x_2) \to (x_0^1, x_0^2)} k = k \quad (1c)$$

For example in equation 1a, $f(x_1, x_2) = x_1$ and $L = x_0^1$. Suppose $\epsilon > 0$ is chosen and let $\delta = \epsilon$. Then using the definition of limit we see that

$$0 < \sqrt{(x_1 - x_0^1)^2 + (x_2 - x_0^2)^2} < \delta = \epsilon$$
$$\Rightarrow \sqrt{(x_1 - x_0^1)^2} < \epsilon$$
$$\Rightarrow |x_1 - x_0^1| < \epsilon, \quad \sqrt{a^2} = |a|$$
$$\Rightarrow |f(x_1, x_2) - x_0^1| < \epsilon, \quad x_1 = f(x_1, x_2)$$
That is
\[ |(f(x_1, x_2) - x_1^0)| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} < \delta \]
So
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} f(x_1, x_2) = \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} x_1 = x_1^0 \]

2.2. Properties of limits.

**Theorem 1** (Properties of Limits of Functions with Domain in $\mathbb{R}^2$).

The following rules hold if $L$, $M$, and $k$ are real numbers and

\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} f(x_1, x_2) = L \]
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} g(x_1, x_2) = M \]
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} [f(x_1, x_2) + g(x_1, x_2)] = L + M \] \hspace{1cm} (2a)
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} [f(x_1, x_2) - g(x_1, x_2)] = L - M \] \hspace{1cm} (2b)
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} [f(x_1, x_2)g(x_1, x_2)] = LM \] \hspace{1cm} (2c)
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} [kf(x_1, x_2)] = kL \] \hspace{1cm} (2d)
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} \left[ \frac{f(x_1, x_2)}{g(x_1, x_2)} \right] = \frac{L}{M}, \quad M \neq 0 \] \hspace{1cm} (2e)

If $r$ and $s$ are integers with no common factors, and $s \neq 0$ then
\[ \lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} [f(x_1, x_2)]^{\frac{r}{s}} = L^{\frac{r}{s}} \quad \text{(provided } L^{\frac{r}{s}} \text{ is defined)} \] \hspace{1cm} (2f)

If $s$ is an even number, then it is assumed that $L > 0$

3. Continuity

A function $f(x_1, x_2)$ is continuous at the point $(x_1^0, x_2^0)$ if

1. $f$ is defined at $(x_1^0, x_2^0)$
2. $\lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} f(x_1, x_2)$ exists
3. $\lim_{(x_1, x_2) \rightarrow (x_1^0, x_2^0)} f(x_1, x_2) = f(x_1^0, x_2^0)$

The intuitive meaning of continuity is that if the point $(x_1, x_2)$ changes by a small amount, then the value of $f(x_1, x_2)$ changes by a small amount. The means that the surface that is the graph of a continuous function has no hole or break.

Sums, differences, products and quotients of continuous functions are continuous on their domains.
4. Definition of Partial Differentiation

Let \( f \) be a function with domain an open set in \( \mathbb{R}^n \) and range in \( \mathbb{R}^1 \), i.e. \( y = f(x_1, x_2, \ldots, x_n) \). Now define the partial derivative of \( f \) with respect to \( x_i \) as

\[
\frac{\partial f}{\partial x_i}(x) = f_i(x) = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}
\]

(3)

whenever the limit exists.

The slope of the curve \( z = f(x_1, x_2) \) at the point \((x_0^1, x_0^2, f(x_0^1, x_0^2))\) in the plane \( x_2 = x_0^2 \) is the value of the partial derivative of \( f \) with respect to \( x \) at \((x_0^1, x_0^2)\).

5. Using the Limit Concept to Compute a Partial Derivative

5.1. Procedure.

a: Add the vector \( h \) with zeros in all but one place to the vector \( x \) (\( h \neq 0 \)) and compute \( f(x+h) \).

b: Compute \( f(x) \)

c: Compute the change in the value function: \( f(x+h) - f(x) \).

d: For \( h \neq 0 \), form the quotient \( \frac{f(x+h) - f(x)}{h} \).

e: Simplify the fraction in d as much as possible. Whenever possible, cancel \( h \) from both the numerator and denominator.

f: \( f'(x) \) is the number that \( \frac{f(x+h) - f(x)}{h} \) approaches as \( h \) tends to zero.

5.2. Example. Let the function be

\[
f(x) = 2x_1^3(x_2^2 + 1)
\]

a: \( f(x+h) = 2(x_1 + h)^3(x_2^2 + 1) = 2(x_1^3 + 3x_1^2h + 3h^2x_1 + h^3)(x_2^2 + 1) \)

b: \( f(x) = 2x_1^3(x_2^2 + 1) \)

c: \( f(x+h) - f(x) = 2(3x_1^2h + 3h^2x_1 + h^3)(x_2^2 + 1) \)

d: \( \frac{f(x+h) - f(x)}{h} = 2(3x_1^2h + 3h^2x_1 + h^3)(x_2^2 + 1) \)

e: \( \frac{2(3x_1^2h + 3h^2x_1 + h^3)(x_2^2 + 1)}{h} = 2(3x_1^2 + 3hx_1 + h^2)(x_2^2 + 1) \)

f: As \( h \to 0 \), the expression goes to \( 6x_1^2(x_2^2 + 1) \).

6. Calculating Partial Derivatives

6.1. The intuitive idea of computing a partial derivative. We calculate \( \frac{\partial f}{\partial x_1} \) by differentiating \( f \) with respect to \( x_1 \) in the usual way while treating \( x_2 \) as a constant. Similarly for other partial derivatives.

6.2. Example problems.

a.

\[
f(x_1, x_2) = x_1^2x_2 + 8x_1^2x_2^3 + x_1\ln(x_2)
\]

\[
\frac{\partial z}{\partial x_1} = 2x_1x_2 + 16x_1^3x_2 + \ln(x_2)
\]

\[
\frac{\partial z}{\partial x_2} = x_1^2 + 24x_1^2x_2^2 + \frac{x_1}{x_2}
\]
b.

\( f(x_1, x_2, x_3) = x_1 \sin(x_2 + 3x_3) \)

\[ \frac{\partial f}{\partial x_1} = \sin(x_2 + 3x_3) \]
\[ \frac{\partial f}{\partial x_2} = x_1 \cos(x_2 + 3x_3) \]
\[ \frac{\partial f}{\partial x_3} = 3x_1 \cos(x_2 + 3x_3) \]

\[ \frac{\partial f}{\partial x_1} = 5 + 14x_1 + 2x_2 + 3x_3 \]
\[ \frac{\partial f}{\partial x_2} = 3 + 2x_1 + 10x_2 + 4x_3 \]
\[ \frac{\partial f}{\partial x_3} = 2 + 3x_1 + 4x_2 + 4x_3 \]

c.

\( f(x_1, x_2, x_3) = 50 + 5x_1 + 3x_2 + 2x_3 + 7x_1^2 + 2x_1x_2 + 3x_1x_3 + 5x_2^2 + 4x_2x_3 + 2x_3^2 \)

7. Geometric Interpretation of Partial Derivatives

When the domain of a function is \( \mathbb{R}^2 \) and the graph of the function is in \( \mathbb{R}^3 \), a partial derivative with respect to one of the variables is the slope of a tangent line created when we intersect a vertical plane at a fixed level of the other variable with the surface \( \mathbb{R}^3 \). Consider the function

\[ f(x_1, x_2) = 10x_1^{1/4} x_2^{1/2} \]

which is shown in figure 21 along with a vertical plane at \( x_2 = 10 \).

Now consider figure 22 which highlights the intersection of the vertical plane and the surface representing the function. This line is a graph of the function \( f(x_1, 10) = 10x_1^{1/4} \sqrt{10} \). Figure 23 shows this intersection line alone. The partial derivative of \( f(x_1, x_2) \) with respect to \( x_1 \) represents the slope of the tangent to this curve at a given point. Figure 24 shows the tangent to the curve representing the intersection of the vertical plane at \( x_2 = 10 \) and the surface, while figure 25 shows all of the graphs together.

Figure 26 shows the tangent to the curve representing the intersection of the vertical plane at \( x_1 = 1 \) and the surface \( f(x_1, x_2) = \frac{1}{4}(x_1 - 1)x_2 - (x_1 - 1)^2 - x_2^2 \). Figure 27 shows the tangent to the curve representing the intersection of the vertical plane at \( x_1 = 1 \) and the surface \( f(x_1, x_2) = (x_1 - 2)x_2 + x_2^2 + e^{x_1}x_2^2 \). Figure 28 shows tangents to the curve in both the \( x_1 \) and \( x_2 \) directions for the surface \( f(x_1, x_2) = 4e^{-x_1^2 - x_2^2} + x_2^2 \).

8. Tangent Lines and Planes for Functions with Domain in \( \mathbb{R}^2 \)

The equation for the plane that is tangent to the graph of \( y = f(x_1, x_2) \) at the point \( \{x_1^0, x_2^0, f(x_1^0, x_2^0)\} \) is

\[ y = f(x_1^0, x_2^0) + \frac{\partial f(x_1^0, x_2^0)}{\partial x_1} (x_1 - x_1^0) + \frac{\partial f(x_1^0, x_2^0)}{\partial x_2} (x_2 - x_2^0) \quad (4) \]

Intuitively, this says we approximate the function \( f \) by its value at the point \( \{x_1^0, x_2^0\} \), then move away from it along the plane that is tangent to the surface at that point. This plane has slope \( \frac{\partial f(x_1^0, x_2^0)}{\partial x_1} \) in
Figure 21. Function \( f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2} \) with Vertical Plane at \( x_2 = 10 \)

Figure 22. Intersection of function \( f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2} \) and Vertical Plane at \( x_2 = 10 \)

the \( x_1 \) direction and slope \( \frac{\partial f(x_1^0, x_2^0)}{\partial x_2} \) in the \( x_2 \) direction. Equation 4 is also called a second order Taylor series approximation to the function \( f \) at the point \( \{x_1^0, x_2^0\} \).

A tangent plane contains the tangent lines when we hold \( x_1 \) and \( x_2 \) respectively constant. If we hold \( x_2 \) constant at \( x_2^0 \), then equation 4 reduces to
Figure 23. The function \( f(x_1, 10) = 10x_1^{1/4}10^{1/2} \)

\[ \begin{array}{c}
\end{array} \]

Figure 24. Tangent to the function \( f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2} \) when \( x_2 \) is fixed at 10 and \( x_1 = 4 \)

\[ \begin{array}{c}
\end{array} \]

\[ y = f(x_1^0, x_2^0) + \frac{\partial f(x_1, x_2^0)}{\partial x_1} (x_1 - x_1^0) + \frac{\partial f(x_1^0, x_2)}{\partial x_2} (x_2^0 - x_2) \]

\[ = f(x_1^0, x_2^0) + \frac{\partial f(x_1^0, x_2)}{\partial x_1} (x_1 - x_1^0) \]

(5)

which is just the equation for a tangent line when the domain of the function is R^1.
Figure 25. Tangent to the function \( f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2} \) when \( x_2 \) is fixed at 10 and \( x_1 = 4 \)

Consider figure 24 where we hold \( x_2 \) constant at 10. This looks just like a tangent graph when the domain of the function is the real line. Then consider figure 29 where we hold \( x_1 \) constant at 4 and vary \( x_2 \). If we simplify this graph as with figure 24, we obtain a graph similar to a tangent graph when the domain of the function is the real line and the independent variable is \( x_2 \). This is shown in figure 30. If we rotate this graph as in figure 31, we can see clearly that it represents the value of \( f(x_1, x_2) \) as a function of \( x_2 \).

The plane tangent to the surface \( f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2} \) at the point \( \{x_1=4, x_2=10\} \) is shown in figure 32. The plane tangent to the surface \( f(x_1, x_2) = (x_1 - 2)x_2 + x_2^2 + e^{x_1x_2} \) is shown in figure 33.

9. Higher order partial derivatives

9.1. Second order partial derivatives. When we differentiate a function \( f(x_1, x_2) \) twice, we produce its second order derivatives. These derivatives are usually denoted by

\[
\frac{\partial^2 f}{\partial x_1^2} \quad \text{or} \quad f_{x_1x_1} \quad \text{or} \quad f_{11} \\
\frac{\partial^2 f}{\partial x_2^2} \quad \text{or} \quad f_{x_2x_2} \quad \text{or} \quad f_{22} \\
\frac{\partial^2 f}{\partial x_1 \partial x_2} \quad \text{or} \quad f_{x_1x_2} \quad \text{or} \quad f_{12} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} \quad \text{or} \quad f_{x_2x_1} \quad \text{or} \quad f_{21} (6)
\]

The defining equations are

\[
\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_1} \quad (7)
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_2}
\]

Notice that the order in which the derivatives are taken, \( \frac{\partial^2 f}{\partial x_1 \partial x_2} \), means differentiate first with respect to \( x_2 \), then with respect to \( x_1 \).
FIGURE 26. Tangent to the function $f(x_1, x_2) = \frac{1}{2}(x_1 - 1)x_2 - (x_1 - 1)^2 - x_2^2$ when $x_1$ is fixed at 1 and $x_2 = \frac{1}{2}$.

Here are some example of first and second order partial derivatives.

a. Here is a function and the first and second order partial derivatives.
Figure 27. Tangent to the function \( f(x_1, x_2) = (x_1 - 2)x_2 + x_2^2 + e^{x_1}x_2 \) when \( x_1 \) is fixed at 1 and \( x_2 = \frac{1}{2} \).

\[
f(x, w) = 50x^2w + 3wx
\]

\[
\frac{\partial f}{\partial x} = 100xw + 3w
\]

\[
\frac{\partial f}{\partial w} = 50x^2 + 3x
\]

\[
\frac{\partial^2 f}{\partial x^2} = 100w
\]

\[
\frac{\partial^2 f}{\partial w \partial x} = 100x + 3
\]

\[
\frac{\partial^2 f}{\partial x \partial w} = 100x + 3
\]

\[
\frac{\partial^2 f}{\partial w^2} = 0
\]

b. Here is a function and the first and second order partial derivatives.
Figure 28. Tangents to the function $f(x_1, x_2) = 4e^{-\frac{x_1^2 + x_2^2}{6}} + \frac{x_2^2}{6}$ at \{0.5, 0.3\}

\[ f(x_1, x_2, x_3) = 50 + 5x_1 + 3x_2 + 2x_3 + 7x_1^2 + 2x_1x_2 + 3x_1x_3 + 5x_2^2 + 4x_2x_3 + 2x_3^2 \]

\[
\frac{\partial f}{\partial x_1} = 5 + 14x_1 + 2x_2 + 3x_3 \\
\frac{\partial f}{\partial x_2} = 3 + 2x_1 + 10x_2 + 4x_3 \\
\frac{\partial f}{\partial x_3} = 2 + 3x_1 + 4x_2 + 4x_3 \\
\frac{\partial^2 f}{\partial x_1^2} = 14, \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 2, \quad \frac{\partial^2 f}{\partial x_3 \partial x_1} = 3 \\
\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2, \quad \frac{\partial^2 f}{\partial x_2^2} = 10, \quad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 4 \\
\frac{\partial^2 f}{\partial x_1 \partial x_3} = 3, \quad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 4, \quad \frac{\partial^2 f}{\partial x_3^2} = 4 \]
FIGURE 29. Tangent to the function $f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2}$ when $x_1$ is fixed at 4 and $x_2 = 10$

FIGURE 30. Tangent to the function $f(4, x_2) = 10\sqrt{2}x_2^{1/2}$ when $x_2 = 10$

c. Here is another function and the first and second order partial derivatives.
Figure 31. Rotated graph of tangent to the function \( f(4, x_2) = 10\sqrt{2}x_2^{1/2} \) when \( x_2 = 10 \)

Figure 32. Plane tangent to the function \( f(x_1, x_2) = 10x_1^{1/4}x_2^{1/2} \) at the point \( x_1 = 4 \) and \( x_2 = 10 \)

\[
f(x_1, x_2, x_3) = 50x_1^{0.2}x_2^{0.3}x_3^{0.4}
\]

\[
\frac{\partial f}{\partial x_1} = 10x_1^{-0.8}x_2^{0.3}x_3^{0.4}
\]

\[
\frac{\partial f}{\partial x_2} = 15x_1^{0.2}x_2^{-0.7}x_3^{0.4}
\]

\[
\frac{\partial f}{\partial x_3} = 20x_1^{0.2}x_2^{0.3}x_3^{-0.6}
\]

\[
\frac{\partial^2 f}{\partial x_1^2} = -8x_1^{-1.8}x_2^{0.3}x_3^{0.4} , 
\frac{\partial^2 f}{\partial x_2^2} = 3x_1^{-0.8}x_2^{-0.7}x_3^{0.4} , 
\frac{\partial^2 f}{\partial x_3^2} = 4x_1^{-0.8}x_2^{0.3}x_3^{-0.6}
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} = 3x_1^{-0.8}x_2^{-0.7}x_3^{0.4} , 
\frac{\partial^2 f}{\partial x_1 \partial x_3} = 4x_1^{-0.8}x_2^{0.3}x_3^{-0.6}
\]

\[
\frac{\partial^2 f}{\partial x_2 \partial x_3} = 6x_1^{0.2}x_2^{-0.7}x_3^{-0.6} , 
\frac{\partial^2 f}{\partial x_3^3} = -12x_1^{0.2}x_2^{0.3}x_3^{-1.6}
\]
Figure 33. Plane tangent to the function \( f(x_1, x_2) = (x_1 - 2)x_2 + x_2^2 + e^{x_1 x_2} \) when \( x_1 = 1 \) and \( x_2 = \frac{1}{2} \).

9.2. Young’s theorem. As should be obvious from the examples, \( \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_2} \). This is more generally stated as Young’s theorem. If \( f(x_1, x_2, \ldots, x_n) \) and its partial derivatives \( f_1, f_2, \ldots, f_11, f_12, \) are defined throughout an open region containing a point \((x_1^0, x_2^0, x_3^0, \ldots, x_n^0)\) and are all continuous at \((x_1^0, x_2^0, x_3^0, \ldots, x_n^0)\) then \( \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \).

9.3. Higher order derivatives. We can compute higher order partial derivatives just as we computed higher order derivatives by simply differentiating again. Here is an example. Consider the function

\[
f(x_1, x_2) = 10x_1^{1/4} x_2^{1/2}
\]

The first order partial derivatives are

\[
\frac{\partial f}{\partial x_1} = \frac{5}{2} x_1^{-3/4} x_2^{1/2}
\]
\[
\frac{\partial f}{\partial x_2} = 5x_1^{1/4} x_2^{-1/2}
\]

The second order partial derivatives are

\[
\frac{\partial^2 f}{\partial x_1^2} = -\frac{15}{8} x_1^{-7/4} x_2^{1/2}
\]
\[
\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{5}{4} x_1^{-3/4} x_2^{-1/2}
\]
\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{5}{4} x_1^{-3/4} x_2^{-1/2}
\]
\[
\frac{\partial^2 f}{\partial x_2^2} = -\frac{5}{2} x_1^{1/4} x_2^{-3/2}
\]

The third order partial derivatives are
10. Definition of the Gradient and Hessian of a Function of \( n \) Variables

10.1. Gradient of \( f \). The gradient of a function of \( n \) variables \( f(x_1, x_2, \ldots, x_n) \) is defined as follows.

\[
\nabla f(x) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)
\]

(8)

10.2. Hessian matrix of \( f \). The Hessian matrix of a function of \( n \) variables \( f(x_1, x_2, \ldots, x_n) \) is as is the \( n \times n \) matrix of second order partial derivatives. It is symmetric due to Young’s theorem and looks as follows

\[
\nabla^2 f(x) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_i \partial x_j}
\end{bmatrix}
\]

(9)

\( i, j = 1, 2, \ldots, n \)

11. Evaluating the Double Integral

11.1. Definitions. Consider a closed and bounded set in \( \mathbb{R}^2 \) denoted by \( Q \). We will assume that \( Q \) is a basic region. A basic region is a connected set in which the total boundary consists of a finite number of continuous arcs of the form \( x_2 = \phi(x_1), x_1 = \psi(x_2) \). Let \( f \) be a real-valued function \( f \) that is continuous on \( Q \). We want to define the double integral

\[
\frac{\partial^2 f}{\partial x_1^2} = \frac{105}{32} x_1^{-11/4} x_2^{1/2}
\]

\[
\frac{\partial^2 f}{\partial x_2 \partial x_1 \partial x_1} = -\frac{15}{16} x_1^{-7/4} x_2^{-1/2}
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2 \partial x_1} = -\frac{15}{16} x_1^{-7/4} x_2^{-1/2}
\]

\[
\frac{\partial^2 f}{\partial x_2 \partial x_2 \partial x_1} = -\frac{5}{8} x_1^{-3/4} x_2^{-3/2}
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2 \partial x_2} = -\frac{5}{8} x_1^{-3/4} x_2^{-3/2}
\]

\[
\frac{\partial^2 f}{\partial x_2^2} = \frac{15}{4} x_1^{1/4} x_2^{-5/2}
\]
\[ \iint_Q f(x_1, x_2) \, dx_1 dx_2 \]  

(10)

Geometrically we can think of the double integral as the volume of the solid that is bounded below by Q and above by \( f(x_1, x_2) \). To define the integral we enclose Q by a rectangle R with sides parallel to the coordinate axis. We extend \( f \) to all of R by setting \( f \) equal to zero outside of Q. This extended function, which we continue to call \( f \), is bounded on R, and is continuous on all of R except possibly at the boundary of Q. In such a case \( f \) is integrable on R, that is, there exists a unique number \( I \) such that

\[ L_f(P) \leq I \leq U_f(P) \]  

(11)

for all partitions \( P \) of R. We define \( L_f(P) \) and \( U_f(P) \) based on nonoverlapping rectangles that partition R.

If we partition \( x_1 \) into \( m \) sections and \( x_2 \) into \( n \) sections then we have \( m \times n \) nonoverlapping rectangles \( R^{ij} \).

\[ R^{ij} : x_1^{i-1} \leq x_1 \leq x_1^i, \quad x_2^{j-1} \leq x_2 \leq x_2^j \]  

(12)

where \( 1 \leq i \leq m, \ 1 \leq j \leq n \). For each rectangle, \( R^{ij} \), \( f \) takes on a maximum value and minimum value. It will take a zero value for points in a given rectangle that are outside of Q. We know it takes on a maximum and minimum value because \( f \) is continuous and \( R^{ij} \) is closed and bounded. We will denote the maximum value that \( f \) takes on \( R^{ij} \) by \( M^{ij} \) and the minimum value that \( f \) takes on \( R^{ij} \) by \( m^{ij} \). We then compute area of the rectangular cube \( (B^{ij}) \) with the rectangle \( R^{ij} \) as its base and \( M^{ij} \) as its height as

\[ Area(B^{ij}) = M^{ij} \times Area(R^{ij}) = M^{ij} (x_1^i - x_1^{i-1}) (x_2^j - x_2^{j-1}) = M^{ij} \Delta x_1^i \Delta x_2^j \]  

(13)

We then compute \( U_f(P) \) as

\[ U_f(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} M^{ij} \Delta x_1^i \Delta x_2^j \]  

(14)

and then compute \( L_f(P) \) as

\[ L_f(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} m^{ij} \Delta x_1^i \Delta x_2^j \]  

(15)

Note that

\[ \iint_Q 1 \, dx_1 dx_2 = \iint_Q dx_1 dx_2 \]  

(16)

gives the volume of solid of constant height 1 erected over Q or

\[ \text{area of Q} = \iint_Q dx_1 dx_2 \]  

(17)
11.2. Properties.

(a) Linearity

\[ \iint_{Q} [\alpha f(x_1, x_2) + \beta g(x_1, x_2)] \, dx_1 \, dx_2 = \alpha \iint_{Q} f(x_1, x_2) \, dx_1 \, dx_2 + \beta \iint_{Q} g(x_1, x_2) \, dx_1 \, dx_2 \]  

(18)

(b) Order

if \( f \geq 0 \) on \( Q \) then \( \iint_{Q} f(x_1, x_2) \, dx_1 \, dx_2 \geq 0 \)

if \( f \leq g \) on \( Q \) then \( \iint_{Q} f(x_1, x_2) \, dx_1 \, dx_2 \leq \iint_{Q} g(x_1, x_2) \, dx_1 \, dx_2 \)  

(19)

(c) Additivity

If \( Q \) is broken up into a finite number of non-overlapping basic regions \( Q_1, Q_2, \ldots, Q_n \), then

\[ \iint_{Q} f(x_1, x_2) \, dx_1 \, dx_2 = \iint_{Q_1} f(x_1, x_2) \, dx_1 \, dx_2 + \iint_{Q_2} f(x_1, x_2) \, dx_1 \, dx_2 + \cdots + \iint_{Q_n} f(x_1, x_2) \, dx_1 \, dx_2 \]  

(20)

(d) Mean Value Theorem

If \( f \) and \( g \) are continuous functions of a basic region and if \( g \) is non-negative on \( Q \), then there exists a point \( (x_0^1, x_0^2) \) in \( Q \) for which

\[ \iint_{Q} f(x_1, x_2) g(x_1, x_2) \, dx_1 \, dx_2 = f(x_0^1, x_0^2) \iint_{Q} g(x_1, x_2) \, dx_1 \, dx_2 \]  

(21)

For the case of \( g(x_0^1, x_0^2) = 1 \), we obtain

\[ \iint_{Q} f(x_1, x_2) \, dx_1 \, dx_2 = f(x_0^1, x_0^2) \iint_{Q} \, dx_1 \, dx_2 \]  

\[ = f(x_0^1, x_0^2) \times \text{area of } Q \]  

(22)

11.3. Evaluation of double integrals by repeated integrals. Difficulty in evaluating a double integral

\[ \iint_{Q} f(x_1, x_2) \, dx_1 \, dx_2 \]

can come from two sources: the integrand \( f \) or from the base region \( Q \). In the section we consider a technique for evaluating double integrals of continuous functions of over regions of types we call Type I and Type II.

11.3.1. Type I Regions. The projection of \( Q \) onto the \( x_1 \) axis is a closed interval \([a,b]\) and \( Q \) consists of all points \((x_1, x_2)\) with

\[ a \leq b \text{ and } \phi_1(x_1) \leq x_2 \leq \phi_2(x_1) \]

Consider figure 34 which describe a regions of Type I.

For a region of Type I, we compute the double integral iteratively as
Figure 34. Region of Type I

\[
\begin{align*}
\int \int_{Q} f(x_1, x_2) dx_1 dx_2 &= \int_{a}^{b} \left( \int_{\phi_1(x_1)}^{\phi_2(x_1)} f(x_1, x_2) dx_2 \right) dx_1 \\
&= \int_{a}^{b} \left( \int_{\phi_1(x_1)}^{\phi_2(x_1)} f(x_1, x_2) dx_2 \right) dx_1
\end{align*}
\] (23)

We first calculate

\[
\int_{\phi_1(x_1)}^{\phi_2(x_1)} f(x_1, x_2) dx_2
\] (24)

by integrating \(f(x_1, x_2)\) with respect to \(x_2\) from \(x_2 = \phi_1(x_1)\) to \(x_2 = \phi_2(x_1)\). The resulting expression is a function of \(x_1\) alone, which we then integrate with respect to \(x_1\) from \(x_1 = a\) to \(x_1 = b\).

11.3.2. Type II Regions. The projection of \(Q\) onto the \(x_2\) axis is a closed interval \([c, d]\) and \(Q\) consists of all points \((x_1, x_2)\) with

\[c \leq d \text{ and } \psi_1(x_2) \leq x_1 \leq \psi_2(x_2)\]

Consider figure 35 which describe a regions of Type II. For a region of Type II, we compute the double integral iteratively as

\[
\int \int_{Q} f(x_1, x_2) dx_1 dx_2 = \int_{c}^{d} \left( \int_{\psi_1(x_2)}^{\psi_2(x_2)} f(x_1, x_2) dx_1 \right) dx_2
\] (25)

We first calculate
by integrating $f(x_1, x_2)$ with respect to $x_1$ from $x_1 = \psi_1(x_2)$ to $x_1 = \psi_2(x_2)$. The resulting expression is a function of $x_2$ alone, which we then integrate with respect to $x_2$ from $x_2 = c$ to $x_2 = d$.

11.4. Examples.

11.4.1. Evaluation over a rectangle. Consider the following integral

$$\int_Q f(x_1, x_2)dx_1dx_2 = \int_{x_1^1}^{x_1^2} \int_{x_2^1}^{x_2^2} x_1^2x_2^2 dx_2dx_1$$

First evaluate

$$\int_{x_1^1}^{x_1^2} x_1^2x_2^2 dx_2$$

This will give
\[ \int_1^2 x_1^2 x_2 \, dx_2 = \left( \frac{1}{2} x_1^2 x_2^2 \right) \Big|_1^2 = \frac{1}{2} x_1^2 x_2^2 \bigg|_1^2 - \frac{1}{2} x_1^2 \bigg|_1^2 = [2 x_1^2] - \left[ \frac{1}{2} x_1^2 \right] \]
\[ = \frac{3}{2} x_1^2 \]
\]

Now integrate this function from 0 to 3 as follows

\[ \int_0^3 \frac{3}{2} x_1^2 \, dx_1 = \left( \frac{x_1^3}{2} \right) \bigg|_0^3 = \left[ \frac{27}{2} \right] - \left[ 0 \right] \]
\[ = \frac{27}{2} \]
\]

Now consider the following integral

\[ \iint_Q f(x_1, x_2) \, dx_1 \, dx_2 = \int_1^2 \int_0^3 x_1^2 x_2 \, dx_1 \, dx_2 \]
\]

First evaluate

\[ \int_0^3 x_1^2 x_2 \, dx_1 \]
\]

This will give

\[ \int_0^3 x_1^2 x_2 \, dx_1 = \left( \frac{1}{3} x_1^3 x_2 \right) \bigg|_0^3 = \left[ \frac{1}{3} \cdot 3^3 x_2 \right] - \left[ \frac{1}{3} \cdot 0^3 x_2 \right] \]
\[ = [9 x_2] - 0 \]
\[ = 9 x_2 \]
\]

Now integrate this function from 1 to 2 as follows
\[
\int_1^2 9x_2 \, dx_2 = \left( \frac{9x_2^3}{2} \right)_{1}^{2} \\
= \left[ \frac{36}{2} \right] - \left[ \frac{9}{2} \right] \\
= \frac{27}{2}
\]

(34)

11.4.2. Evaluation over a general region 1. Consider the following integral

\[
\int_{Q} f(x_1, x_2) \, dx_1 \, dx_2 = \int_{-1}^{1} \int_{2x_1^2}^{x_1^2+1} (x_1 + 2x_2) \, dx_2 \, dx_1
\]

(35)

First evaluate

\[
\int_{2x_1^2}^{x_1^2+1} (x_1 + 2x_2) \, dx_2
\]

(36)

This will give

\[
\int_{2x_1^2}^{x_1^2+1} (x_1 + 2x_2) \, dx_2 = (x_1x_2 + x_2^2) \bigg|_{2x_1^2}^{x_1^2+1}
\]

\[
= (x_1(x_1^2 + 1) + (x_1^2 + 1)^2) - (x_1(2x_1^2) + (2x_1^2)^2)
\]

\[
= (x_1^3 + x_1 + x_1^4 + 2x_1^2 + 1) - (2x_1^3 + 4x_1^4)
\]

\[
= -3x_1^4 + x_1^3 + 2x_1^2 + x_1 + 1
\]

(37)

Now integrate this function from -1 to 1 as follows

\[
\int_{-1}^{1} (-3x_1^4 - x_1^3 + 2x_1^2 + x_1 + 1) \, dx_1
\]

\[
= \left( \frac{3x_1^5}{5} - \frac{x_1^4}{4} + \frac{2x_1^3}{3} + \frac{x_1^2}{2} + x_1 \right)_{-1}^{1}
\]

\[
= \left( \frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1 \right) - \left( \frac{3}{5} - \frac{1}{4} + \frac{-2}{3} + \frac{1}{2} - 1 \right)
\]

\[
= \left( \frac{36}{60} - \frac{15}{60} + \frac{40}{60} + \frac{30}{60} + \frac{60}{60} \right) - \left( \frac{-36}{60} - \frac{15}{60} + \frac{-40}{60} + \frac{30}{60} - \frac{60}{60} \right)
\]

\[
= \frac{-72}{60} + \frac{80}{60} + \frac{120}{60}
\]

\[
= \frac{128}{60}
\]

\[
= \frac{32}{15}
\]

(38)

Consider a graph of the function in equation 35 in figure 36
Figure 36. Integral as area under a curve over a region