REVIEW OF SIMPLE UNIVARIATE CALCULUS

1. APPROXIMATING CURVES WITH LINES

1.1. **The equation for a line.** A linear function of a real variable x is given by

$$y = f(x) = ax + b, \quad \text{a and b are constants} \tag{1}$$

The graph of linear equation is a straight line. The number a is called the slope of the function and the number b is called the y-intercept.

1.2. **The point slope formula for a line.** Consider a point $P = (x_1, y_1)$ and a line with slope a. The line with slope a passing through the point $P(x_1, y_1)$ can be determined by picking an arbitrary point on the line denoted (x, y) and then using the formula for the slope of a line to get the linear equation through that point. The slope a is equal to the change in y over the change in x or

$$a = \frac{y - y_1}{x - x_1}, \; x \neq x_1 \tag{2}$$

We can then solve the equation for either $y \; or \; y - y_1$ as follows.

$$\begin{aligned} a =& \frac{y - y_1}{x - x_1}, \; x \neq x_1 \\ \Rightarrow y - y_1 =& \; a(x - x_1) \\ \Rightarrow y =& \; a(x - x_1) + y_1 \\ =& \; ax + (y_1 - ax_1) \end{aligned} \tag{3}$$

1.3. **Using a line to approximate a curve.** If we want to approximate a curve with a line at a particular point, one way to do so is pick a line going through that point that has the same slope as the curve at that point. This is called the tangent line. In figure 1 the line approximating the curve f at the point (x,f(x)) is shown. We know the tangent line passes through (x, f(x)) and hence all we need to determine is the slope. To do this choose a small number, $h \neq 0$, and on the graph of f, mark the point (x + h, f (x + h)). The line that passes through this point has slope $\frac{f(x+h)-f(x)}{h}$. We can see this in figure 2.

This line, of course, has a slope different from the tangent line. If h were larger, the approximation would be worse, while if h were smaller, the approximation would be better.

1.4. **Intuitive definition of derivative of f at the point [z,f(z)].** The slope of the tangent to the curve y = f(x) at the point [z, f(z)] is called the derivative of f with respect to x at the point z. The derivative is written as f′(x) or $\frac{df(x)}{dx}$ or $\frac{dy}{dx}$.

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FIGURE 1. Tangent to a Curve

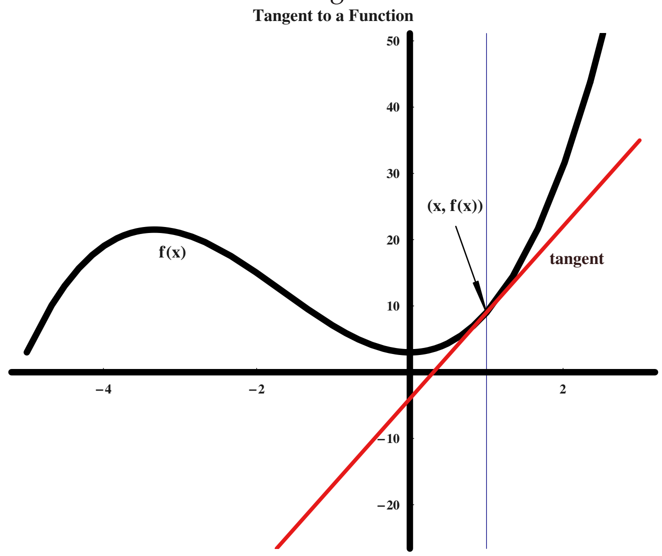
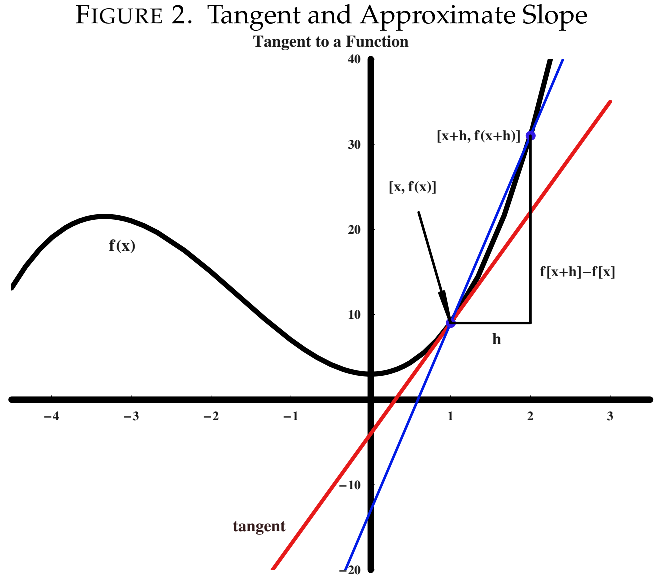


FIGURE 2. Tangent and Approximate Slope



1.5. **Equation of the tangent line to a graph given the definition of a derivative.** The equation for the tangent to the graph of y = f(x) at the point [a, f(a)] is

$$\begin{aligned} y - f(a) &= f'(a)(x-a) \\ \Rightarrow y &= f(a) + f'(a)(x-a) \end{aligned} \tag{4}$$

Intuitively, this says we approximate the function f by its value at the point a, then move away from a along a line with slope f′(x). Equation 4 is also called a first order Taylor series approximation to the function f at the point a.

2. A BRIEF DISCUSSION OF LIMITS

2.1. **Definition of a function.** A function f is a set of ordered pairs (x, y), no two of which have the same first member. That is, if (x, y) ∈ f and (x, z) ∈ f, then y = z. The definition requires that for every x in the domain of f, there is exactly one y such that (x, y) ∈ f. It is customary to call y the value of f at x and write

$$y = f(x) \tag{5}$$

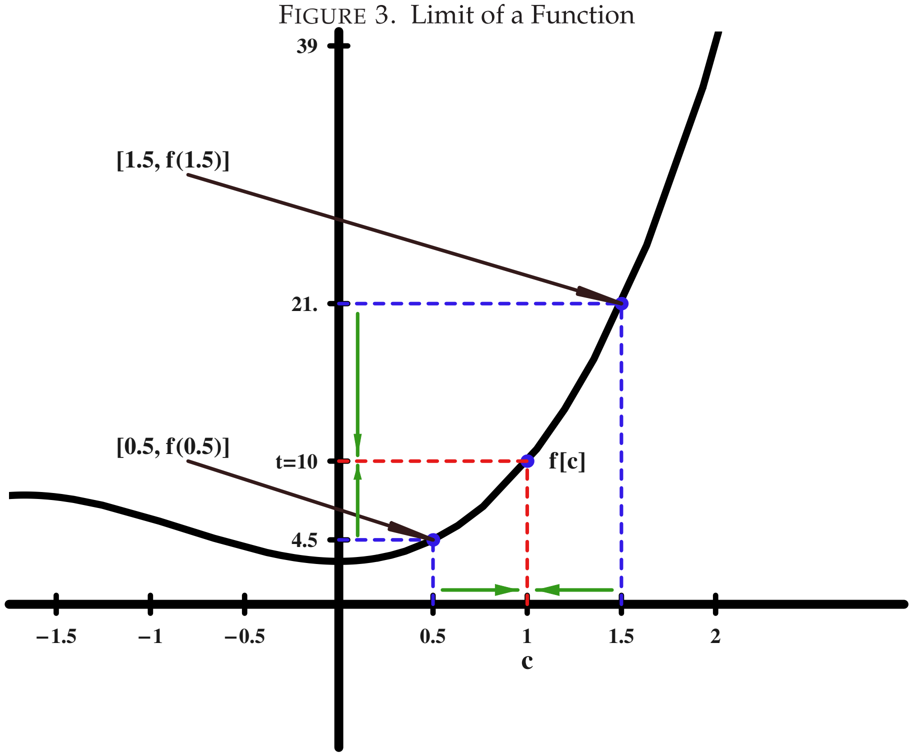
In R^2, a function passes the vertical line test, i.e., a vertical line passes through the graph of the function only once. For example the mapping y = x^2 is a function because for each value of y there is a distinct value of x. The unit circle $x^2 + y^2$ = 1 is not a function because at zero y can equal +1 and -1.

2.2. **Idea of limit.** Consider a number t and a function defined near the number c, but not necessarily defined at c itself. A rough translation of

$$\lim_{x \to c} f(x) = t \tag{6}$$

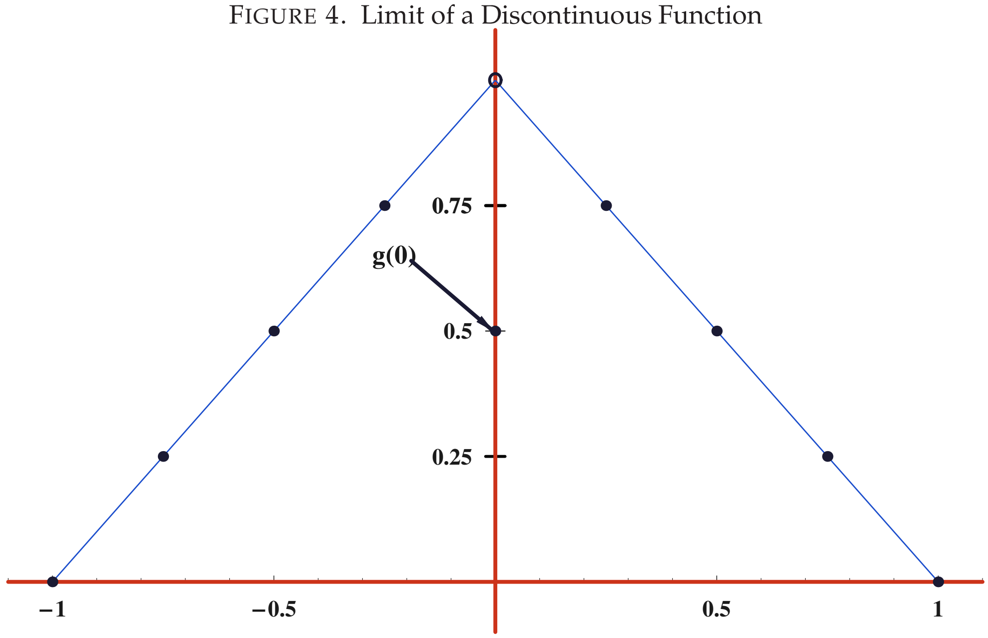
might be ”as x approaches c, f(x) approaches t” or ”for x close to c, f(x) is close to t.” Consider the example in figure 3 where the function is well behaved at all points.

FIGURE 3. Limit of a Function



As x gets close to c, f(x) gets close to t. Now consider the case where f is defined peculiarly at c as in figure 4. The limit is still t. Notice that even though f(c) is a peculiar point, the limit of f(·) at c is t.

FIGURE 4. Limit of a Discontinuous Function



Now consider the function

$$f(x) = \frac{1}{x - c}, \tag{7}$$

For c=5, the graph is contained in figure 5. Notice that the limit of the function does not exist or goes to infinity as x goes to 5.

2.3. **Formal definition of limit.**

$$\lim_{x \to t} f(x) = t, \text{ if and only if} \begin{cases} \text{for} & each\ \epsilon \text{ there exists } \delta > 0 \text{ such that} \\ \text{if} & 0 < |x - c| < \delta \text{ then } |f(x) - t| < \epsilon\,. \end{cases} \tag{8}$$

For f to have a limit at c, it must be defined at all numbers sufficiently close to c, but it need not be defined at c. Even if it is defined at c, its value at c is irrelevant. The choice of δ depends on the previous choice of ϵ. We do not require that there exist a number ϵ which "works" for all δ but, rather, that for each δ there exists a ϵ which works for it.

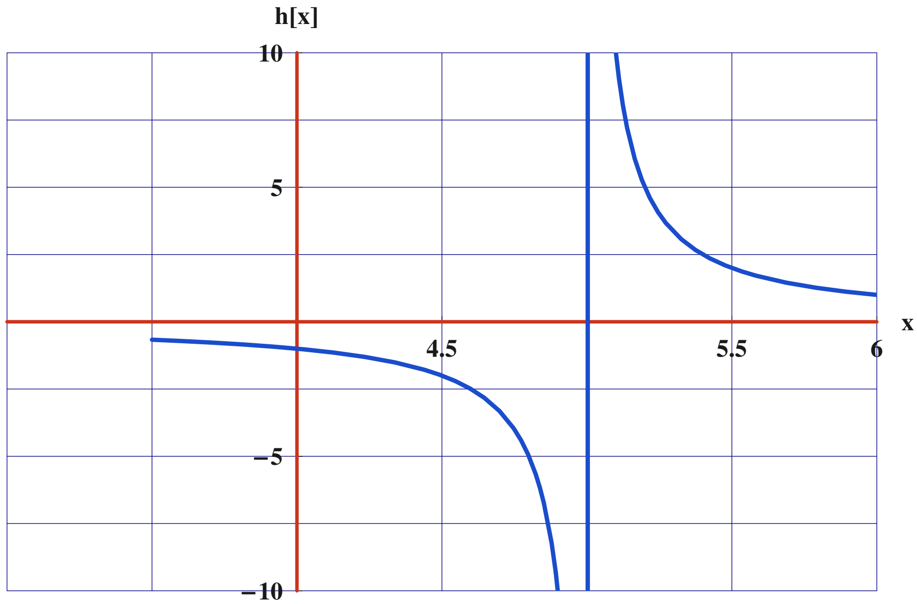
2.4. **Some examples of computing limits.**

2.4.1. *Example 1.* Show that $\lim_{x \to 2}$ (2x - 1) = 3.

We need to find a $\epsilon > 0$, $\delta > 0$ pair such that

$$\text{if } 0 < |x - 2| < \delta, \text{ then } |(2x - 1) - 3| < \epsilon$$

FIGURE 5. Limit that Does Not Exist at a Point



To find the appropriate level of δ, we need to find a connection between the two expressions

$$|x-2| \text{ and } |(2x-1)-3|$$

First rewrite the second expression as

$$|(2x-1)-3| = |2x-4| = 2|x-2|$$

To make $|(2x-1)-3|$ less than ϵ, we need only make $|x-2|$ twice as small. So we can choose $\delta = \frac{1}{2}\,\epsilon$. To see that this works notice that if $0 < |x-2| < \frac{1}{2}\,\epsilon$, then $2|x-2| < \epsilon$. But this implies that $|(2x-1)-3| < \epsilon$. This is shown graphically in figure 6.

2.4.2. *Example 2.* Show that $\lim_{x\to -1}(2-3x) = 5$.

We need to find a $\epsilon > 0$, $\delta > 0$ pair such that

$$\text{if } 0 < |x-(-1)| < \ \delta \text{ then } |(2-3x)-5| < \ \epsilon$$

To find the appropriate level of δ, we need to find a connection between the two expressions

$$|x-(-1)| \text{ and } |(2-3x)-5|$$

First rewrite the second expression as

$$|(2-3x)-5| = |-3x-3| = 3|-x-1| = 3|x+1|$$

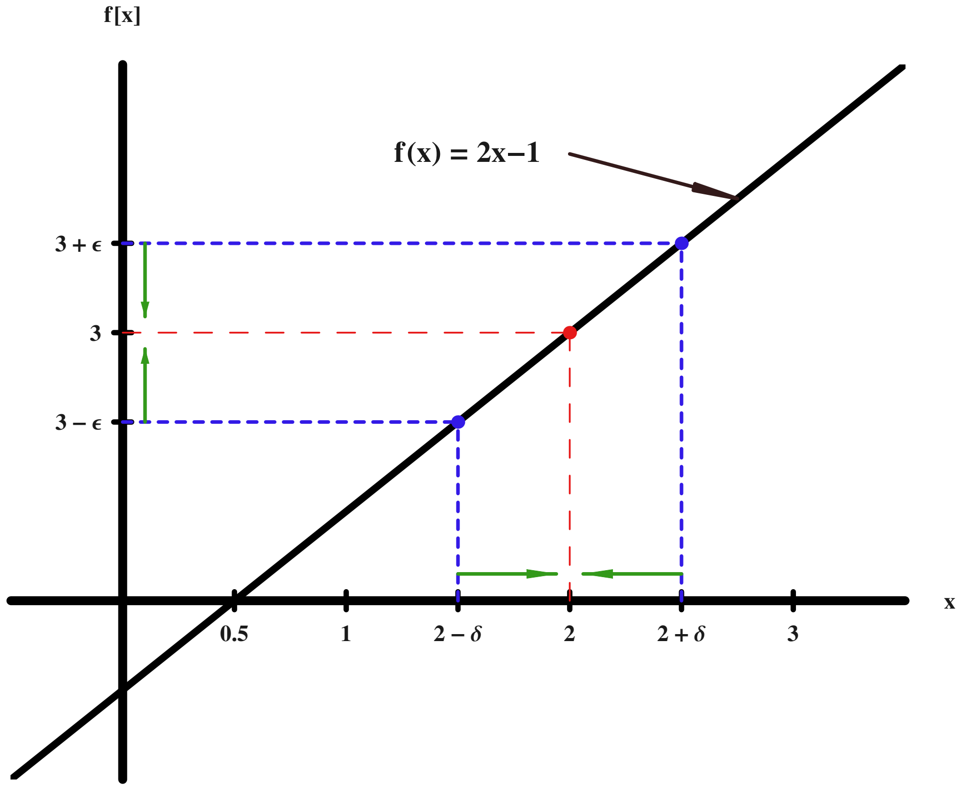
and the first expression as

$$|(x-(-1))| = |x+1|$$

From these two expressions we can conclude that

$$|(2-3x)-5| = 3|x-(-1)|$$

FIGURE 6. Limit of 2x-1 as x $\to$ 2



To make $|(2-3x)-5|$ less than ϵ, we need only make $|x-(-1)|$ three times as small. So we can choose $\delta = \frac{1}{3}\,\epsilon$. To see that this works notice that if $0 < |x-(-1)| < \frac{1}{3}\,\epsilon$, then $3|x-(-1)| < \epsilon$. But this implies that $|(2-3x)-5| < \epsilon$. This is shown graphically in figure 7.

2.5. **Uniqueness of limits.** If $\lim_{x\to c}$ *f(x)* = *t* and $\lim_{x\to c}$ *f(x)* = *m then t* = *m*.

2.6. **Some rules for limits.** If $\lim_{x\to c}$ f(x) = A and $\lim_{x\to c}$ g(x) = B, then

$$\lim_{x\to a}[f(x)+g(x)] = A+B \tag{9a}$$

$$\lim_{x\to a}[f(x)-g(x)] = A-B \tag{9b}$$

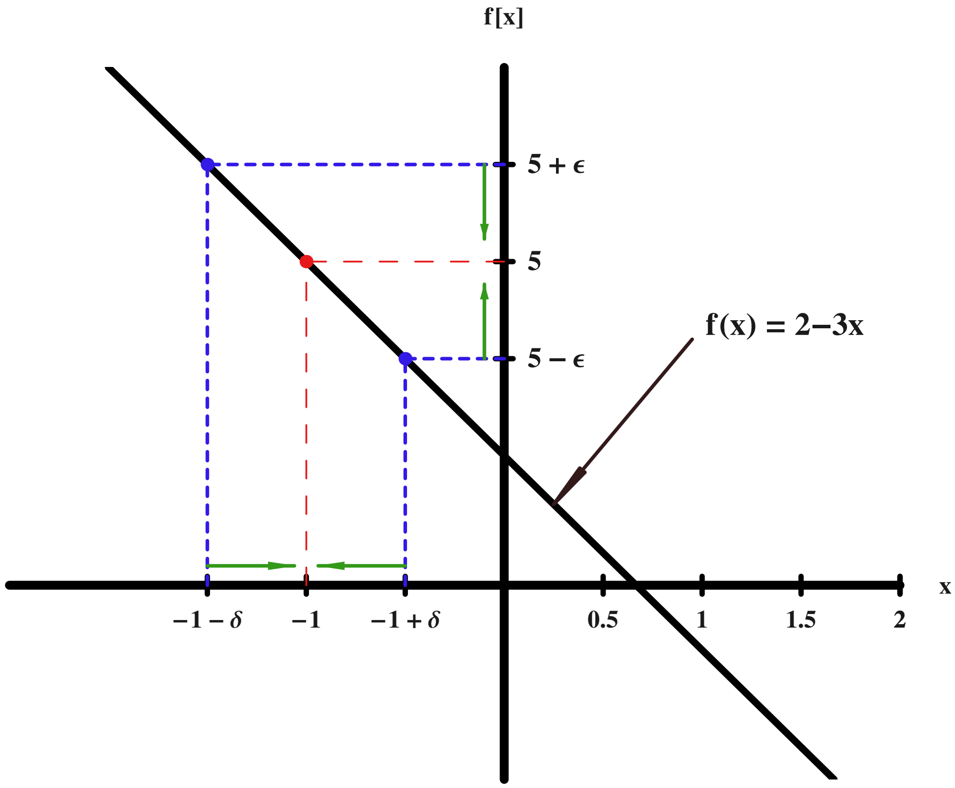
$$\lim_{x\to a}[f(x)g(x)] = A.B \tag{9c}$$

$$\lim_{x\to a}[f(x)/g(x)] = A/B \text{ (provided B} \neq 0) \tag{9d}$$

$$\lim_{x\to a}[f(x)]^{\frac{p}{q}} = A^{\frac{p}{q}} \text{ (provided A}^{\frac{p}{q}}\text{is defined)} \tag{9e}$$

2.7. **Equality of functions and equality of limits.** If the functions d and g are equal for all x close to a (but not necessarily at x = a), then

FIGURE 7. Limit of 2-3x as x → -1



$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x), \qquad \text{whenever either limit exists.}$$

3. Formal definition of the derivative and rules for differentiation

3.1. **Definition.** Let f be a function defined on an open interval (a, b), and assume that c ∈ (a, b). Then f is said to be differentiable at c whenever the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{10}$$

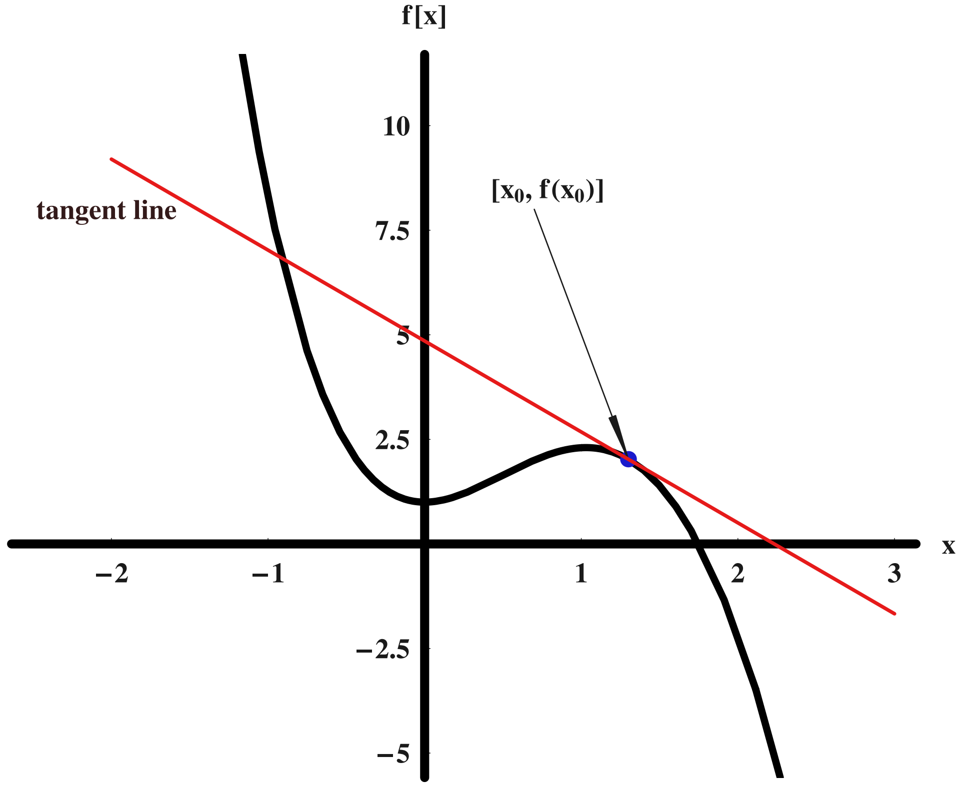
exists. The limit, denoted by f′(c), is called the derivative of f at c. Another way to express the definition is

$$\frac{df(x)}{dx}(x_0) = f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{11}$$

Where the derivative is defined, it is equal to the slope of the tangent to the curve f(x) at the point x_0 as shown in figure 8.

3.2. **Using the limit concept to compute a derivative.**

FIGURE 8. The Derivative of f at x_0 is the Slope of the Tangent Line at x_0



3.2.1. *Procedure.*

a: Add h to a ($h \neq 0$) and compute f(a+h).
b: Compute f(a)
c: Compute the change in the value function: f(a+h) - f(a).
d: For $h \neq 0$, form the quotient $\frac{f(a+h) - f(a)}{h}$.
e: Simplify the fraction in d as much as possible. Whenever possible, cancel h from both the numerator and denominator.
f: $f'(a)$ is the number that $\frac{f(a+h) - f(a)}{h}$ approaches as h tends to zero.

3.2.2. *Example.* Let the function be $f(x) = x^3$.

a: $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ha^2 + h^3$
b: $f(a) = x^3$
c: $f(a+h) - f(a) = 3a^2h + 3ha^2 + h^3$
d: $\frac{f(a+h) - f(a)}{h} = \frac{3a^2h + 3ha^2 + h^3}{h}$
e: $\frac{3a^2h + 3ha^2 + h^3}{h} = 3a^2 + 3ah + h^2$
f: As $h \to 0$, the expression goes to $3a^2$.

3.3. **Some useful rules of differentiation.**

3.3.1. *Constant functions.* The derivative of a constant is zero.

$$\begin{aligned} f(x) &= a \\ \frac{df(x)}{dx} &= f'(x) = 0 \\ f(x) &= 3 \\ \frac{df(x)}{dx} &= 0 \end{aligned} \tag{12}$$

3.3.2. *Additive and multiplicative constants.*

$$\begin{aligned} y &= b + f(x) \\ \frac{dy}{dx} &= f'(x) \\ y &= a f(x) \\ \frac{dy}{dx} &= a f'(x) \end{aligned} \tag{13}$$

Consider the example

$$\begin{aligned} \text{If } y &= 5 + 4f(x) \\ \text{then } \frac{dy}{dx} &= 4f'(x) \end{aligned} \tag{14}$$

3.3.3. *Power rule.*

$$\begin{aligned} \text{If } f(x) &= x^n \\ \text{then } \frac{df(x)}{dx} &= f'(x) = nx^{n-1} \end{aligned} \tag{15}$$

Consider as an example

$$\begin{aligned} \text{If} f(x) &= x^4 \\ \frac{df(x)}{dx} &= f'(x) = 4x^3 \end{aligned} \tag{16}$$

3.3.4. *Sum and difference rule.* The derivative of a sum is the sum of the derivatives. The same holds true for differences. So let H(x) = f(x) + g(x). Then H′(x) = f′(x) + g′(x). Specifically

$$\begin{aligned} \frac{d[f(x) + g(x)]}{dx} &= \frac{df(x)}{dx} + \frac{dg(x)}{dx} \\ &= f'(x) + g'(x) \end{aligned} \tag{17}$$

Consider as an example

$$\begin{aligned} y &= 3x^2 + 5x + 3 \\ f'(x) &= 6x + 5 \end{aligned} \tag{18}$$

3.3.5. *Product Rule.* Let H(x) = f(x)g(x). Then H′(x)= d[f(x).g(x)]

$$\begin{aligned} H'(x) &= \frac{d[f(x).g(x)]}{dx} = f(x).\frac{dg(x)}{dx} + g(x).\frac{df(x)}{dx} \\ &= f(x).g'(x) + g(x).f'(x) \end{aligned} \tag{19}$$

We sometimes say that derivative of the product of two functions is equal to the derivative of the first times the second, plus the second times the derivative of the first.

Consider the following example where f(x) = (3x+2) and g(x) = x^2 and H(x) = x^2(3x + 2) = $3x^3$ + $2x^2$.

$$\begin{aligned} H'(x) &= f(x) \cdot \frac{dg(x)}{dx} + g(x) \cdot \frac{df(x)}{dx} \\ &= (3x + 2)(2x) + (x^2)(3) \\ &= 6x^2 + 4x + 3x^2 \\ &= 9x^2 + 4x \end{aligned} \tag{20}$$

Taking the derivative directly we obtain

$$\begin{aligned} H(x) &= 3x^3 + 2x^2 \\ \Rightarrow H'(x) &= 9x^2 + 4x \end{aligned} \tag{21}$$

3.3.6. *Quotient Rule.* If f and g are differentiable at x and g(x) $\neq$ 0, then H(·) = $\frac{f(\cdot)}{g(\cdot)}$ is differentiable at x, and

$$\begin{aligned} H'(x) &= \frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x) \cdot \frac{df(x)}{dx} - f(x) \cdot \frac{dg(x)}{dx}}{[g(x)]^2} \\ &= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2} \end{aligned} \tag{22}$$

Consider the following example where f(x) = (3x - 5) and g(x) = (x - 2).

$$\begin{aligned} H(x) &= \frac{3x - 5}{x - 2} \\ H'(x) &= \frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x) \cdot \frac{df(x)}{dx} - f(x) \cdot \frac{dg(x)}{dx}}{(g(x))^2} \\ &= \frac{(x - 2)(3) - (3x - 5)(1)}{(x - 2)^2} \\ &= \frac{3x - 6 - 3x + 5}{(x - 2)^2} \\ &= \frac{-1}{(x - 2)^2} \end{aligned} \tag{23}$$

3.4. **Second and higher order derivatives.** The derivative of a function f is called the first derivative of f. If f′ is also differentiable, then we can differentiate f′ in turn. We call f″ the second derivative of f. We use the following notation for the second derivative

$$\frac{d\left(\frac{df(x)}{dx}\right)}{dx} = (f')' = f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d^2\, y}{dx^2} \tag{24}$$

Consider the following example

$$\begin{aligned} f(x) &= (3x + 1)\,(x^2 + 5) \\ \frac{df(x)}{dx} &= (3)(x^2 + 5) + (3x + 1)\,(2x) \\ &= 3x^2 + 15 + 6x^2 + 2x \\ &= 9x^2 + 2x + 15x \\ \frac{d\left(\frac{df(x)}{dx}\right)}{dx} &= 18x + 2 \end{aligned} \tag{25}$$

The derivative of $\frac{d^2 f(x)}{d\,x^2}$ (f′′) is called the third derivative of f. We use the notation f′′′ or f $^{(3)}$ or $\frac{d^3 f(x)}{d\,x^3}$ for this derivative. For the nth derivative we use the notation f $^{(n)}$ or $\frac{d^n f(x)}{d\,x^n}$.

3.5. **The chain rule for composite functions.**

3.5.1. *Definition.* Let f be defined on an open interval S, and let g be defined on f(S), and consider the composite function g∘f defined on S by the equation

$$(g \circ f)(x) = g[f(x)] \tag{26}$$

Assume that there is a point c in S such that f(c) is an interior point of f(S). If f is differentiable at c and if g is differentiable at f(c), then g∘f is differentiable at c and we have

$$(g \circ f)'(c) = g'\,[f(c)] \cdot f'(c) \tag{27}$$

This can also be written

$$\begin{aligned} \frac{dg[f(x)]}{dx} &= \frac{dg[f(x)]}{d\,f\,(x)} \cdot \frac{df(x)}{dx} \\ &= g'\,(f\,(x)) \cdot f'\,(x) \end{aligned} \tag{28}$$

3.5.2. *Examples.*

1: Example 1

$$\begin{aligned}
f(x) &= x^2 + 2x \\
g(x) &= 3x^2 \\
g(f(x)) &= 3\,(x^2 + 2x)^2 \\
&= 3\,(x^4 + 4x^3 + 4x^2) \\
&= 3x^4 + 12x^3 + 12x^2 \\
\frac{dg[f(x)]}{dx} &= \frac{dg[f(x)]}{df(x)} \cdot \frac{df(x)}{dx} \\
&= 6\,(x^2 + 2x)\,(2x + 2) \\
&= 6\,(2x^3 + 2x^2 + 4x^2 + 4x) \\
&= 12x^3 + 36x^2 + 24x
\end{aligned} \tag{29}$$

If we simply take the derivative of $(g \circ f)$ we obtain

$$\begin{aligned}
g(f(x)) &= 3x^4 + 12x^3 + 12x^2 \\
\frac{dg[f(x)]}{dx} &= 12\,x^3 + 36\,x^2 + 24\,x
\end{aligned} \tag{30}$$

2: Example 2

As another example let y = u^3 and let u = $[1 - (2 + 3x)^2]$. We can then find dy/dx as

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\,\left[1 - (2 + 3x)^2\right]^3 \\
&= 3\,[\,1 - (2 + 3x)^2]^2\,\frac{d}{dx}\,[1 - (2 + 3x)^2\,] \\
&= 3\,[\,1 - (2 + 3x)^2]^2\,[(-2)(2 + 3x)\,\frac{d}{dx}\,(3x)\,] \\
&= 3\,[\,1 - (2 + 3x)^2]^2\,[(-6)(2 + 3x)\,] \\
&= 3\,[\,1 - (2 + 3x)^2]^2\,[(-12 - 18x)\,] \\
&= [\,1 - (2 + 3x)^2]^2\,(-36 - 54x)
\end{aligned} \tag{31}$$

3.6. **Exponential and logarithmic functions.**

3.6.1. *Exponential functions - e^x.*

$$\begin{aligned} f(x) &= e^x \\ \Rightarrow \frac{df(x)}{dx} &= \frac{d(e^x)}{dx} = e^x \end{aligned} \tag{32}$$

3.6.2. *Logarithmic functions - log (x).*

$$\begin{aligned} f(x) &= log\,(x) \\ \Rightarrow \frac{df(x)}{dx} &= \frac{d(log(x))}{dx} = \frac{1}{x} \end{aligned} \tag{33}$$

3.6.3. *p^x and $p^{f(x)}$.*

$$\begin{aligned} \frac{d(p^x)}{dx} &= p^x \; log\, p \\ \frac{d(p^{f(x)})}{dx} &= p^{f(x)} \; log\, p \; \frac{df(x)}{dx} \end{aligned} \tag{34}$$

For example

$$\begin{aligned} \frac{d(4^x)}{dx} &= 4^x \, log\, 4 \\ \frac{d(5^{3x^2})}{dx} &= 5^{3x^2} \; log\, 5 \; (6x) \end{aligned}$$

3.6.4. *Exponential functions and the chain rule.*

$$\frac{d[e^{g(x)}]}{dx} = e^{g(x)} \cdot \frac{d[g(x)]}{dx} \tag{35}$$

For example

$$\frac{d[e^{2x^3}]}{dx} = e^{2x^3} \cdot (6x^2)$$

3.6.5. *Logarithms and the chain rule.*

$$\frac{d[log(g(x))]}{dx} = \frac{\frac{d[g(x)]}{dx}}{g(x)} = \frac{g'(x)}{g(x)} \tag{36}$$

For example,

$$\frac{d[log(2x^2 + 3x)]}{dx} = \frac{4x + 3}{2x^2 + 3x}$$

3.7. **Implicit Differentiation.**

3.7.1. *The Idea of Implicit Differentiation.* Sometimes we cannot solve an equation for y as a function of x, that is we cannot find an explicit formula for x. We may be able to compute dy/dx in some cases, however. Consider first an example where $x^3 - 3xy^2 + y^3 = 0$. While an explicit formula for y is not available, we can differentiate both sides of the equation using the chain rule to obtain

$$\begin{aligned} x^3 - 3xy^2 + y^3 &= 0 \\ \Rightarrow 3x^2 - 3x\left(2y\frac{dy}{dx}\right) - 3y^2 + 3y^2\frac{dy}{dx} &= 0 \end{aligned} \tag{37}$$

Then we can collect terms in $\frac{dy}{dx}$ and rearrange to obtain

$$\begin{aligned} 3x^2 - 3y^2 &= (6xy - 3y^2)\frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{3x^2 - 3y^2}{6xy - 3y^2} \\ &= \frac{x^2 - y^2}{2xy - y^2} \end{aligned} \tag{38}$$

At the point (2, -1) the derivative is (-3/5).

3.7.2. *Procedure to find dy/dx for an implicit equation in y and x.*

1: Differentiate each side of the equation with respect to x, considering y as a function of x.
2: Solve the resulting equation for dy/dx.

3.8. **Linear approximations and differentials.**

3.8.1. *Linear approximations.* The linear approximation to f about the point a is given by

$$f(x) \approx f(a) + f'(a)(x - a) \; (x \; close \; to \; a) \tag{39}$$

This is also the equation for the tangent at the point a.

3.8.2. *Definition of a differential.* Consider a differentiable function y = f(x) and let dx denote an arbitrary change in the variable x. The expression f′(x) dx is called the **differential** of y = f(x), and it is denoted by dy (or df), so that

$$dy = f'(x)\,dx\,. \tag{40}$$

Note that dy is proportional to dx with f′(x) the factor of proportionality. Note that if x changes by dx, then the corresponding change in y = f(x) is

$$\Delta y = f(x + dx) - f(x)\,. \tag{41}$$

Now consider the linear approximation to a function in equation 39. Replace x by x + dx and a by x as follows

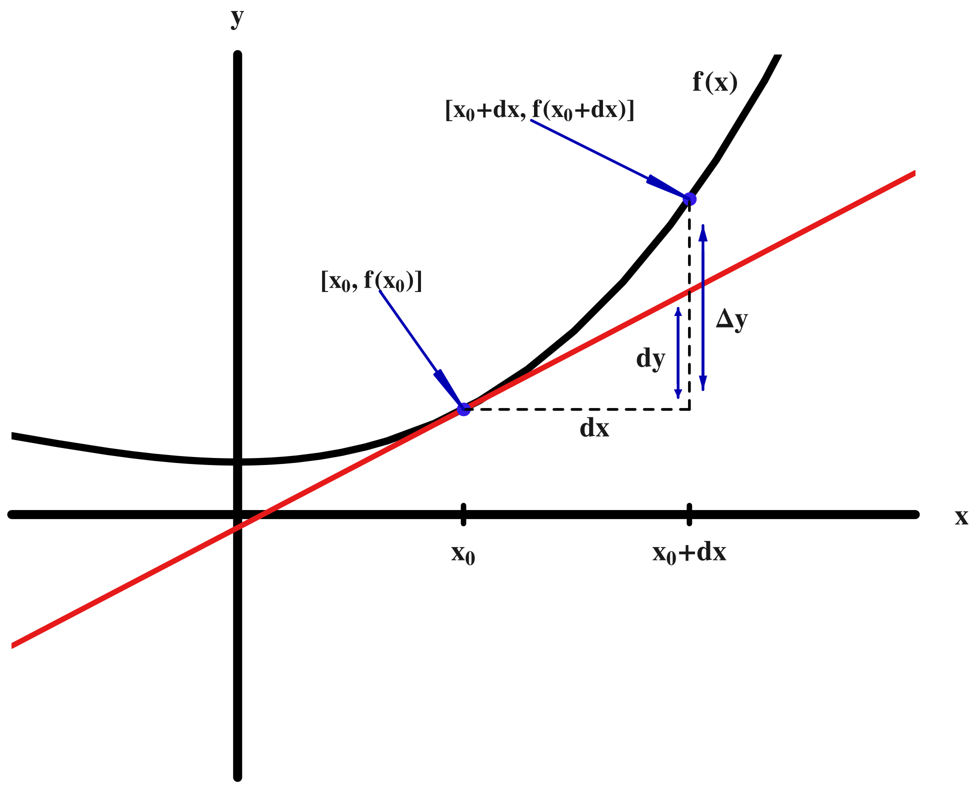
$$\begin{aligned} f(x + dx) &\approx f(x) + f'(x)(x + dx - x) \; (x + dx \text{ close to } x) \\ \Rightarrow f(x + dx) - f(x) &\approx f'(x)(dx) \; (x + dx \; close \; to \; x) \end{aligned} \tag{42}$$

Now replace $f(x + dx) - f(x)$ with Δy from equation 41 and $f'(x)(dx)$ with dy from 40 to obtain

$$\begin{aligned} f(x + dx) - f(x) &\approx f'(x)\,(dx)\;(x + dx \text{ close to } x) \\ \Rightarrow \Delta\, y &\approx f'(x)(dx) = dy\;(x + dx \text{ close to } x) \\ \Rightarrow \Delta\, y &\approx dy = f'(x)(dx)\;(x + dx \text{ close to } x) \end{aligned} \tag{43}$$

The differential dy is not the actual increment in y as x is changed to x + dx, but rather the change in y that would occur if y continued to change at the fixed rate f´(x) as x changes to x + dx. figure 9 illustrates the difference between Δy and dy.

FIGURE 9. Difference Between Δy and dy



3.9. **Rules for differentials.**

1: $d(a\,f + b\,g) = a\,df + b\,dg$
2: $d(f \cdot g) = g\,df + f\,dg$
3: $d\left(\frac{f}{g}\right) = \frac{g df - f\,dg}{g^2}, \quad g \neq 0$

3.10. **Example problems.** For each of the following find dy.

a: $y = 3x^2 + 7x - 5$
b: $y = -x\,(x^2 + 3)$
c: $y = (x - 8)\,(7x + 4)$
d: $y = \frac{x}{x^2 + 1}$

3.11. **Economic Example.** Consider the following simple macro model

$$\begin{aligned} Y &= C + I \\ C &= f(Y) \\ N &= g(Y) \end{aligned} \tag{44}$$

where Y is income, C is consumption, I is investment and N is employment. Using differentials we can find how Y and N change as we change I. First find the differential for each equation to obtain

$$\begin{aligned} dY &= d\,C + dI \\ dC &= f'(Y)dY \\ dN &= g'(Y)\,dY \end{aligned} \tag{45}$$

Now substitute for dC in the differential of the national income identity and solve for dY to obtain

$$\begin{aligned} dY &= d\,C + dI \\ &= f'(Y)\,dY + dI \\ \Rightarrow dY\,(1 - f'(Y)) &= dI \\ \Rightarrow dY &= \frac{1}{1 - f'(Y)}\,dI \end{aligned} \tag{46}$$

Similarly we can obtain dN as

$$\begin{aligned} dN &= g'(Y)\,dY \\ &= \frac{g'(Y)}{1 - f'(Y)}\,dI \end{aligned} \tag{47}$$

Now if $g'(Y) > 0$ and $0 \leq f'(Y) \leq 1$, then as investment increases so does employment.

4. ANTIDERIVATIVES

4.1. **Definition of an Antiderivative.** A function F is called the **antiderivative** of f on an interval I if F´(x) = dF/dx = f(x) for all x in I. If F is an antiderivative of f on an interval I, then the most general antiderivative of F on I is F(x) + c, where c is an arbitrary constant.

4.2. **Examples.**

1: Let F(x) = 4x and f(x) = F′(x) = 4. Then a particular antiderivative of f(x) = 4 is F(x) = 4x and a general antiderivative of f(x) = 4x +c. For example G(x) = 4x+5 has derivative G′(x) = 4 as does H(x) = 4x+22.

2: Let F(x) = ln x and f(x) = 1/x. Then a particular antiderivative of f(x) = 1/x is F(x) = ln x for x $>$ 0. f(x) is not defined for x = 0.

4.3. **Antidifferentiation Formulas.**

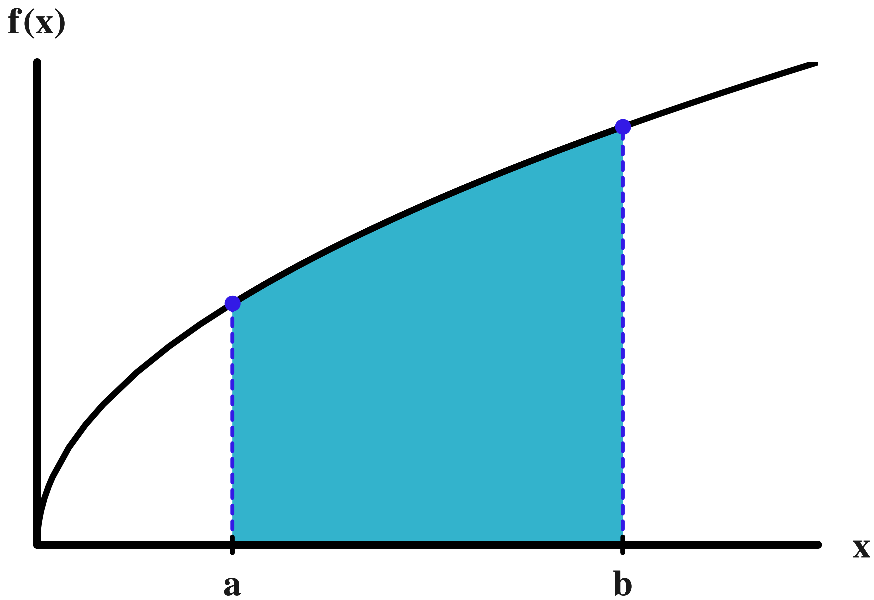
TABLE 1. **Antidifferentiation Formulas**

Function f(x)	Antiderivative F(x)
c f(x)	c F(x)
f(x) + g(x)	F(x) + G(x)
x^n, x $\neq$ -1	$\frac{x^{n+1}}{n+1}$
1/x	ln $\|x\|$
e^x	e^x

5. INTEGRATION

5.1. **Intuition.** The area under a curve f(x) between x = a and x = b is called the integral of f between a and b. In figure10 the area beneath the curve between a and b is the value of the integral of f(x) evaluated from a to b.

FIGURE 10. Integration and Area



5.2. **Riemann integral (Rudin** [1, p. 120]**).** Let P = $\{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of the closed interval [a, b] where a = $x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n$ = b and let $\Delta x_i = x_i - x_{i-1}$. Suppose f is a real bounded function defined on [a,b]. Corresponding to each partition P of [a, b] define the following

$$\begin{aligned} M_i &= \sup \; f(x) \; (x_{i-1} \leq x \leq x_i), \\ m_i &= \inf \; f(x) \; (x_{i-1} \leq x \leq x_i), \\ U(P, f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \\ L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} m_i (x_i - x_{i-1}). \end{aligned} \tag{48}$$

Consider the function defined in figure 11 over the interval [a,b]. The lower sum $(L(P,\, f) = \sum_{i=1}^{n} m_i \Delta x_i \ = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$) uses the lowest value of f(x) in any interval. In figure 12, the interval [a,b] is divided into ten parts. The approximation to the area is better than when the partition is six.

FIGURE 11. Lower Sum with Six Divisions

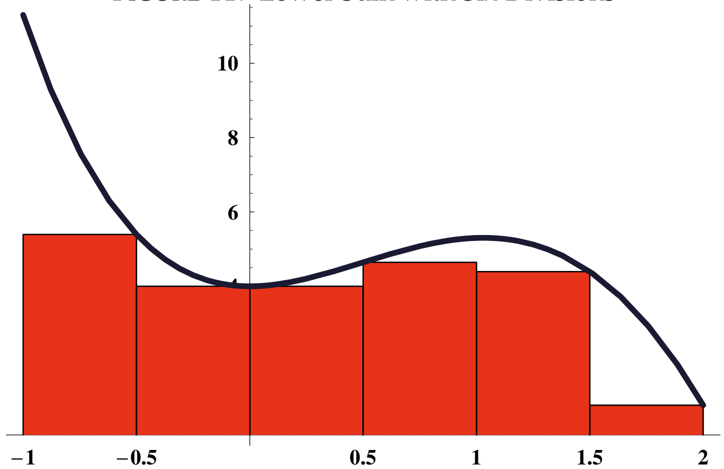
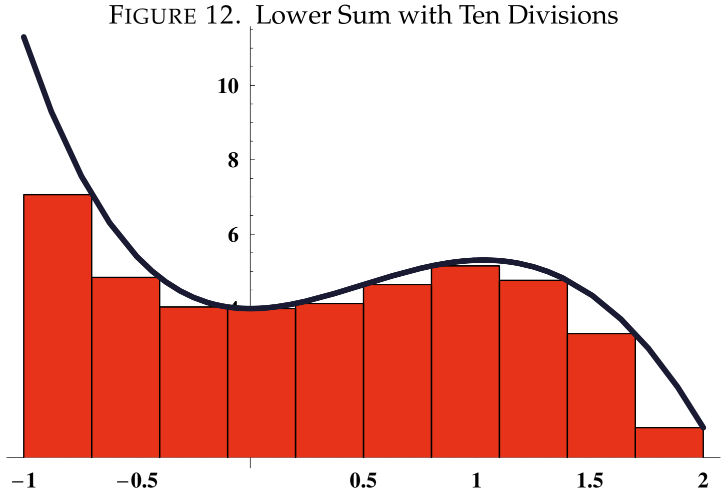


FIGURE 12. Lower Sum with Ten Divisions



The upper sum $(L(P,\, f) \ = \ \sum_{i=1}^{n} \ M_i \, \Delta x_i \ = \ \sum_{i=1}^{n} \ M_i (x_i \ - \ x_{i-1})$) uses the highest value of f(x) in any interval. In figure 13, the interval [a,b] is divided into six parts while in figure 14, it is divided into ten parts.

FIGURE 13. Upper Sum with Six Divisions

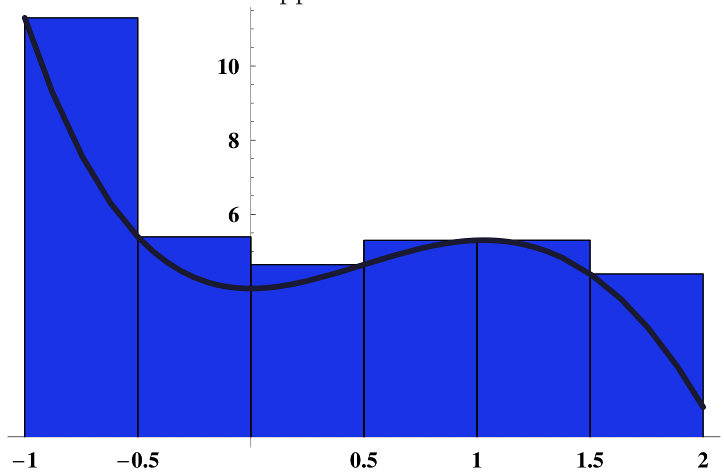
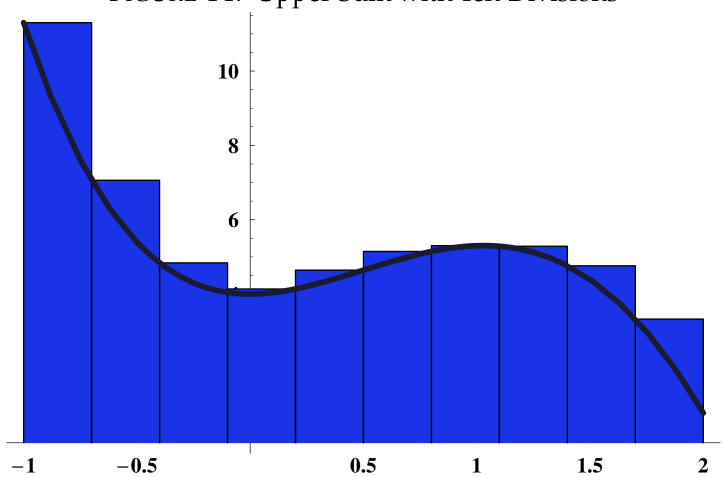


FIGURE 14. Upper Sum with Ten Divisions



Now we can define the upper and lower Riemann integrals of f over [a, b] as

$$\begin{aligned} \int_a^{-b} f(x)\,dx &= \inf\ U(p,f)\,, \\ \int_{-a}^{b} f(x)\,dx &= \sup\ L(P,f)\,, \end{aligned} \tag{49}$$

where the inf and sup are taken over all partitions P of [a,b]. If the upper and lower integrals are equal we say that f is Riemann integrable on [a,b] and write the common value of equation 49 as

$$\int_a^b f(x)\,dx \tag{50}$$

5.3. **Riemann-Stieltjes integral (Rudin** [1, p. 122]**).** Let α be a monotonically increasing function on [a, b]. Corresponding to each partition P of [a, b] define

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \tag{51}$$

where it is clear that $\Delta\alpha_i \geq 0$. Now define

$$\begin{aligned} U(P, f, \alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i\,, \\ L(P, f, \alpha) &= \sum_{i=1}^{n} m_i\, \Delta\alpha_{i.} \end{aligned} \tag{52}$$

where M_i and m_i are defined as in 48. Now define the following upper and lower integrals as

$$\begin{aligned} \int_a^{-b} f\, d\alpha &= \inf\; U(p, f, \alpha), \\ \int_{-\,a}^{b} f\, d\alpha &= \sup\; L(P, f, \alpha), \end{aligned} \tag{53}$$

where the inf and sup are taken over all partitions P of [a,b]. If the upper and lower integrals are equal we write the common value of equation 53 as

$$\int_a^b f\, d\alpha \tag{54}$$

or sometimes as

$$\int_a^b f(x)\, d\alpha(x). \tag{55}$$

The idea is that we consider the upper sum for all possible partitions and take its lowest value. Then we consider the lower sum for all possible partitions and take its highest value. These two values should be the same and are the integral of f(x) over the interval [a, b].

5.4. **Fundamental theorem of calculus.** If f is Riemann integrable on [a,b] and if there is a differentiable function F on [a,b] such that F′ = dF/dx = f then

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{56}$$

Consider for example the function F(x) = 2x where a = 3 and b = 5. In this case F′ = f = 2 and

$$\begin{aligned} \int_a^b f(x)\, dx &= F(b) - F(a) \\ \int_3^5 2\, dx &= F(5) - F(3) \\ &= 2(5) - 2(3) \\ &= 10 - 6 = 4\,. \end{aligned} \tag{57}$$

Or consider for example the function F(x) = $3x^2$ + 2x where a = 1 and b = 4. In this case F′ = f = 6x + 2 and

$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a)\\ \int_1^4 (6x + 2)\,dx &= F(4) - F(1)\\ &= [3\,(4^2) + 2(4)] - [3\,(1^2) + 2(1)]\\ &= [48 + 8] - [3 + 2]\\ &= 56 - 5 = 51\,.\end{aligned} \tag{58}$$

5.5. **Properties of the definite integral.**

1: $\int_a^b c\,dx = c\,(b-a)$, where c is any constant

2: $\int_a^b [\,f\,(x) + g(x)\,]\,dx = \int_a^b f\,(x)\,dx + \int_a^b g(x)\,dx$

3: $\int_a^b c\,f\,(x)\,dx = c\,\int_a^b f\,(x)\,dx$

4: $\int_a^b [\,f\,(x) - g(x)\,]\,dx = \int_a^b f\,(x)\,dx - \int_a^b g(x)\,dx$

5.6. **Notation used in evaluating definite integrals.**

$$G\,(x)\,|_a^b = G(b) - G(a) \tag{59}$$

So for example

$$\begin{aligned}\int_a^b f(x)\,dx &= F(x)\,|_a^b = F(b) - F(a)\\ \int_1^4 (\,6x + 2\,)\,dx &= \left(3x^2 + 2x\right)\,|_1^4\\ &= \left[\,3\,(4^2\,) + 2(4)\,\right] - \left[\,3\,(1^2\,) + 2(1)\,\right]\\ &= [\,48 + 8\,] - [3 + 2]\\ &= 56 - 5 = 51\,.\end{aligned} \tag{60}$$

5.7. **Some useful integration formulas.**

a: powers

$$\int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}|\,_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \quad n \neq -1 \tag{61}$$

b: reciprocals

$$\int_a^b \frac{1}{x}\,dx = \ln(x)|\,_a^b = \ln(b) - \ln(a) = \ln(b/a) \tag{62}$$

c: exponentials

$$\int_a^b e^x\,dx = e^x|\,_a^b = e^b - e^a \tag{63}$$

5.8. **Indefinite Integrals.** Recall that a function F is called an antiderivative of f on an interval I if F´(x) = f(x) for all x in I. An alternative notation for the antiderivative of f is called the indefinite integral and is written

$$\text{antiderivative of } f(x) \;=\; \int f(x)\; dx \tag{64}$$

Operationally we then have

$$\int f(x)\; dx \;=\; F(x) \;\text{ means } F'(x) \;=\; f(x) \tag{65}$$

Note that the definite integral $\int_a^b f(x)\; dx$ is a number while the indefinite integral $\int f(x)\; dx$ is a function. Note that we also have the following

$$\int_a^b f(x)\; dx \;=\; \int f(x)\; dx \;|_a^b \tag{66}$$

5.9. **Differentiation of Integral Expressions.**

5.9.1. *Some fundamental rules.* The derivative of an indefinite integral with respect to the variable of integration is just the integrand

$$\frac{d}{dx} \int f(x)\; dx \;=\; f(x) \tag{67}$$

Similarly the integral of a derivative of a function with respect to a variable is just that function plus a constant

$$\int F'(x)\; dx \;=\; F(x) \;+\; c \tag{68}$$

For a definite integral where F´(x) = f(x), this gives

$$\int_a^t f(x)\; dx \;=\; F(x)\;|_a^t \;=\; F(t) \;-\; F(a) \tag{69}$$

5.9.2. *Differentiation of an integral expression.* First consider the case where the lower limit of integration (a) is fixed

$$\begin{aligned} \frac{d}{dt} \int_a^t f(x)\; dx \;&= \frac{d}{dt}\; [\,F(t) \;-\; F(a)\,] \\ &= F'\,(t\,) \\ &= f\,(t\,) \end{aligned} \tag{70}$$

Similarly when the upper limit of integration is fixed

$$\begin{aligned} \int_t^a f(x)\; dx \;&= F(x)\;|_t^a \;=\; F(a) \;-\; F(t) \\ \frac{d}{dt} \int_t^a f(x)\; dx \;&= \frac{d}{dt}\; [F(a) \;-\; F(t)] \\ &= -\; F'(t) \\ &= -\; f(t) \end{aligned} \tag{71}$$

And in general if a(t) and b(t) are differentiable and f(x) is continuous we obtain

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x)\ dx \ = \ f[b(t)]\, b'(t) \ - \ f\ [a(t)]\, a'(t) \tag{72}$$

We can prove this by letting F′(x) = f(x) and writing $\int_u^v f(x)\ dx \ = \ F(v) \ - \ F(u)$. Now let u = b(t) and v = b(t), write out the definition of the integral and then differentiate

$$\begin{aligned} \int_{a(t)}^{b(t)} f(x)\ dx &= F\,[b(t)] \ - \ F[a(t)] \\ \frac{d}{dx}\left[\int_{a(t)}^{b(t)} f(x)\ dx\right] &= \frac{d}{dx}\ [F[b(t)] \ - \ F[a(t)]] \\ &= F'[b(t)]\, b'(t) \ - \ F'[a(t)\, a'(t)] \\ &= f[b(t)]\, b'(t) \ - \ f[a(t)]\, a'(t) \end{aligned} \tag{73}$$

5.10. **Integration Table.** Here is a table listing integrals for a variety of functions.

$\int [f(x) \ + \ g(x)]\, dx \ = \ \int f(x)\, dx \ + \ \int g(x)\, dx$	$\int_a^b f(x)\ dx \ = \ \int_a^c f(x)\, dx \ + \ \int_c^b f(x)\, dx$
$\int c\ f(x)\ dx \ = \ c\ \int f(x)\, dx$	$\int \ \sin(x)\ dx \ = \ - \ \cos(\ x\ ) \ + \ c$
$\int x^{n\ dx} \ = \ \frac{x^{n\,+\,1}}{n\,+\,1} \ + \ c\,(n \ \neq \ -1)$	$\int \ \cos(x)\ dx \ = \ \sin(x) \ + \ c$
$\int \frac{1}{x}\ dx \ = \ \ln\|x\| \ + \ c$	$\int \ \frac{1}{x^2\,+\,1}\ dx \ = \ \tan^{-1}(x) \ + \ c$
$\int e^x\ dx \ = \ e^x \ + \ c$	$\int \ \sec^2(x)\ dx \ = \ \tan(x) \ + \ c$
$\int \ \ln\ x\ dx \ = \ x\ \ln x \ - \ x \ + \ c$	$\int \ csc^2(\ x\ )\ dx \ = \ -\cot(x) \ + \ c$
$\int \ e^{a\,x}\ dx \ = \frac{1}{a} e^{a\,x} \ + \ c$	$\int \ \sec(x) \ \ \tan(x)\ dx \ = \ \sec(x) \ + \ c$
$\int \ p^x\ dx \ = \ \frac{p^x}{\ln\ p} \ + \ c$	$\int \ csc(x) \ \ \cot(x)\ dx \ = \ -\,csc(x) \ + \ c$
	$\int \ \frac{1}{\sqrt{1\,-\,x^2}}\ dx \ = \ \sin^{-1}(x) \ + \ c$

5.11. **Integration using substitution.**

5.11.1. *Formal Definiion of Integration by Substitution.* Consider the function y = F(g(x)). Using the chain rule we can compute the derivative dy/dx as follows.

$$\frac{dy}{dx} \ = \frac{d}{dx}\,[F(g(x))] \ = \ F'(g(x))\, g'(x) \tag{74}$$

Now consider the indefinite integral of the rightmost expression in equation 74.

$$\int F'(g(x))\, g'(x)\ dx \ = F(g(x)) \ + \ c \tag{75}$$

Now assume that F′ is an antiderivative for a function f. We then have

$$\int f(g(x))\, g'(x)\ dx \ = \ \int F'(g(x))\, g'(x)\ dx \ = F(g(x)) \ + \ c \tag{76}$$

By making the substitution u = g(x) we can often calculate the integral $\int f(g(x))\, g'(x)\ dx$ by computing $\int f(u)\, du \ = \ F(u) \ + \ c$ and then making the substitution u = g(x). Specifically

$$\begin{gathered} \int f(g(x))\, g'(x)\ dx \ = \int f(u)\ du \\ u \ = \ g(x)\,,\ du \ = g'(x)\ dx \end{gathered} \tag{77}$$

The idea is to write the original integral in an alternative form making the substitution u = g(x) and du = g′(x) dx, computing the integral $\int f(u)\,du = F(u) + c$, and then substituting back in for u.

5.11.2. *Example 1.* Consider the following example. Find

$$\int \frac{1}{(3 + 2x)^2}\, dx \tag{78}$$

Let u = 3 + 2x. Then du = 2 dx. We can then write the following

$$\begin{aligned} \int \frac{1}{(3 + 2x)^2}\, dx &= \frac{1}{2} \int \frac{2}{(3 + 2x)^2}\, dx \\ &= \frac{1}{2} \int \frac{1}{u^2}\, du \\ &= \left(\frac{1}{2}\right) \frac{-1}{u} + c \\ &= \left(\frac{-1}{2}\right) \frac{1}{(3 + 2x)} + c \\ &= -\left(\frac{1}{2\,(3 + 2x)}\right) + c \\ &= \left(\frac{-1}{2}\right) (3 + 2x)^{-1} + c \end{aligned} \tag{79}$$

We can check the answer by differentiating

$$\begin{aligned} \frac{d}{dx} \left(\frac{-1}{2}\right) (3 + 2x)^{-1} + c &= \left(\frac{-1}{2}\right) (-1)\,(3 + 2x)^{-2}\,(2) \\ &= \left(\frac{2}{2}\right) (3 + 2x)^{-2} \\ &= \frac{1}{(3 + 2x)^2} \end{aligned} \tag{80}$$

5.11.3. *Example 2.* Consider another example. Find

$$\int x \sqrt{a^2 + b^2 x^2}\, dx \tag{81}$$

Let u = a^2+ b^2x^2. Then du = $2b^2$x dx. We can then write the following

$$\begin{aligned} \int x \sqrt{a^2 + b^2 x^2} &= \frac{1}{2\,b^2} \int (2\,b^2\,x) \sqrt{a^2 + b^2 x^2}\, dx \\ &= \frac{1}{2\,b^2} \int \sqrt{u}\, du \\ &= \frac{1}{2\,b^2} \left(\frac{2}{3}\right) u^{3/2} + c \\ &= \frac{1}{3\,b^2}\, u^{3/2} + c \\ &= \frac{1}{3\,b^2}\, (a^2 + b^2 x^2)^{3/2} + c \end{aligned} \tag{82}$$

We can check the answer by differentiating

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{3\,b^2}\,(a^2\,+\,b^2\,x^2)^{3/2}]\,+\,c\right] &= \left(\frac{3}{2}\right)\left(\frac{1}{3\,b^2}\right)\,(a^2\,+\,b^2\,x^2\,)^{1/2}\,2\,b^2\,x \\ &= \left(\frac{3}{2}\right)\left(\frac{2\,b^2\,x}{3\,b^2}\right)\,(a^2\,+\,b^2\,x^2\,)^{1/2} \\ &= x\,(a^2\,+\,b^2\,x^2\,)^{1/2} \\ &= x\,\sqrt{a^2\,+\,b^2\,x^2} \end{aligned} \tag{83}$$

5.12. **Integration by parts.**

5.12.1. *Procedure for Integration by Parts.* Consider the function h(x) = f(x)g(x). Using the product rule and differentiating we obtain

$$\frac{d}{dx}h(x)\;=\;\frac{d}{dx}\;f(x)\,g(x)\;=f(x)\,g'(x)\;+\;f'(x)\,g(x) \tag{84}$$

Now integrate both sides of the expression as follows

$$\begin{aligned} \int\frac{d}{dx}h(x)\,dx\;=\;\int\frac{d}{dx}\;f(x)\,g(x)\,dx\;&=\int\;f(x)\,g'(x)\,dx\;+\;\int\;f'(x)\,g(x)\,dx \\ \Rightarrow\;f(x)\,g(x)\;+\;c\;&=\int\;f(x)\,g'(x)\,dx\;+\;\int\;f'(x)\,g(x)\,dx \\ \Rightarrow\;\int\;f(x)\,g'(x)\,dx\;&=f(x)\,g(x)\;-\;\int\;f'(x)\,g(x)\,dx\;+\;c \\ \Rightarrow\;\int\;f(x)\,g'(x)\,dx\;&=f(x)\,g(x)\;-\;\int\;f'(x)\,g(x)\,dx \end{aligned} \tag{85}$$

We can use this information to compute $\int\;f(x)\,g'(x)\,dx$ by computing $\int\;f'(x)\,g(x)\,dx$ when the second integral is easier to compute. In practice we typically make the following substitutions

$$\begin{aligned} u\;&=f(x), & dv\;&=\;g'(x)\,dx \\ du\;&=f'(x)\,dx, & v\;&=\;g(x) \end{aligned} \tag{86}$$

5.12.2. *Example 1 of Integration by Parts.* Consider the following example. Find

$$\int\;x\,e^x\,dx \tag{87}$$

Set

$$\begin{aligned} u\;&=x, & dv\;&=\;e^x\,dx \\ du\;&=dx, & v\;&=\;e^x \end{aligned} \tag{88}$$

Then

$$\begin{aligned} \int x\, e^x\, dx &= \int u\, dv \\ &= uv \; - \int v\, du \\ &= x\, e^x \; - \int e^x\, dx \\ &= x\, e^x \; - \; e^x + \; c \end{aligned} \tag{89}$$

5.12.3. *Example 2 of Integration by Parts.* Find

$$\int \frac{x\, e^x}{(x \; + \; 1)^2} \; dx \tag{90}$$

Set

$$\begin{aligned} u \; &= x e^x, & dv \; &= \; \frac{1}{(x \; + \; 1)^2} \, dx \\ du \; &= (x e^x \; + \; e^x) dx, & v \; &= \; \frac{-1}{x \; + \; 1} \end{aligned} \tag{91}$$

Then

$$\begin{aligned} \int \frac{x\, e^x}{(x \; + \; 1)^2} \, dx \; &= \int u\, dv \\ &= uv \; - \int v\, du \\ &= \frac{-\, x\, e^x}{x \; + \; 1} \; - \int \frac{-1}{x \; + \; 1} \; (x\, e^x \; + \; e^x \,) \, dx \\ &= \frac{-\, x\, e^x}{x \; + \; 1} \; - \int \frac{-1}{x \; + \; 1} e^x \; (x \; + \; 1 \,) \, dx \\ &= \frac{-\, x\, e^x}{x \; + \; 1} \; + \int e^x\, dx \\ &= \frac{-\, x\, e^x}{x \; + \; 1} \; + \; e^x \; + \; c \\ &= \frac{-\, x\, e^x}{x \; + \; 1} \; + \; \frac{e^x \,(x \; + \; 1)}{x \; + \; 1} \; + \; c \\ &= \frac{e^x}{x \; + \; 1} \; + \; c \end{aligned} \tag{92}$$

We can check the answer by differentiating

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^x}{x \; + \; 1} \; + \; c \right] \; &= \frac{(x + \, 1)\, e^x \; - \; e^x}{(x \; + \; 1)^2} \\ &= \frac{x\, e^x}{(x \; + \; 1)^2} \end{aligned} \tag{93}$$

REFERENCES

[1] Rudin, Walter. *Principles of Mathematical Analysis, 3^{rd} Edition.* New York: McGraw Hill, 1976.