VARIOUS TOOLS FOR COMPARATIVE STATICS

1. The chain rule (or total derivative) for composite functions of several variables

1.1. **Chain rule for functions of two variables.** When y = f (x_1, x_2) with x_1 = g(t) and x_2 = h(t), then

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} \\ &= \frac{\partial f}{\partial x_1}\frac{dg(t)}{dt} + \frac{\partial f}{\partial x_2}\frac{dh(t)}{dt} \end{aligned} \tag{1}$$

This is usually called the total derivative of y with respect to t.

1.2. **Example.** Let the function y = f(x_1, x_2) be given by

$$y = f(x_1, x_2) = x_1^2 + x_2^3$$

with

$$\begin{aligned} x_1(t) &= t^2 + 2t + 1 \\ x_2(t) &= 3t \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2x_1, \ \frac{\partial f}{\partial x_2} = 3x_2^2 \\ \frac{dx_1}{dt} &= 2t + 2, \ \frac{dx_2}{dt} = 3 \\ \Rightarrow \frac{dy}{dt} &= (2x_1)(2t + 2) + (3x_2^2)(3) \\ &= (2t^2 + 4t + 2)(2t + 2) + (27t^2)(3) \\ &= 4t^3 + 4t^2 + 8t^2 + 8t + 4t + 4 + 81t^2 \\ &= 4t^3 + 93t^2 + 12t + 4 \end{aligned}$$

If we multiply out the expression for y = f(x_1, x_2) substituting x_1(t) and x_2(t) we obtain

$$\begin{aligned} y = f(x_1, x_2) &= x_1^2 + x_2^3 \\ &= (t^2 + 2t + 1)^2 + (3t)^3 \\ &= t^4 + 2t^3 + t^2 + 2t^3 + 4t^2 + 2t + t^2 + 2t + 1 + 27t^3 \end{aligned}$$

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Taking the derivative with respect to t we obtain

$$\begin{aligned}\frac{df}{dt} &= 4t^3 + 6t^2 + 2t + 6t^2 + 8t + 2 + 2t + 2 + 81t^2 \\ &= 4t^3 + 93t^2 + 12t + 4\end{aligned}$$

1.3. **In-class exercises.** Find the total derivative of each of the following with respect to t.

(1)

$$\begin{aligned} y &= f(x_1, x_2) = x_1^2 + x_2^3 \\ x_1(t) &= t^2 \\ x_2(t) &= 2t \end{aligned}$$

(2)

$$\begin{aligned} y &= f(x_1, x_2) = x_1^2 + x_2^2 \\ x_1(t) &= t^2 + 2t \\ x_2(t) &= 2t + 1 \end{aligned}$$

(3)

$$\begin{aligned} y &= f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 \\ x_1(t) &= t^2 + 2t + 3 \\ x_2(t) &= 2t - t^2 \end{aligned}$$

(4)

$$\begin{aligned} y &= f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 \\ x_1(t) &= t^2 + 2t \\ x_2(t) &= 2t \\ x_3(t) &= t^2 - 5t \end{aligned}$$

(5)

$$\begin{aligned} y &= f(x_1, x_2) = \frac{x_1^2 + x_2^2}{x_1 + x_2} \\ x_1(t) &= t^2 + 2t \\ x_2(t) &= 2t + 1 \end{aligned}$$

2. Directional Derivatives

2.1. **Idea.** If y = f(x_1, x_2), the partial derivatives, $\frac{\partial f}{\partial x_1}$ $\frac{\partial f}{\partial x_2}$ measure the rates of change of f(x_1, x_2), in the directions of the x_1 - axis and the x_2 - axis, respectively. We can also measure the rate of change of the function in other directions. Consider a particular point in the domain of f and denote it (x_1^0, x_2^0) . Any non-zero vector (h, k) is then a direction in which we move away from the point (x_1^0, x_2^0) in a straight line to points of the form

$$(x_1, x_2) = (x_1(t), x_2(t)) = \left(x_1^0 + th,\ x_2^0 + tk\right) \tag{2}$$

Given any initial point (x_1^0, x_2^0) and any direction (h, k), define the directional function g by

$$g(t) = f(x_1^0 + th, x_2^0 + tk) \tag{3}$$

The derivative of this function is

$$\begin{aligned} \frac{dg}{dt} &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} \\ &= \frac{\partial f}{\partial x_1}(x_1^0 + th, x_2^0 + tk)h + \frac{\partial f}{\partial x_2}(x_1^0 + th, x_2^0 + tk)k \end{aligned} \tag{4}$$

Now let t = 0 so that we are at the point (x_1^0, x_2^0). Then we obtain

$$\frac{dg}{dt}(0) = \frac{\partial f}{\partial x_1}(x_1^0, x_2^0)h + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)k \tag{5}$$

If the vector (h, k) has length 1, the derivative of f in the direction (h, k) is called the directional derivative of f in the direction of (h, k) at (x_1^0, x_2^0). Specifically, the directional derivative of f(x_1, x_2) at (x_1^0, x_2^0) in the direction of the unit vector (h, k) is

$$D_{h,k}\, f(x_1^0, x_2^0) = \frac{\partial f}{\partial x_1}(x_1^0, x_2^0)h + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)k \tag{6}$$

Note that when the length of (h, k) is one, a move away from (x_1^0, x_2^0) in the direction (h, k) changes the value of f by approximately $D_{h,k}f(x_1^0, x_2^0)$. Also notice that the directional derivative is the product of the gradient of f and the vector (h, k).

2.2. **Example.** Consider the function f(x_1, x_2) with the following direction and initial point.

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + 3x_2^2 \\ Direction &= (2, 5) \\ Point &= (1, 1) \end{aligned}$$

First normalize the direction vector. Because the length of the vector is $\sqrt{29}$ we can normalize it as $\left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$. Then find the gradient of f as

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + 3x_2^2 \\ \frac{\partial f}{\partial x_1} &= 2x_1 \\ \frac{\partial f}{\partial x_2} &= 6x_2 \end{aligned}$$

Evaluated at (1,1) we obtain

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2 \\ \frac{\partial f}{\partial x_2} &= 6 \end{aligned}$$

The directional derivative is then given by

$$\begin{pmatrix} 2 & 6 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{29}}, \\ \frac{5}{\sqrt{29}} \end{pmatrix} = \frac{34}{\sqrt{29}}$$

3. More general chain rules

3.1. **General form of the chain rule.** Let y = f(x_1, x_2, ... x_n) and let x_1 = g_1 (t_1, t_2, ..., t_m), x_2 = g_2 (t_1, t_2, ..., t_m), ... , x_n= g_n (t_1, t_2, ..., t_m) where t is an m-vector of other variables upon which the x vector depends. Then the following holds

$$\begin{aligned} \frac{\partial y}{\partial t_j} &= \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_j}, \quad j = 1, 2, \cdots, n \\ &= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_j}, \quad j = 1, 2, \cdots, n \end{aligned} \tag{7}$$

3.2. **Example.** Consider the function y = f(x_1, x_2) along with the auxiliary functions x_1(z, w) and x_2 (z, w)

$$\begin{aligned} y = f(x_1, x_2) &= 3x_1 + 2x_1 x_2^2 \\ x_1(z, w) &= 5z + 2zw \\ x_2(z, w) &= zw^2 + 3w \end{aligned}$$

where t_1 = z and t_2 = w from equation 7 . We can find the partial derivative of y with respect to z using equation 7 as follows.

$$\begin{aligned} \frac{\partial y}{\partial z} &= \frac{\partial f(x_1, x_2)}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{\partial x_2}{\partial z} \\ &= (3 + 2x_2^2)(5 + 2w) + (4x_1x_2)(w^2) \\ &= (3 + 2(z^2w^4 + 6zw^3 + 9w^2))(5 + 2w) + 4(5z + 2zw)(zw^2 + 3w)w^2 \\ &= (3 + 2z^2w^4 + 12zw^3 + 18w^2)(5 + 2w) + (20zw^2 + 8zw^3)(zw^2 + 3w) \\ &= 15 + 6w + 10z^2w^4 + 4z^2w^5 + 60zw^3 + 24zw^4 + 90w^2 + 36w^3 \\ &+ 20z^2w^4 + 60zw^3 + 8z^2w^5 + 24zw^4 \\ &= 15 + 6w + 90w^2 + 36w^3 + 120zw^3 + 48zw^4 + 30z^2w^4 + 12z^2w^5 \end{aligned}$$

We can also find the partial of y with respect to w as

$$\begin{aligned} \frac{\partial y}{\partial w} &= \frac{\partial f(x_1, x_2)}{\partial x_1}\frac{\partial x_1}{\partial w} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{\partial x_2}{\partial w} \\ &= (3 + 2x_2^2)(2z) + (4x_1x_2)(2zw + 3) \\ &= (3 + 2(z^2w^4 + 6zw^3 + 9w^2))(2z) + 4(5z + 2zw)(zw^2 + 3w)(2zw + 3) \\ &= (3 + 2z^2w^4 + 12zw^3 + 18w^2)(2z) + (20z + 8zw)(zw^2 + 3w)(2zw + 3) \\ &= (6z + 4z^3w^4 + 24z^2w^3 + 36zw^2) + (20z^2w^2 + 60zw + 8z^2w^3 + 24zw^2)(2zw + 3) \\ &= 6z + 4z^3w^4 + 24z^2w^3 + 36zw^2 + 40z^3w^3 + 120z^2w^2 + 16z^3w^4 + 48z^2w^3 \\ &+ 60z^2w^2 + 180zw + 24z^2w^3 + 72zw^2 \\ &= 6z + 180zw + 108zw^2 + 180z^2w^2 + 96z^2w^3 + 40z^3w^3 + 20z^3w^4 \end{aligned}$$

4. Linear Approximations and Differentials

4.1. **Differentials.** Consider a function y = f(x_1, x_2, ... , x_n). If dx_1, dx_2, ... , dx_n are arbitrary real numbers (not necessarily small), we define the differential of y = f(x_1, x_2, ... , x_n) as

$$dy \;=\; \frac{\partial f}{\partial x_1} dx_1 \;+\; \frac{\partial f}{\partial x_2} dx_2 \;+\; \ldots \;+\; \frac{\partial f}{\partial x_n} dx_n \tag{8}$$

When x_i is changed to x_i + dx_i. then the actual change in the value of the function is the **increment**

$$\Delta y \;=\; f\,(x_1 \,+\, dx_1,\; x_2 \,+\, dx_2, \cdots,\; x_n \,+\, dx_n\,) \;-\; f\,(x_1, x_2, \cdots, x_n) \tag{9}$$

If dx_i is small in absolute value, the Δy can be approximated by dy

$$\Delta y \;\approx\; dy \;=\; \frac{\partial f}{\partial x_1} dx_1 \;+\; \frac{\partial f}{\partial x_2} dx_2 \;+\; \cdots \;+\; \frac{\partial f}{\partial x_n} dx_n \tag{10}$$

4.2. **Rules for differentials.**

1: dc = 0 (c is a constant)
2: d(cx^n) = cnx^{n-1} dx
3: $d\,(a\,f \;+\; b\,g) \;=\; a\,df \;+\; b\,dg$ (a and b are constants)
4: $d\,(f\,g) \;=\; g\,df \;+\; f\,dg$
5: $d\left(\frac{f}{g}\right) \;=\; \frac{g\,df \;-\; f\,dg}{g^2}, \quad g \;\neq\; 0$
6: $d\,(f\,g\,h) \;=\; g\,h\;df \;+\; f\,h\;dg \;+\; f\,g\;dh$
7: If y = g[f(x_1, x_2, ... , x_n)] then dy = g'[f(x_1, x_2, ... , x_n)] df.

4.3. **Differentials and systems of equations.**

4.3.1. *Idea.* We can find partial derivatives of implicit systems using differentials. We take the total differential of both sides of each equation, set all differentials of variables that are not changing equal to zero, and then divide each equation by the differential of the one exogenous variable that is changing. We then solve the resulting system for the various partial derivatives.

4.3.2. *Example 1.* Consider the system

$$\begin{aligned} \phi_1(x_1,\; x_2,\; p, w_1, w_2) \;&=14\,p \;-\; 2\,p\,x_1 \;-\; w_1 \;=\; 0 \\ \phi_2(x_1,\; x_2\; p, w_1, w_2) \;&=11\,p \;-\; 2\,p\,x_2 \;-\; w_2 \;=\; 0 \end{aligned} \tag{11}$$

The total differential of each equation is

$$\begin{aligned} 14\,dp \;-\; 2\,p\,dx_1 \;-\; 2\,x_1\,dp \;-\; d\,w_1 \;&=0 \\ 11\,dp \;-\; 2\,p\,dx_2 \;-\; 2\,x_2\,dp \;-\; d\,w_2 \;&=0 \end{aligned} \tag{12}$$

Now set dw_1 = dw_2 = 0 and divide each equation by dp

$$\begin{aligned} 14 \;-\; 2\,p\,\frac{dx_1}{dp} \;-\; 2\,x_1 \;&=0 \\ 11 \;-\; 2\,p\,\frac{dx_2}{dp} \;-\; 2\,x_2 \;&=0 \end{aligned} \tag{13}$$

Solving we obtain

$$
\begin{aligned}
2\,p\,\frac{\partial x_1}{\partial p} &= 14 \;-\; 2\,x_1 \\
\Rightarrow \;\; \frac{\partial x_1}{\partial p} &= \frac{14 \;-\; 2x_1}{2\,p} \;=\; \frac{7 \;-\; x_1}{p} \\
2\,p\,\frac{\partial x_1}{\partial p} &= 11 \;-\; 2\,x_1 \\
\Rightarrow \;\; \frac{\partial x_2}{\partial p} &= \frac{11 \;-\; 2x_1}{2\,p} \;=\; \frac{5.5 \;-\; x_2}{p}
\end{aligned}
\tag{14}
$$

4.3.3. *Example 2.* Consider the following macroeconomic model:

$$
\begin{aligned}
Y &= C \;+\; I \;+\; G \\
C &= f\,(Y \;-\; T) \\
I &= h\,(r) \\
r &= m\,(M)
\end{aligned}
\tag{15}
$$

The variables are defined as follows: Y is national income, C is consumption, I is investment, G is public expenditure, T is tax revenue, r is the interest rate, and M is money supply. There are seven variables and four equations so we can potentially solve for 4 endogenous variables in terms of 3 exogenous variables. If we assume that f, h, and m are differentiable functions with $0 < f' < 1, h' < 0$, and $m' < 0$, then these equations determine Y, C, I, and r as differentiable functions of M, T, and G. We can also find the differentials of Y, C, I, and r in terms of the differentials of M, T, and G. The total differential of the system is

$$dY = dC \;+\; dI \;+\; dG \tag{16a}$$

$$dC = f'(Y \;-\; T)\,(dY \;-\; dT) \tag{16b}$$

$$dI = h'(r)\; dr \tag{16c}$$

$$dr = m'(M)\; dM \tag{16d}$$

We need to solve this system for the differential changes dY, dC, dI, and dr in terms of the differential changes dM, dT, and dG in the exogenous policy variables M, T, and G. From equations 16c and 16d, we can find dI and dr as follows

$$
\begin{aligned}
dr &= m'(M)\; dM \\
dI &= h'(r)\; m'(M)\; dM
\end{aligned}
\tag{17}
$$

Inserting the expression for dI from equation 17 into the first two equations in 16 gives

$$
\begin{aligned}
dY \;-\; dC &= h'(r)\; m'(M)\; dM \;+\; dG \\
f'(Y \;-\; T)\; dY \;-\; dC &= f'(Y \;-\; T)\; dT
\end{aligned}
\tag{18}
$$

This gives two equations to determine the two unknowns dY and dC in terms of dM, dG, and dT. We can write this in matrix form as follows

$$
\begin{bmatrix} 1 & -1 \\ f'(Y \;-\; T) & -1 \end{bmatrix}
\begin{bmatrix} d\,Y \\ d\,C \end{bmatrix}
=
\begin{bmatrix} c\; h'(r)\; m'(M)\; dM \;+\; dG \\ f'(Y \;-\; T)\; dT \end{bmatrix}
\tag{19}
$$

We can use Cramer's rule to solve this system. The determinant of the coefficient matrix is given by

$$D \;=\; \begin{vmatrix} 1 & -1 \\ f'(Y \;-\; T) & -1 \end{vmatrix} \;=\; (-1) \;-\; (-f'(Y \;-\; T)) \;=\; f'(Y \;-\; T) \;-\; 1 \tag{20}$$

First solving for dY we obtain

$$dY \;=\; \frac{\begin{vmatrix} h'(r)\, m'(M)\, dM \;+\; dG & -1 \\ f'(Y \;-\; T)\, dT & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ f'(Y \;-\; T) & -1 \end{vmatrix}} \;=\; \frac{\begin{vmatrix} h'(r)\, m'(M)\, dM \;+\; dG & -1 \\ f'(Y \;-\; T)\, dT & -1 \end{vmatrix}}{f'(Y \;-\; T) \;-\; 1}$$

$$\begin{aligned} \Rightarrow\; dY \;&=\; \frac{-\,h'(r)\, m'(M)\, dM \;-\; dG \;+\; f'(Y \;-\; T)\, dT}{f'(Y \;-\; T) \;-\; 1} \\ &=\; \frac{-\,h'(r)\, m'(M)}{f'(Y \;-\; T) \;-\; 1}\, dM \;-\; \frac{1}{f'(Y \;-\; T) \;-\; 1}\, dG \;+\; \frac{f'(Y \;-\; T)}{f'(Y \;-\; T) \;-\; 1}\, dT \\ &=\; \frac{h'm'}{1 \;-\; f'}\, dM \;-\; \frac{f'}{1 \;-\; f'}\, dT \;+\; \frac{1}{1 \;-\; f'}\, dG \end{aligned} \tag{21}$$

Then solving for dC we obtain

$$dC \;=\; \frac{\begin{vmatrix} 1 & h'(r)\, m'(M)\, dM \;+\; dG \\ f'(Y \;-\; T) & f'(Y \;-\; T)\, dT \end{vmatrix}}{\begin{vmatrix} 1 & h'(r)\, m'(M)\, dM \;+\; dG \\ f'(Y \;-\; T) & f'(Y \;-\; T)\, dT \end{vmatrix}} f'(Y \;-\; T) \;-\; 1$$

$$\begin{aligned} \Rightarrow\; dC \;&=\; \frac{f'(Y \;-\; T)\, dT - h'(r)\, m'(M)\, dM\, f'(Y \;-\; T) \;-\; dG\, f'(Y \;-\; T)}{f'(Y \;-\; T) \;-\; 1} \\ &=\; \frac{-\,h'(r)\, m'(M)\, f'(Y \;-\; T)}{f'(Y \;-\; T) \;-\; 1}\, dM \;-\; \frac{f'(Y \;-\; T)}{f'(Y \;-\; T) \;-\; 1}\, dG \;+\; \frac{f'(Y \;-\; T)}{f'(Y \;-\; T) \;-\; 1}\, dT \\ &=\; \frac{f'\, h'\, m'}{1 \;-\; f'}\, dM \;-\; \frac{f'}{1 \;-\; f'}\, dT \;+\; \frac{f'}{1 \;-\; f'}\, dG \end{aligned} \tag{22}$$

We have now found the differentials dY, dC, dI, and dr as linear functions of dM, dT, and dG. If we set dM and dG equal to zero, then

$$\begin{aligned} dY \;&=\; -\;\frac{f'}{1 \;-\; f'}\, dT \\ \Rightarrow\; \frac{\partial Y}{\partial T} \;&=\; -\;\frac{f'}{1 \;-\; f'} \end{aligned} \tag{23}$$

Similarly ∂r/∂T=0 and ∂I/∂T=0. Because we assumed that 0 < f' < 1, ∂Y/∂T =-f′(1 - f′)< 0. If dM, dT, and dG are small in absolute value, then

$$\Delta Y \;=\; Y(M_0 \;+\; dM,\; T_0 \;+\; dT,\; G_0 \;+\; dG) \;-\; Y(M_0,\; T_0,\; G_0) \;\approx\; dY$$

References

[1] Hadley, G. *Linear Algebra*. Reading, MA: Addison-Wesley, 1961.
[2] Sydsaeter, Knut. *Topics in Mathematical Analysis for Economists*. New York: Academic Press, 1981.