# FUNCTIONS AND EQUATIONS

## 1. SETS AND SUBSETS

1.1. **Definition of a set.** A set is any collection of objects which are called its elements. If x is an element of the set S, we say that x belongs to S and write

$$x \varepsilon S.$$

If y does not belong to S, we write

$$y \notin S.$$

The simplest way to represent a set is by listing its members. We use the notation

$$A = \{1, 2, 4\}$$

to denote the set whose elements are the numbers 1, 2 and 4. We can also describe a set by describing its elements instead of listing them. For example we use the notation

$$C = \{x : x^2 + 2x - 3 = 0\}$$

to denote the set of all solutions to the equation $x^2 + 2x - 3 = 0$. The set is {-1, 3}.

1.2. **Subsets.**

1.2.1. *Definition of a subset.* If all the elements of a set X are also elements of a set Y, then X is a subset of Y and we write

$$X \subseteq Y$$

where $\subseteq$ is the set-inclusion relation. Consider the following five sets.

$$\begin{aligned} A &= \{1, 2, 4\} \\ B &= \{1, 2, 4, 5\} \\ C &= \{1, 2, 4, 5\} \\ D &= \{1, 2, 4, 5, 8, 10\} \\ E &= \{1, 2, 3, 4, 5, 8, 10\} \end{aligned}$$

The following relations are all true.

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$$
\begin{aligned}
&A \subseteq B, \quad A \subseteq C, \quad A \subseteq D, \quad A \subseteq E \\
&B \subseteq C, \quad B \subseteq D, \quad B \subseteq E \\
&C \subseteq B, \quad C \subseteq D, \quad C \subseteq E \\
&D \subseteq E
\end{aligned}
$$

1.2.2. *Definition of a proper subset.* If all the elements of a set X are also elements of a set Y, but not all elements of Y are in X, then X is a **proper** subset of Y and we write

$$X \subset Y$$

where $\subset$ is the proper set-inclusion relation. The following relations are all true.

$$
\begin{aligned}
&A \subset B, \quad A \subset C, \quad A \subset D, \quad A \subset E \\
&B \subset D, \quad B \subset E \\
&C \subset D, \quad C \subset E \\
&D \subset E
\end{aligned}
$$

1.2.3. *Definition of equality of sets.* Two sets are equal if they contain exactly the same elements, and we write

$$X = Y.$$

For example B = C.

1.2.4. *Examples of sets.*

- **a:** all corn farmers in Iowa
- **b:** all firms producing steel
- **c:** the set of all consumption bundles that a given consumer can afford
- **d:** the set of all combinations of outputs that can be produced by a given set of inputs

1.3. **Set operations.**

1.3.1. *Intersections.* The intersection, W, of two sets X and Y is the set of elements that are in both X and Y. We write

$$W = X \cap Y = \{x : x \in X \, and \, x \in Y\}.$$

The definition is symmetric: that is, $A \cap B = B \cap A$. Also, $A \cap B \subseteq A$ and $A \cap B = A \iff A \subseteq B$. We have $(A \cap B) \cap C = A \cap (B \cap C)$ and write this as $A \cap B \cap C$.

1.3.2. *Empty or null sets.* The empty set or the null set is the set with no elements. The empty set is written $\emptyset$. For example, if the sets A and B contain no common elements we can write

$$A \cap B = \emptyset$$

and these two sets are said to be disjoint.

1.3.3. *Unions.* The union of two sets A and B is the set of all elements in one or the other of the sets. We write

$$V = A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

We have

$$\begin{aligned} A \cup B &= B \cup A \\ A \cup (B \cup C) &= (A \cup B) \cup C = A \cup B \cup C \\ A &\subseteq A \cup B \\ A &= A \cup B \iff B \subseteq A \end{aligned}$$

1.3.4. *Distributive laws.*

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

Consider the following three sets.

$$\begin{aligned} A &= \{1, 2, 4\} \\ B &= \{1, 2, 3, 5\} \\ C &= \{2, 4, 6\} \end{aligned}$$

Then we have

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5\} \\ A \cup C &= \{1, 2, 4, 6\} \\ B \cup C &= \{1, 2, 3, 4, 5, 6\} \\ A \cup B \cup C &= \{1, 2, 3, 4, 5, 6\} \\ A \cap B &= \{1, 2\} \\ A \cap C &= \{2, 4\} \\ B \cap C &= \{2\} \\ A \cap B \cap C &= \{2\} \end{aligned}$$

We also have

$$\begin{aligned} A \cap (B \cup C) &= \{1, 2, 4\} \\ (A \cap B) \cup (A \cap C) &= \{1, 2, 4\} \\ A \cup (B \cap C) &= \{1, 2, 4\} \\ (A \cup B) \cap (A \cup C) &= \{1, 2, 4\} \end{aligned}$$

1.3.5. *Complements.* The complement of a set X is the set of elements of the universal set U that are not elements of X, and is written $X^C$. Thus

$$X^C = \{x : \in U : x \notin X\}.$$

We have

$$\begin{aligned} \emptyset^C &= U \\ U^C &= \emptyset \\ (A^C)^C &= A \\ A \cup A^C &= U \\ A \cap A^C &= \emptyset \\ A \subseteq B &\iff B^C \subseteq A^C \end{aligned}$$

1.3.6. *De Morgan's laws.*

$$\begin{aligned} (A \cup B)^C &= A^C \cap B^C \\ (A \cap B)^C &= A^C \cup B^C \end{aligned}$$

Consider the following three sets.

$$\begin{aligned} U &= \{1, 2, 3, 4, 5\} \\ A &= \{1, 2, 4\} \\ B &= \{1, 2, 3, 5\} \end{aligned}$$

Then

$$\begin{aligned} A \cap B &= \{1, 2\} \\ A \cup B &= \{1, 2, 3, 4, 5\} = U \\ A^C &= \{3, 5\} \\ B^C &= \{4\} \\ (A \cap B)^C &= \{3, 4, 5\} \\ (A \cup B)^C &= \emptyset \end{aligned}$$

1.3.7. *Set difference or relative complement.* If A and B are two subsets of X, then define the set difference B \ A, or the relative complement of A in B, as the set of elements in B that are not in A. Thus

$$B \setminus A = \{x : x \in B \text{ and } x \notin A\}$$

We have $B \setminus A = B \cap A^C$.

Consider the following three sets.

$$\begin{aligned} X &= \{1,\, 2,\, 3,\, 4,\, 5\} \\ A &= \{1,\, 2,\, 4\} \\ B &= \{1,\, 3,\, 5\} \end{aligned}$$

Then

$$\begin{aligned} A \setminus B &= \{2,\, 4\} \\ B \setminus A &= \{3,\, 5\} \end{aligned}$$

## 2. NUMBERS

### 2.1. **Natural numbers.** The natural numbers are the elements of the set

$$Z_+ \;=\; \{1,\, 2,\, 3,\, \ldots\}.$$

A group of three dots, . . . , called an ellipsis, indicates that the numbers continue in the indicated pattern. If the ellipsis contains nothing to the right, it is assumed that the pattern continues forever.

The natural numbers are closed under addition meaning that for numbers a and b that are elements of $Z_+$, then a + b $\varepsilon$ $Z_+$. Similarly, a $\times$ b $\varepsilon$ $Z_+$ and we say that the natural numbers are closed under multiplication.

### 2.2. **Prime numbers.** A prime number is a natural number greater than one that is divisible only by itself and 1, i.e., {2,3,5,7,11,13,17,19,23,. . . }. The first 25 prime numbers are as follows.

TABLE 1. Prime Numbers

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 |
| 13 | 17 | 19 | 23 | 29 |
| 31 | 37 | 41 | 43 | 47 |
| 53 | 59 | 61 | 67 | 71 |
| 73 | 79 | 83 | 89 | 97 |

### 2.3. **Composite numbers.** A composite number is a natural number greater than one that is not prime.

$$\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30 \ldots\}$$

### 2.4. **Integers.** The integers are the elements of the set

$$Z \;=\; \{\cdots,\; -3,\; -2,\; -1,\; 0,\; 1,\; 2,\; 3,\; \cdots\}.$$

The integers are closed under addition, subtraction and multiplication. The positive integers are the same as the natural numbers.

### 2.5. **Rational numbers.** The rational numbers are the elements of the set

$$Q \;=\; \left\{ \frac{a}{b} : \, a \,\varepsilon\, Z \,,\; b \,\varepsilon\, Z - \{0\} \right\}.$$

The expression $\frac{a}{b}$ is called a fraction in the sense that a represents a portion of b. We call a the numerator of the fraction and b the denominator of the fraction. An integer *n* is also a rational number because n = $\frac{n}{1}$. The rational numbers are closed under addition, subtraction, multiplication and division.

2.6. **Irrational numbers.** Numbers that cannot be written as ratios or quotients of integers are called irrational numbers. Examples include $\sqrt{2}$, $\sqrt{5}$, $\sqrt{12}$, $\pi$ . To see why $\sqrt{2}$ is not a rational number see Appendix A.

2.7. **Real numbers.**

2.7.1. *Definition of the real numbers.* The union of the sets of rational and irrational numbers is the set of real numbers. We can think of the set of real numbers, R, as extending along a line to infinity in each direction having no breaks or gaps.

2.7.2. *Writing real numbers as decimal fractions.* We can write any real number as an infinite decimal fraction. For example we can write 354.651 as $3 \times 10^2$+ $5 \times 10^1$+$4 \times 10^0$+ $\frac{6}{10^1}$+ $5 \times 10^{-2}$+$1 \times 10^{-3}$. In general a real number can be written as x = $\pm m.\beta_1\beta_2\beta_3\ldots$, where m is an integer and $\beta_n$ ( n = 1, 2, . . . , ) is an infinite series of digits, each in the range 0 to 9.

Rational numbers can be written as finite or recurring decimal fractions, that is, after a certain place in the decimal representation, it either stops or continues to repeat a finite sequence of digits. For example,

$$\begin{aligned} \frac{3}{4} &= 0.75 \\ \frac{2}{3} &= 0.6\overline{6} \\ \frac{4}{11} &= 0.36\overline{36} \\ \frac{4}{13} &= 0.307692\overline{307692} \end{aligned}$$

Irrational numbers are non-repeating decimal fractions. For example,

$$\begin{aligned} \pi &= 3.1415926535898\ldots \\ \sqrt{2} &= 1.4142135623731\ldots \\ e &= 2.7182818284590\ldots \\ 10^{1/3} &= 2.1544346900319 \end{aligned}$$

Numbers of the form $n^{1/m}$ are irrational unless n is the $m^{th}$ power of an integer. For example, $625^{1/4}$ = 5, i.e., 625 = $5^4$.

2.7.3. *Addition.* One can view addition as a process of *counting on* or *counting up*. Start with one of the numbers say m and view it as set with m objects. Then start counting the elements of the set containing n objects starting with m+1 for the first element of the second set. Consider the following example where n = 3 and m = 5. Label the first set $A_1$ and the second set $A_2$.

$$\begin{aligned} A_1 &= \{1, 2, 3\} \\ A_2 &= \{1, 2, 3, 4, 5\} \end{aligned}$$

Counting the objects in both sets we obtain 8 so that 3 added to 5 gives 8. We often use the word *plus* for addition and use the sign +. Then 3 + 5 = 8.

2.7.4. *Multiplication.* Multiplication is the process of adding numbers together *multiple* times. For example, if we add 3 and 3 (3 + 3) we have two groups of 3, or 3 two times. If we add four and four and four (4+4+4) we have three groups of 4, or 4 3 times. We sometimes use the word *times* to represent multiplication. We use the symbol × or the symbol * or the symbol · to represent multiplication. So 3 times 2 is written $3 \times 2$ or 3 * 2 or $3 \cdot 2$. When the context is clear, we can represent multiplication by writing two numbers next to each other with a space between them as in 3 2 = 6. In such cases we often enclose the numbers in parentheses as in (3)(2) = 6. It is useful to memorize or at least be able to construct a table listing the first 20 or numbers multiplied by each other.

TABLE 2. Multiplication Table

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **2** | **4** | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 |
| **3** | 6 | **9** | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 | 39 | 42 | 45 | 48 | 51 | 54 | 57 | 60 |
| **4** | 8 | 12 | **16** | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 | 52 | 56 | 60 | 64 | 68 | 72 | 76 | 80 |
| **5** | 10 | 15 | 20 | **25** | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 | 96 | 100 |
| **6** | 12 | 18 | 24 | 30 | **36** | 42 | 48 | 54 | 60 | 66 | 72 | 78 | 84 | 90 | 96 | 102 | 108 | 114 | 120 |
| **7** | 14 | 21 | 28 | 35 | 42 | **49** | 56 | 63 | 70 | 77 | 84 | 91 | 98 | 105 | 112 | 119 | 126 | 133 | 140 |
| **8** | 16 | 24 | 32 | 40 | 48 | 56 | **64** | 72 | 80 | 88 | 96 | 104 | 112 | 120 | 128 | 136 | 144 | 152 | 160 |
| **9** | 18 | 27 | 36 | 45 | 54 | 63 | 72 | **81** | 90 | 99 | 108 | 117 | 126 | 135 | 144 | 153 | 162 | 171 | 180 |
| **10** | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | **100** | 110 | 120 | 130 | 140 | 150 | 160 | 170 | 180 | 190 | 200 |
| **11** | 22 | 33 | 44 | 55 | 66 | 77 | 88 | 99 | 110 | **121** | 132 | 143 | 154 | 165 | 176 | 187 | 198 | 209 | 220 |
| **12** | 22 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | **144** | 156 | 168 | 180 | 192 | 204 | 216 | 228 | 240 |
| **13** | 26 | 39 | 52 | 65 | 78 | 91 | 104 | 117 | 130 | 143 | 156 | **169** | 182 | 195 | 208 | 221 | 234 | 247 | 260 |
| **14** | 28 | 42 | 56 | 70 | 84 | 98 | 112 | 126 | 140 | 154 | 168 | 182 | **196** | 210 | 224 | 238 | 252 | 266 | 280 |
| **15** | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 210 | **225** | 240 | 255 | 270 | 285 | 300 |
| **16** | 32 | 48 | 64 | 80 | 96 | 112 | 128 | 144 | 160 | 176 | 192 | 208 | 224 | 240 | **256** | 272 | 288 | 304 | 320 |
| **17** | 34 | 51 | 68 | 85 | 102 | 119 | 136 | 153 | 170 | 187 | 204 | 221 | 238 | 255 | 272 | **289** | 306 | 323 | 340 |
| **18** | 36 | 54 | 72 | 90 | 108 | 126 | 144 | 162 | 180 | 198 | 216 | 234 | 252 | 270 | 288 | 306 | **324** | 342 | 360 |
| **19** | 38 | 57 | 76 | 95 | 114 | 133 | 152 | 171 | 190 | 209 | 228 | 247 | 266 | 285 | 304 | 323 | 342 | **361** | 380 |
| **20** | 40 | 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 | 220 | 240 | 260 | 280 | 300 | 320 | 340 | 360 | 380 | **400** |

Notice that across the second two or down the second column is counting by twos and that across the fifth row or down the fifth column is counting by fives.

Also notice the following.

1. All multiples of 2 are even numbers.
2. The sum of the digits in all multiples of 3 is a lesser multiple of 3.
3. All multiples of 5 end in 0 or 5.
4. All multiples of 6 are even numbers and the sum of the digits is a lesser multiple of 3.
5. The sum of the digits in all multiples of 9 is 9 or a multiple of 9.
6. All multiples of 10 end in 0.
7. The items in the diagonal of the table representing each number multiplied by itself are called squares. Note that they differ by 2,5,7,9,11,13,15,17,19, 21, 23, 25, 27, 29, 31, 33, 35, 37, and 39.

8. The table is symmetric.

2.7.5. *Field properties of addition and multiplication of real numbers.* For any real numbers a, b, c:

TABLE 3. Field Properties of Real Numbers

| # | | Addition – **A** | Multiplication – **M** |
|---|---|---|---|
| 0 | Closure | a + b $\varepsilon$ R | ab $\varepsilon$ R. |
| 1 | Commutative | a + b = b + a | ab = ba |
| 2 | Associative | a + (b +c) = (a + b) + c | a(bc) = (ab)c |
| 3 | Identity | There is a number 0 with a + 0 = 0 + a = a. | There is a number 1 with (a)(1) = (1)(a) = a. |
| 4 | Inverse | There is a number -a with a + -a = -a + a = 0. | If a $\neq$ 0, there is a number $\frac{1}{a}$ with (a)$\left(\frac{1}{a}\right)$ = $\left(\frac{1}{a}\right)$(a) = 1. $\frac{1}{a}$ is also written as $a^{-1}$ and $aa^{-1}$=1. |
| D | Distributive | a(b + c) = ab + ac | |

2.7.6. *Definition of subtraction.* For any real numbers a and b

a - b = a + -b.

2.7.7. *Definition of division.* For the real numbers a and b with b $\neq$ 0,

a $\div$ b = a/b = $\frac{a}{b}$ = (a)$\left(\frac{1}{b}\right)$ = (a)(1/b).

2.7.8. *Some uniqueness theorems for real numbers derived from the field properties and the definitions.*

TABLE 4. Uniqueness theorems for real numbers

| Theorem # | Description | Condition | Result |
|---|---|---|---|
| Theorem 1a | Uniqueness of 0 | If z and a are elements of R such that z + a = a then | z = 0 |
| Theorem 1b | Uniqueness of 1 | If u and b $\neq$ 0 are elements of R such that ub = b then | u = 1 |
| Theorem 2a | Uniqueness of -a | If a and b are elements of R such that a + b = 0 then | b = -a |
| Theorem 2b | Uniqueness of $\frac{1}{a}$ | If a $\neq$ 0 and b are elements of R such that ab = 1 then | $b = \frac{1}{a}$ |

## 3. Operations with Real Numbers

3.1. **Summary of results.** Using the field properties in section 2.7.5, one can state the following properties of real numbers.

Table 5. Operations with Real Numbers

| # | Property | Example | Notes |
|---|---|---|---|
| 1 | a - b = a +(-b) | 6 - 2 = 6 + (-2) = 4 | Subtraction |
| 2 | a - (-b) = a + b | 4 - (-3) = 4 + 3 = 7 | |
| 3 | -a = (-1) a | -4 = (-1)4 | |
| 4 | a(b+c) = ab + ac | 3 (2+5)= (3)(2) + (3)(5) = 21 | Distributive |
| 5 | a(b-c) = ab - ac | 3 (1-4)= (3)(1) - (3)(4) = -9 | |
| 6 | -(a+b) = (-1)(a+b) = -a-b | -(6+3) = -6-3 = -9 | Distributive |
| 7 | -(a-b) = -a+b | -(4-3) = -4+3 = -1 | |
| 8 | -(-a) = a | -(-2) = 2 | |
| 9 | a(0) = 0 | 3(0) = 0 | |
| 10 | (-a)(b) = -(ab) = a(-b) | (-3)(4) = -(3·4) = (3)(-4)=-12 | |
| 11 | (-a)(-b) = ab | (-3)(-5) = 3 $\times$ 5 = 15 | |
| 12 | $\frac{a}{1}$ = a | $\frac{2}{1}$ = 2 | |
| 13 | $\frac{a}{b}$ = a $\left(\frac{1}{b}\right)$ | $\frac{2}{5}$ = 2$\left(\frac{1}{5}\right)$ | |
| 14 | $\frac{a}{-b}$ = -$\left(\frac{a}{b}\right)$ = $\frac{-a}{b}$ | $\frac{2}{-9}$ = -$\left(\frac{2}{9}\right)$ = $\frac{-2}{9}$ | |
| 15 | $\frac{-a}{-b} = \frac{a}{b}$ | $\frac{-2}{-7} = \frac{2}{7}$ | |
| 16 | $\frac{0}{a}$ = 0 when $a \neq 0$ | $\frac{0}{7}$ = 0 | |
| 17 | $\frac{a}{a}$ = 1 when $a \neq 0$ | $\frac{2}{2}$ = 1, $\frac{-5}{-5}$ = 1 | |
| 18 | a$\left(\frac{b}{a}\right)$ = b | 2$\left(\frac{7}{2}\right)$ = 7 | |
| 19 | a·$\frac{1}{a}$ = 1 when a $\neq$ 0 | 2 ·$\frac{1}{2}$ = 1 | |
| 20 | $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ | $\frac{2}{3} \cdot \frac{4}{5} = \frac{2\cdot 4}{3\cdot 5} = \frac{8}{15}$ | |
| 21 | $\frac{ab}{c}$ = $\left(\frac{a}{c}\right)$b = a $\left(\frac{b}{c}\right)$ | $\frac{2\cdot 7}{3} = \frac{2}{3} \cdot 7 = 2 \cdot \frac{7}{3}$ | |
| 22 | $\frac{a}{bc} = \left(\frac{a}{b}\right)\left(\frac{1}{c}\right) = \left(\frac{1}{b}\right)\left(\frac{a}{c}\right)$ | $\frac{2}{3\cdot 7} = \frac{2}{3} \cdot \frac{1}{7} = \frac{1}{3} \cdot \frac{2}{7}$ | |
| 23 | $\frac{a}{b} = \left(\frac{a}{b}\right)\left(\frac{c}{c}\right) = \left(\frac{ac}{bc}\right)$ when c $\neq$ 0 | $\frac{2}{7} = \left(\frac{2}{7}\right)\left(\frac{5}{5}\right) = \frac{2\cdot 5}{7\cdot 5}$ | |
| 24 | $\frac{a}{b(-c)} = \frac{a}{(-b)(c)} = \frac{-a}{bc} = \frac{-a}{(-b)(-c)}$ = - $\frac{a}{bc}$ | $\frac{2}{3(-5)} = \frac{2}{(-3)(5)} = \frac{2}{15} = \frac{-2}{(-3)(-5)}$ = - $\frac{2}{3(5)}$ = - $\frac{2}{15}$ | |
| 25 | $\frac{a(-b)}{c} = \frac{(-a)b}{c} = \frac{ab}{-c} = \frac{(-a)(-b)}{-c}$ = - $\frac{ab}{c}$ | $\frac{2(-3)}{5} = \frac{(-2)(3)}{5} = \frac{2(3)}{-5} = \frac{(-2)(-3)}{-5} = \frac{2(3)}{5}$ = -$\frac{6}{5}$ | |
| 26 | $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ | $\frac{2}{9} + \frac{3}{9} = \frac{2+3}{9} = \frac{5}{9}$ | |
| 27 | $\frac{a}{c}$ - $\frac{b}{c} = \frac{a-b}{c}$ | $\frac{2}{9}$ - $\frac{3}{9} = \frac{2-3}{9} = \frac{-1}{9}$ | |
| 28 | $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ | $\frac{4}{5} + \frac{2}{3} = \frac{4\cdot 3+5\cdot 2}{5\cdot 3} = \frac{22}{15}$ | |
| 29 | $\frac{a}{b}$ - $\frac{c}{d} = \frac{ad-bc}{bd}$ | $\frac{4}{5}$ - $\frac{2}{3} = \frac{4\cdot 3-5\cdot 2}{5\cdot 3} = \frac{2}{15}$ | |
| 30 | $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ | $\frac{\frac{2}{3}}{\frac{7}{5}} = \frac{2}{3} \div \frac{7}{5} = \frac{2}{3} \cdot \frac{5}{7} = \frac{2\cdot 5}{3\cdot 7} = \frac{10}{21}$ | |
| 31 | $\frac{a}{\frac{b}{c}} = a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{ac}{b}$ | $\frac{2}{\frac{3}{5}} = 2 \div \frac{3}{5} = 2 \cdot \frac{5}{3} = \frac{2\cdot 5}{3} = \frac{10}{3}$ | |
| 32 | $\frac{\frac{a}{b}}{c} = \frac{a}{b} \div c = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$ | $\frac{\frac{2}{3}}{5} = \frac{2}{3} \div 5 = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{3\cdot 5} = \frac{2}{15}$ | |

### 3.2. **Working with fractions.**

3.2.1. *Properties of fractions.*

1. If a, b, c, and d are real numbers and b and d are not zero then *Equality of Fractions* implies that

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

2. If a, b, and x are real numbers and b and x are not zero then the *Fundamental Property of Fractions* implies that

$$\frac{ax}{bx} = \frac{a}{b}$$

3.2.2. *Multiplying fractions.* To multiply two fractions, multiply the numerators and denominators separately, i.e.,

$$\frac{x}{y} \times \frac{a}{b} = \frac{x}{y} \cdot \frac{a}{b} = \frac{x}{y}\frac{a}{b} = \frac{xa}{yb}$$

For example,

$$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$$

3.2.3. *Dividing fractions.* To divide two fractions, use the inverse multiplication rule, i.e.,

$$\frac{\frac{x}{y}}{\frac{a}{b}} = \frac{x}{y}\frac{b}{a} = \frac{xb}{ya}$$

For example,

$$\frac{\frac{3}{4}}{\frac{1}{2}} = \left(\frac{3}{4}\right)\left(\frac{2}{1}\right) = \frac{6}{4} = \frac{3}{2}$$

3.2.4. *Simplifying fractions.* The *Fundamental Property of Fractions* allows us to write any fraction in an infinite number of ways by multiplying the numerator and denominator by the same number and then repeating this process for a different number. For example

$$\frac{1}{2} = \frac{2}{4} = \frac{20}{40} = \frac{50}{100} = \frac{2.5}{5} = \frac{.25}{.50}$$

$$\frac{7}{13} = \frac{14}{26} = \frac{35}{65} = \frac{175}{325} = \frac{2.3\overline{3}}{4.3\overline{3}} = \frac{1}{1.857142\overline{857142}}$$

A fraction which has the smallest possible integers in both the numerator and denominator is said to be in reduced form. The process of eliminating these integer *multipliers* in the numerator and denominator of a fraction is called simplifying the fraction or writing the fraction in reduced form. For example, the fraction $\frac{20}{40}$ can be written in a different form by eliminating the *multiplier* 5 as follows

$$\frac{20}{40} = \frac{5 \times 4}{5 \times 8} = \frac{5}{5} \times \frac{4}{8} = 1 \times \frac{4}{8} = \frac{4}{8}$$

The *multiplier* 5 is often called a factor. The fraction $\frac{4}{8}$ can then be written in a different form by eliminating the *multiplier* 4 as follows

$$\frac{4}{8} = \frac{4 \times 1}{4 \times 2} = \frac{4}{4} \times \frac{1}{2} = 1 \times \frac{1}{2} = \frac{1}{2}$$

Notice that the product of the two factors we eliminated from $\frac{20}{40}$ is 5 $\times$ 4 = 20. This implies we could find the reduced form of $\frac{20}{40}$ in one step as follows

$$\frac{20}{40} = \frac{20\times 2}{20\times 2} = \frac{20}{20}\times\frac{1}{2} = 1\times\frac{1}{2} = \frac{1}{2}$$

We can *reduce* a fraction by finding common multipliers or factors in the numerator and denominator of a fraction. A *fail safe* way to do this is to factor a number into the primes of which it is made. We can factor 12 as $2 \times 2 \times 3$. And 2 and 3 are both prime.

**Definition 1** (Prime Factorization)**.** The *prime factorization* of a number is a group of prime numbers that multiplied together is equal to that number.

For example 357 = $3 \times 7 \times 17$ and 1729 = $7 \times 13 \times 19$. The prime factors in a number may well repeat as in 192 = $2 \times 2 \times 2 \times 2 \times 2 \times 3$.

To find the prime factorization of a number we can use the following algorithm.

**Definition 2** (Algorithm for Prime Factorization)**.**

1. If the number is divisible by 2, then write down 2 as a factor and proceed with the prime factorization of the quotient.
2. If the number is not divisible by 2, then see if the number is divisible by 3. If the number is divisible by 3, then write down 3 as a factor and proceed with the prime factorization of the quotient.
3. If the number is not divisible by 2 or 3, then see if the number is divisible by 5. If the number is divisible by 5, then write down 5 as a factor and proceed with the prime factorization of the quotient.
4. If the number is not divisible by 2 or 3 or 5, then see if the number is divisible by 7. If the number is divisible by 7, then write down 7 as a factor and proceed with the prime factorization of the quotient.
5. Continue in this fashion until the quotient is a prime number or the prime factor you are trying is the smallest prime less than the square root of the smallest perfect square larger than the number being factored.

Consider the prime factorization of 693.

1. 693 is not divisible by 2, so proceed with 3.
2. The sum of the digits in 693 is 18 so 693 is divisible by 3. Forming the quotient we obtain $\frac{693}{3}$ = 231.
3. 231 is not divisible by 2, so proceed with 3.
4. The sum of the digits in 693 is 6 so 231 is divisible by 3. Forming the quotient we obtain $\frac{231}{3}$ = 77.
5. 77 is not divisible by 2, so proceed with 3.
6. 77 is not divisible by 3, so proceed with 5.
7. 77 is not divisible by 5, so proceed with 7.
8. 77 is divisible by 7. The quotient is 11 so we are finished because 11 is a prime number.

The prime factorization of 693 is then $3 \times 3 \times 7 \times 11$.

Finding the prime factorizations of the numerator and denominator of a fraction and then canceling like terms will always reduce a fraction to its simplest form, i.e. this will always identify the factors that are common in the numerator and denominator of a fraction. In many cases, however, it may be easier to look for larger common factors than those in the prime factorization. Consider the following example

$$\frac{420}{378} = \frac{2\times 2\times 3\times 5\times 7}{2\times 3\times 3\times 3\times 7} = \frac{\not{2}\times 2\times \not{3}\times 5\times \not{7}}{\not{2}\times \not{3}\times 3\times 3\times \not{7}} = \frac{2\times 5}{3\times 3} = \frac{10}{9}$$

We might do this faster by noticing that the numerator and denominator are divisible by 2 because they are even and divisible by 3 because the sum of the digits is divisible by 3 (6 and 18). This would then give

$$\frac{420}{378} = \frac{6 \times 70}{6 \times 63} = \frac{\not{6} \times 70}{\not{6} \times 63} = \frac{70}{63}$$

Remembering our seven times table we can simplify as

$$\frac{420}{378} = \frac{70}{63} = \frac{7 \times 10}{7 \times 9} = \frac{10}{9}$$

When using this approach the following facts or rules are useful.

1. All even numbers have 2 as a factor.
2. If the last two digits of a number divide by 4, the whole number divides by 4. For example the last two digits of 124 divide by 4 and 124/4 = 31.
3. If the last three digits of a number divide by 8, the whole number divides by 8. For example the last three digits of 3128 divide by 8 and 3128/8 = 391. Similarly for 16, 32, . . .
4. If a number ends in 5 or 0, it is divisible by 5.
5. If the last two digits of a number are divisible by 25, then the whole number is divisible by 25. For example, the last two digits of 675 are divisible by 25 so 675 is divisible by 25, i.e., 675/25 = 27. Similarly for 625, 3125, . . .
6. All numbers ending in zero are divisible by 10. Numbers ending in two zeroes are divisible by 100 and so on.
7. If the sum of the digits in a number is divisible by 3, the whole number is divisible by 3.
8. If the sum of the digits in a number is divisible by 9, the whole number is divisible by 9.
9. If the sum of the digits in a number is divisible by 3 and the number is even, the whole number is divisible by 6.
10. For 11, add alternate digits, if the sums are identical, differ by 11, or by multiple of 11, then the number is divisible by 11. Consider 121 where 1+ 1 = 2 and 121/11=11. Or 28,347 where 2 + 3 + 7 = 8 + 4 so that 28347/11 = 2577. Or consider 869 where 8 + 9 = 17 and 17 - 6 = 11. Therefore 869/11= 79.

A useful fact to remember in factoring a number is that if it is not a prime number, it will have at least one factor less than the square root of the number which is the smallest perfect square greater than the number in question. Consider, for example, 139. None of the rules we listed seem to help in factoring 139. Because 144 is the smallest perfect square greater than 139, we know that the only primes numbers we need to try are 2, 3, 5, 7, and 11. We can easily eliminate 2, 3, and 5. We can eliminate 11 because $10 \neq 3$ and 10 - 3 is not divisible by 11. Or we can just remember our eleven times table. So we can try 7.

$$\begin{array}{r} 19 \\ 7\overline{)\,139} \\ \underline{70} \\ 69 \\ \underline{63} \\ 6 \end{array}$$

We find that $\frac{139}{7} = 19\,\frac{6}{7}$ . This does not divide evenly so we know that 139 is a prime number.

3.2.5. *Adding and subtracting fractions.* To add or subtract two fractions, the denominators must be the same (common denominator). If the denominators are the same, then the addition is completed by adding the numerators, i.e.,

$$\frac{x}{y} + \frac{a}{y} = \frac{x+a}{y}$$

For example

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

If the denominators in the fractions we want to add are not the same, we must rewrite them so that they have the same denominator. For example

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

or

$$\frac{3}{24} + \frac{5}{16} = \frac{6}{48} + \frac{15}{48} = \frac{21}{48}$$

But the $\frac{21}{48}$ is not in reduced form, i.e.,

$$\frac{21}{48} = \frac{\not{3} \times 7}{\not{3} \times 16} = \frac{7}{16}$$

If we first reduce $\frac{3}{24}$ to $\frac{1}{8}$, we obtain

$$\frac{1}{8} + \frac{5}{16} = \frac{2}{16} + \frac{5}{16} = \frac{7}{16}$$

So making sure all fractions in a product are in reduced from will help us obtain a sum that is in reduced form. But how do we find common denominators. In the examples so far, a reasonable choice has seemed obvious. Is there a more systematic way? One way that will always work is to simply multiply the denominators together to obtain the common denominator and then multiply the numerators by the denominators of the other fractions. For example

$$\begin{aligned}\frac{1}{8} + \frac{5}{16} &= \frac{16}{16 \times 8} + \frac{8 \times 5}{16 \times 8} \\ &= \frac{16}{128} + \frac{40}{128} = \frac{56}{128} \\ &= \frac{14}{32} = \frac{7}{16}\end{aligned}$$

Of course it would have been easier to convert $\frac{1}{8}$ to $\frac{2}{16}$ at the beginning and obtain

$$\frac{1}{8} + \frac{5}{16} = \frac{2}{16} + \frac{5}{16} = \frac{7}{16}$$

Multiplying all the denominators together will often lead to a very large number for the denominator. An alternative approach is to find the least common denominator among the denominators of the fractions being added or subtracted.

**Definition 3** (Least Common Denominator(LCD))**.** The least common denominator for a group of fractions is the smallest number that can be divided by all the denominators.

For example, 6 is the least common denominator for 2 and 3, and 4 is the least common denominator of 2 and 4, while 48 is the least common denominator of 12 and 16. Notice that 48 is much less than $12 \times 16 = 192$.

The most straightforward way to find the least common denominator for a group of fractions is to first find the prime factorization of each denominator. We then create the least common denominator by using each factor the greatest number of times that it appears in any one denominator. For example, consider the least common denominator for 12 and 16. First find the prime factorization of each.

$$12 = 2 * 2 * 3$$
$$16 = 2 * 2 * 2 * 2$$

The least common denominator is then

$$2 * 2 * 2 * 2 * 3 \ = \ 48$$

Consider a slightly more complicated example where we want to add $\frac{1}{2}$, $\frac{5}{12}$, $\frac{7}{16}$, $\frac{13}{20}$ and $\frac{6}{35}$. First find the prime factorization of each number.

$$12 = 2 * 2 * 3$$
$$16 = 2 * 2 * 2 * 2$$
$$20 = 2 * 2 * 5$$
$$35 = 5 * 7$$

The least common denominator is then (2*2*2*2)*3*5*7=1680. Notice that (2*2*2*2), (2*2*3), (2*2*5), and (5*7) are all contained in this expression. Now multiply each fraction by the appropriate factor so that the denominators are all 1680.

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2} * \frac{2}{2} * \frac{2}{2} * \frac{2}{2} * \frac{3}{3} * \frac{5}{5} * \frac{7}{7} \ = \ \frac{840}{1680} \\
\frac{5}{12} &= \frac{5}{12} * \frac{2}{2} * \frac{2}{2} * \frac{5}{5} * \frac{7}{7} \ = \ \frac{700}{1680} \\
\frac{7}{16} &= \frac{7}{16} * \frac{3}{3} * \frac{5}{5} * \frac{7}{7} \ = \ \frac{735}{1680} \\
\frac{13}{20} &= \frac{13}{20} * \frac{2}{2} * \frac{2}{2} * \frac{3}{3} * \frac{7}{7} \ = \ \frac{1092}{1680} \\
\frac{6}{35} &= \frac{6}{35} * \frac{2}{2} * \frac{2}{2} * \frac{2}{2} * \frac{2}{2} * \frac{3}{3} \ = \ \frac{288}{1680}
\end{aligned}$$

Now add and reduce

$$\begin{aligned}
\frac{1}{2} + \frac{5}{12} + \frac{7}{16} + \frac{13}{20} + \frac{6}{35} &= \frac{840}{1680} + \frac{700}{1680} + \frac{735}{1680} + \frac{1092}{1680} + \frac{288}{1680} \\
&= \frac{840 \ + \ 700 \ + \ 735 \ + \ 1092 \ + \ 288}{1680} \\
&= \frac{3655}{1680} \ = \ \frac{5 \times 731}{5 \times 360} \\
&= \frac{731}{336} \\
&= 2\,\frac{59}{336}
\end{aligned}$$

3.2.6. *Separating Fractions.* Given the rule for adding and subtracting fractions we can always write a sum over a denominator with a single term as a sum of fractions with the numerator of each coming from the respectively terms to added in the numerator of the original fraction and the denominator being the common denominator. Consider the following example.

$$\frac{80 + 23 + 7 + 16}{40} = \frac{80}{40} + \frac{23}{40} + \frac{7}{40} + \frac{16}{40}$$
$$= \frac{126}{40}$$

The is no simple way to reduce fractions with more than one term in the denominator is in $\frac{15}{3+x}$. On the other hand, $\frac{3+x}{15} = \frac{3}{15} + \frac{x}{15} = \frac{1}{5} + \frac{x}{15}$.

### 3.3. **Rules of exponents.**

3.3.1. *Product rule for exponents.* If m and n are natural numbers then

$$x^m x^n = x^{m+n}$$

Note that the product rule applies to exponential expressions with the same base. A product of two powers with different bases such as $x^3y^4$, cannot be simplified.

Consider the following examples.

$$5^3\,5^2 = (5*5*5)*(5*5) = 5*5*5*5*5 = 5^5$$
$$x^p\,x^{2p} = x^p * x^p * x^p = x^{3p}$$

3.3.2. *Power rules for exponents.* If m and n are natural numbers then

$$(x^m)^n = x^{mn}$$
$$(xy)^n = x^n y^n$$
$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}, \quad y \neq 0$$

For example,

$$(4^2)^3 = 16^3 = 16*16*16 = 256*16 = 4096 = 4^6 = 4*4*4*4*4*4 = 64*64 = 4096$$
$$(4*5)^3 = 20^3 = 20*20*20 = 400*20 = 8000 = 4^3*5^3 = 64*125 = 8000$$

3.3.3. *Zero rule for exponents.*

$$x^0 = 1, \quad x \neq 0$$

3.3.4. *Negative exponents.* If n is an integer and $x \neq 0$, then

$$x^{-n} = \frac{1}{x^n}$$
$$\frac{1}{x^{-n}} = x^n$$

For example $3^{-2} = \frac{1}{9}$.

3.3.5. *Quotient rule for exponents.* If m and n are integers and x $\neq 0$, then

$$\frac{x^m}{x^n} = x^{m-n}$$

For example,

$$\frac{3^4}{3^2} = \frac{3*3*3*3}{3*3} = \frac{81}{9} = 9 = 3^{4-2} = 3^2 = 9$$

Notice that when the numerator and denominator contain the same number, but to different powers, we can cancel terms just like we did in simplifying fractions.

Or consider the following example.

$$\frac{3^4 * x^2 * y^3}{3 * x * y^2} = 3^3 * x * y = 27xy$$

3.3.6. *A fraction to a negative power.* If n is a natural number, then

$$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n, \quad x \neq 0 \text{ and } y \neq 0$$

For example $\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4} = \frac{3^2}{2^2} = \left(\frac{3}{2}\right)^2$.

3.3.7. *Some more examples.*

1. $\frac{x^3}{yx^2} = \frac{x}{y}$
2. $(2x)^{-1} = \frac{1}{2x}$
3. $2x^{-1} = \frac{2}{x}$
4. $\frac{(3x)^3 2^4}{9x4x^2} = \frac{12}{x}$
5. $\frac{3^3*2^2*5^3}{4^2*3^{-1}} = \frac{3^4*5^3}{2^2}$
6. $\frac{4^2*6^2}{3^3*2^3} = \frac{8}{3}$
7. $\frac{x_1^3 x_2^{1/2}}{x_1^2} = \frac{x_1 x_2^{1/2}}{1} = x_1 x_2^{1/2}$

8. $\frac{t^p t^{q-1}}{t^r t^{s-1}} = t^{p+q-r-s}$

9. $\frac{1}{4}\frac{Ax_1^{1/4}x_2^{1/2}}{x_1} = \frac{1}{4}Ax_1^{-3/4}x_2^{1/2}$

10. $\alpha_2 \frac{x_1^{\alpha_2}x_2^{\alpha_2}}{x_2} = \alpha_2 x_1^{\alpha_2}x_2^{\alpha_2 - 1}$

3.4. **Order of operations.** If an expression does not have parentheses or grouping symbols, follow these steps,

1. Find the values of any exponential expressions
2. Perform all multiplications and/or divisions, working from left to right.
3. Perform all additions and/or subtractions, working from left to right.

## 4. Expressions and equations

4.1. **Variables.** A variable is a symbol that can be replaced by any one of a set of numbers or other objects. For example, we might pick a variable denoted by r. Because r is a variable, it can then be replaced by various members of a given set such as the integers. For example, if r is a variable representing the age of an arbitrary person in years, where fractions of years are rounded down to the next integer, r could be replaced by any of the numbers {0,1,2,3,..., 121, 122..., } where the upper bound on the set is a bit uncertain. (If one likes Biblical dating, 969 might be a reasonable number.)

4.2. **Combining variables and numbers to form expressions.**

4.2.1. *Definition of an expression.* When numbers and variables are combined, the result is called an algebraic expression, or simply an expression.

4.2.2. *Some examples.*

**a:** 12 less than twice a number is written as 2x - 12

**b:** Five times the same number is written as 5x

**c:** The product of output (y) and price (p) less the cost of producing that output c(y) is written py - c(y).

4.2.3. *Evaluating an expression.* Substituting for the variables and calculating a result is called evaluating an expression. For example, consider the expression

$$\frac{1}{2}\, b\, h.$$

If b is 5 and h is 6, then the expression is evaluated as $\frac{1}{2}$(5)(6) = 15.

4.2.4. *Factoring expressions.* Just as we factored numbers, we can factor algebraic expressions. For example 63 = 3 × 3 × 7. And $5x^2 + 15x = 5x(x+3)$. here are some more examples.

1. $12x^2 - 14x - 6 = (6x+2)(2x-3)$
2. $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2$
3. $4u^2 + 8u + 4 = (2u+2)^2 = 4(u+1)^2$
4. $2x^2y - 6x + 6xy - 18 = (xy - 3)(2x + 6)$

4.2.5. *The verbs of algebra.* The most common verbs used in algebra are
= equal
$>$ greater than
$<$ less than
$\geq$ greater than or equal to
$\leq$ less than or equal to
$\neq$ not equal to
$\approx$ approximately equal to

4.2.6. *Combining expressions and verbs to form sentences.*

(1) Definition of a sentence. We can combine expressions and verbs to form sentences.
(2) Sentences that are assumed to be true are either definitions of terms or postulates.
(3) Sentences that are either true or false are called statements or propositions. A true proposition might be "All individuals who can breathe on their own are alive." A false proposition might be "All individuals who can breathe on their own are alert and able to speak."
(4) An open proposition is one that is true for some values of the variables but not others. For example, the proposition $x^2$ - 1 = 0 is true for some values of x but not all values of x.
(5) Statements that are shown to be true using definitions and postulates are called theorems.

4.2.7. *Equations.* An equation is a sentence stating that two expressions are equal. For example we might have the equation

$$\frac{1}{2}b + c = \frac{1}{2}c + d.$$

An equation is an open proposition in that it may be true for some values of the variables but not others.

4.2.8. *Formulas.* A formula is a sentence stating that a single variable is equal to an expression with one or more variables on the other side. Thus

$$c = 2d - b$$

is both an equation and a formula. But a + b = b + a is an equation but not a formula. If we evaluate the expression in a formula and substitute the result for the variable on the other side, the equation will be true for those values of the variables.

4.2.9. *Examples of equations.* Here are some examples of typical equations one might encounter.

**1:** $2x - 10 = -3x$
**2:** $2x - 16 = x + y$
**3:** $x + y = 5x - 3$
**4:** $\frac{x}{y} = 4$
**5:** $x + 3y = 10x + 21$
**6:** $\frac{1}{2}y - 6x = 4x - 3$
**7:** $\frac{x}{3} + 2y = \frac{14x}{6} + 2$
**8:** $y = 5x - 4y + 25$
**9:** $-14x + 3y = \frac{y}{5} - 28$
**10:** $6x + 5y = 21x - 2y$
**11:** $x + y = 5$
**12:** $\frac{2}{3} + \frac{3x}{4} = \frac{y}{2}$

4.2.10. *Some theorems useful in analyzing equations derived from the field properties and definitions.* Some theorems that are useful in analyzing and simplifying equations are contained in table 6.

TABLE 6. Theorems Useful for Analyzing Equations

| Theorem # | Description | Condition | Result |
|---|---|---|---|
| Theorem 3a | Unique solutions to a + x = b | Let a, b be arbitrary elements of R, then | a + x = b has the unique solution x = (-a) + b |
| Theorem 3b | Unique solutions to ax = b | Let a, b be arbitrary elements of R, then | if a $\neq$ 0, ax = b has the unique solution x = $\frac{1}{a}b$ |
| Theorem 4a | Multiplication property of 0 | For any a $\varepsilon$ R | (a)(0) = 0 |
| Theorem 4b | Multiplication property of -1 | For any a $\varepsilon$ R | (a)(-1) = -a |
| Theorem 4c | Opposite of an opposite | For any a $\varepsilon$ R | -(a) = a |
| Theorem 4d | Product of (-1) and (-1) | For any a $\varepsilon$ R | (-1)(-1) = 1 |
| Theorem 4e | Opposite of a sum | For all a and b $\varepsilon$ R | -(a+b) = -a+-b |
| Theorem 5 | Cancelation for addition | Let x, y, z, be arbitrary elements of R. If x + y = x + z then | y = z |
| Theorem 6a | Product of two factors = 0 | Let a and b be elements of R. We have ab = 0 if and only if | a = 0 or b = 0 or both |
| Theorem 6b | Product of two factors = 0 | Let a and b be elements of R. We have a $\neq$ 0 and b $\neq$ 0 iff | ab $\neq$ 0 |
| Theorem 7a | Reciprocal of a reciprocal | Let a be an element of R. If a $\neq$ 0, then 1/a $\neq$ 0 and | $\frac{1}{\left(\frac{1}{a}\right)} = a$ |
| Theorem 7b | Cancelation for multiplication | Let a, b, c, be elements of R. If ab = ac and a $\neq$ 0, then | b = c |
| Theorem 7c | Reciprocal of a product | Let a and b be elements of R. If a $\neq$ 0 and b $\neq$ 0, | $\frac{1}{ab} = \left(\frac{1}{b}\right)$ |
| Theorem 8 | Multiplication of fractions | Let a, b, c, d, be elements of R. If b $\neq$ 0 and d $\neq$ 0, then | bd $\neq$ 0 and $\frac{a}{b} = \left(\frac{b}{c}\right) = \frac{ac}{bd}$ |
| Theorem 9a | Cancelation for fractions | Let a, b, c, be elements of R. If b $\neq$ 0 and c $\neq$ 0, then | $\left(\frac{a}{b}\right) = \frac{ac}{bc}$ |
| Theorem 9b | Addition of fractions | Let a, b, c, be elements of R. If c $\neq$ 0, then | $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ |
| Theorem 9c | Signs for fractions | Let a and b be elements of R. If b $\neq$ 0, then | -b $\neq$ 0 and $\left(\frac{-a}{b}\right) = \left(\frac{a}{-b}\right) = -\left(\frac{a}{b}\right)$ |
| Theorem 9d | Fractions divided by fractions | Let a, b, c, d, be elements of R. If b $\neq$ 0, c $\neq$ 0, and d $\neq$ 0, then | c/d $\neq$ 0 and $\left(\frac{a/b}{c/d}\right) = \frac{ad}{bc} = \left(\frac{a}{b}\right)\ \left(\frac{d}{c}\right)$ |
| Theorem 9e | Cross multiplication for fractions | Let a, b, c, d, be elements of R. If b $\neq$ 0 and d $\neq$ 0, then | $\left(\frac{a}{b}\right) = \left(\frac{c}{d}\right) = \frac{ad+bc}{bd}$ |

## 5. SOLVING EQUATIONS

5.1. **Simple equations.** Most simple equations can be solved using the theorems from table 6. The theorems on cancelation provide a way to obtain equations equivalent to a given equation by using the following equation solving principle.

5.2. **Basic Principle for Solving Equations.** We obtain equivalent equations if on both sides of the equality sign in an equation we do the following:

(1) add the same number
(2) subtract the same number
(3) multiply by the same number $\neq 0$
(4) divide by the same number $\neq 0$.

5.3. **Examples of simple equations.**

(1) $3x + 10 = x + 4$

(2) $6x - \frac{1}{2}(2x - 3) = 3(1 - x) - \frac{7}{6}(x + 2)$

(3) $\frac{x+2}{x-2} - \frac{8}{x^2 - 2x} = \frac{2}{x}$

(4) $\frac{x}{x-5} + \frac{1}{3} = \frac{-5}{5-x}$

5.4. **Quadratic equations.** An equation in which one or more of the variables is squared (or two variables are multiplied together) and there are no variables raised to the third power is called a quadratic equation.

5.4.1. *Some important identities.*

**a:** $(a + b)^2 = a^2 + 2ab + b^2$
**b:** $(a - b)^2 = a^2 - 2ab + b^2$
**c:** $(a + b)(a - b) = a^2 - b^2$

5.4.2. *Completing the square.* Consider the following two expressions:

$$x^2 + bx \tag{1a}$$

$$x^2 + 2hx + h^2 = (x + h)^2 \tag{1b}$$

An interesting question is "what do we have to do" to expression 1a so that it is equivalent to expression 1b. To find the answer write the expression as follows:

$$x^2 + bx + ? = x^2 + 2hx + h^2 \tag{2}$$

The first terms are equal and the second terms will be equal if b = 2h. This implies that h = 1/2b. It is then clear that $h^2 = \left(\frac{1}{2}b\right)^2$. So we can say:

**Proposition 1** (completing the square)**.** To complete the square in the expression on $x^2$ + bx, add $(1/2b)^2$ to it.

Here are some simple examples of expressions for practice in completing the square

**1:** $x^2 + 8x$
**2:** $x^2 + 6x$
**3:** $x^2 - 3x$
**4:** $x^2 + \frac{b}{a}x$
**5:** $x^x + x$
**6:** $x^2 + y^2 - 2x + 4y$
**7:** $x^2 - \frac{2}{3}x$
**8:** $x^2 - 10x$
**9:** $x^2 - 10x + 20$

**10:** $x^2 + \frac{1}{2}x$
**11:** $x^2 + \frac{1}{2}x - \frac{1}{4}$
**12:** $x^2 - 2x\left(\mu + \sigma^2 t\right)$

5.4.3. *General form of a quadratic equation and some simple examples.* The general form of a quadratic equation is

$$ax^2 + bx + c = 0, \qquad (a \neq 0) \tag{3}$$

If the quadratic polynomial $ax^2 + bx + c$ can be factored, then the equation $ax^2 + bx + c = 0$ can be solved by using Theorem 6a: If the product ab = 0, then a = 0 or b = 0, or both a dn b equal zero.

Consider the equation $x^2 - 4 = 0$. We can factor it as

$$x^2 - 4 = (x - 2)(x + 2)$$

Rewriting the equation using this identity we obtain

$$\begin{aligned}(x - 2)(x + 2) &= 0\\ \Rightarrow (x - 2) &= 0 \text{ or } (x + 2) = 0\\ \Rightarrow x &= 2 \text{ or } x = -2\end{aligned}$$

Now consider the equation $x^2 - 3x - 10 = 0$. We can factor it as

$$x^2 - 3x - 10 = (x - 5)(x + 2)$$

Rewriting the equation using this identity we obtain

$$\begin{aligned}(x - 5)(x + 2) &= 0\\ \Rightarrow (x - 5) &= 0 \text{ or } (x + 2) = 0\\ \Rightarrow x &= 5 \text{ or } x = -2\end{aligned}$$

Now consider the equation $x^2 - 3.5x - 2 = 0$. We can factor it as

$$x^2 - 3.5x - 2 = (x - 4)(x + \frac{1}{2})$$

Rewriting the equation using this identity we obtain

$$\begin{aligned}(x - 4)(x + \frac{1}{2}) &= 0\\ \Rightarrow (x - 4) &= 0 \text{ or } (x + \frac{1}{2}) = 0\\ \Rightarrow x &= 4 \text{ or } x = -\frac{1}{2}\end{aligned}$$

Finally consider the equation $12x^2 - 25x - 7 = 0$. We can factor it as

$$12x^2 - 25x - 7 = (4x + 1)(3x - 7)$$

Rewriting the equation using this identity we obtain

$$
\begin{aligned}
(4x + 1)(3x - 7) &= 0 \\
\Rightarrow (4x + 1) &= 0 \text{ or } (3x - 7) = 0 \\
\Rightarrow x &= -\frac{1}{4} \text{ or } x = \frac{7}{3}
\end{aligned}
$$

5.5. **The quadratic formula.** In many cases a simple solution as above is not obvious. Thus we derive the *quadratic formula*. We do so in 7 steps as follows.

$$ax^2 + bx + c = 0 \, (a \neq 0)$$

(1) Divide both sides by a $\neq$ 0. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \, (a \neq 0)$
(2) Add $\frac{-c}{a}$ to each side. $x^2 + \frac{b}{a}x = \frac{-c}{a}(a \neq 0)$
(3) Complete the square by adding $\left(\frac{1}{2}\frac{b}{a}\right)^2$. $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}(a \neq 0)$
(4) Simplify the lhs as a perfect square. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}(a \neq 0)$
(5) Simplify the rhs by adding fractions. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \, (a \neq 0)$
(6) Take the square root of each side. $x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a} \, (a \neq 0)$
(7) Add $\frac{-b}{2a}$ to both sides. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \, (a \neq 0)$

Notice that if $b^2$ - 4ac is negative, we do not have a real solution to the equation.

5.6. **Examples For Practicing Use of the Quadratic Formula.**

**1:** $2x^2 - 4x - 7 = 0$
**2:** $3x^2 + 11x - 4 = 0$
**3:** $4x^2 + 12x = 0$
**4:** $10x^2 + 13x + 3 = 0$
**5:** $4x^2 - 144 = 0$
**6:** $20x + 100 = 0$
**7:** $6x^2 - 5x - 3 = 0$

## 6. Functions of a real variable

6.1. **Definition.** A function of a real variable x with **domain** D is a rule that assigns a unique real number to each number x in D. Functions are often given letter names such as f, g, F, or $\phi$. We often call x the independent variable or the argument of f. If g is a function and x is a number in D, then g(x) denotes the number that g assigns to x. We sometimes make the idea that F has an argument (we substitute a number for the variable in F) explicit by writing F(·). In the case of two variables we sometimes use y = f(x) for the value of f evaluated at the number x. Note the difference between $\phi$ and $\phi$(x).

6.2. **The domain of a function.** The domain is the set of all values that can be substituted for x in the function f(·). If a function f is defined using an algebraic formula, we normally adopt the convention that the domain consists of all values of the independent variable for which the function gives a meaningful value (unless the domain is explicitly mentioned).

6.3. **The range of a function.** Let g be a function with domain D. The set of all values g(x) that the function assumes is called the range of g. To show that a number, say a, is in the range of a function f, we must find a number x such that f(x) = a. Here are some example functions for which to find the domain and the range.

**1:** $f(x) = x, \quad 0 \leq x \leq 60$
**2:** $g(x) = x^2/20, \quad 0 \leq x \leq 60$
**3:** $h(x) = 12/x^2$
**4:** $\phi(x) = 3x^2$
**5:** $x = \frac{-1}{2} y^2, \ y \geq 0$
**6:** $f(x) = \frac{3x}{x^2 - 4}$
**7:** $g(x) = \sqrt{4 - 3x}$

6.4. **The graph of a function.** When the rule that defines a function f is given by an equation in y and x, the graph of f is the graph of the equation, that is the set of points (x, y) in the xy-plane that satisfies the equation. Another way to say this is that the graph of the function g is the set of all point (x, g(x)), where x belongs to the domain of g.

6.5. **The vertical line test.** As set of points in the xy-plane is the graph of a function if and only if a vertical line intersects the graph in at most one point.

6.6. **Linear functions.**

6.6.1. *Definition.* A linear function of a real variable x is given by

$$y = f(x) = ax + b, \quad \text{a and b are constants.}$$

The graph of a linear equation is a straight line. The number a is called the slope of the function and the number b is called the y-intercept.

6.6.2. *The slope of a straight line given two distinct points on the line.* Consider two distinct points on a non-vertical straight line in the plane denoted by P = $(x_1, y_1)$ and Q = $(x_2, y_2)$. Because the line is not vertical and P and Q are distinct, $x_1 \neq x_2$. The slope of the line is given by

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \ x_1 \neq x_2. \tag{4}$$

6.6.3. *The point slope formula for a line.* Consider a point P = $(x_1, y_1)$ and any line with slope a. The line with slope a passing through the point can be determined as follows. Pick an arbitrary point on the line and denote it as (x, y). Then use the formula for the slope of a line as follows.

$$a = \frac{y - y_1}{x - x_1}, \ x \neq x_1 \tag{5}$$

Now solve the equation for either y or y - $y_1$ as follows.

$$\begin{aligned} a &= \frac{y - y_1}{x - x_1}, \ x \neq x_1 \\ \Rightarrow y - y_1 &= a\,(x - x_1) \\ \Rightarrow y &= a\,(x - x_1) + y_1 \\ &= ax + (y_1 - ax_1) \ . \end{aligned} \tag{6}$$

6.6.4. *Point-point formula for a line.* If we are given two points on a line we can find the equation by first finding the slope and then using the point slope formula. Let the two points on the line be denoted $(x_1, y_1)$ and $(x_2, y_2)$, $x_1 \neq x_2$. The slope is given by

$$a = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2. \tag{7}$$

If we substitute in the point-slope formula we obtain

$$\begin{aligned} y - y_1 &= a\,(x - x_1) \\ &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \\ \Rightarrow y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1 \\ &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x - \left(\frac{x_1\,(y_2 - y_1)}{x_2 - x_1}\right) + y_1 \\ &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \left(\frac{-x_1 y_2 + x_1 y_1 + x_2 y_1 - x_1 y_1}{x_2 - x_1}\right) \\ &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \left(\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}\right). \end{aligned} \tag{8}$$

## 7. Complex Numbers

7.1. **Definition of a complex number.** A complex number is an ordered pair of real numbers denoted by $(x_1, x_2)$. The first member, $x_1$, is called the real part of the complex number; the second member, $x_2$, is called the imaginary part. We define equality, addition, subtraction, and multiplication so as to preserve the familiar rules of algebra for real numbers.

7.1.1. *Equality of complex numbers.* Two complex numbers x = $(x_1, x_2)$ and y = $(y_1, y_2)$ are called equal iff

$$x_1 = y_1, \text{ and } x_2 = y_2. \tag{9}$$

7.1.2. *Sum of complex numbers.* The sum of two complex numbers x + y is defined as

$$x + y = (x_1 + y_1, x_2 + y_2) \tag{10}$$

7.1.3. *Difference of complex numbers.* To subtract two complex numbers, the following rule applies.

$$x - y = (x_1 - y_1, x_2 - y_2) \tag{11}$$

7.1.4. *Product of complex numbers.* The product xy is defined as

$$xy = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1). \tag{12}$$

The properties of addition and multiplication defined satisfy the commutative, associative and distributive laws.

7.2. **The imaginary unit.** The complex number (0,1) is denoted by i and is called the imaginary unit. We can show that $i^2$ = -1 as follows.

$$i^2 = (0,1)(0,1) = (0 - 1, 0 + 0) = (-1, 0) = -1 \tag{13}$$

7.3. **Representation of a complex number.** A complex number x = ($x_1$, $x_2$) can be written in the form

$$x = x_1 + i\, x_2. \tag{14}$$

Alternatively a complex number z = (x,y) is sometimes written

$$z = x + iy. \tag{15}$$

7.4. **Modulus of a complex number.** The modulus or absolute value of a complex number x = ($x_1$, $x_2$) is the nonnegative real number $|x|$ given by

$$|x| = \sqrt{x_1^2 \,+\, x_2^2} \tag{16}$$

7.5. **Complex conjugate of a complex number.** For each complex number z = x + iy, the number $z_{-}$ = x - iy is called the complex conjugate of z. The product of a complex number and its conjugate is a real number. In particular, if z = x + iy then

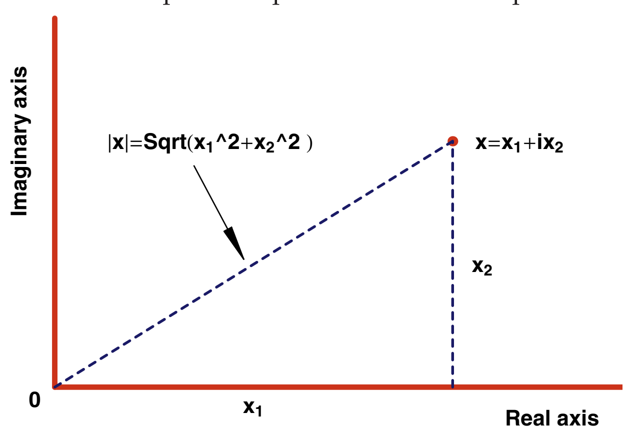
$$z\, z_{-} = (x, y)(x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = x^2 + y^2 \tag{17}$$

Sometimes we will use the notation $\bar{z}$ to represent the complex conjugate of a complex number. So $\bar{z}$ = x - iy. We can then write

$$z\, \bar{z} = (x, y)(x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = x^2 + y^2 \tag{18}$$

7.6. **Graphical representation of a complex number.** Consider representing a complex number in a two dimensional graph with the vertical axis representing the imaginary part. In this framework the modulus of the complex number is the distance from the origin to the point. This is seen clearly in figure 1.

FIGURE 1. Graphical Representation of Complex Number

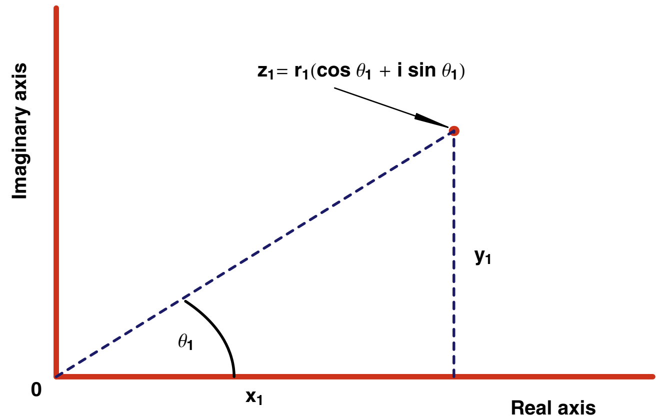



7.7. **Polar form of a complex number.** We can represent a complex number by its angle and distance from the origin. Consider a complex number $z_1 = x_1$ + i $y_1$. Now consider the angle $\theta_1$ which the ray from the origin to the point $z_1$ makes with the x axis. Let the modulus of z be denoted by $r_1$. Then cos $\theta_1 = x_1/r_1$ and sin $\theta_1 = y_1/r_1$. This then implies that

$$\begin{aligned} z_1 &= x_1 \,+\, i\, y_1 \;=\; r_1 \cos\,\theta_1 \,+\, i\, r_1 \sin\,\theta_1 \\ &= r_1\, (\cos\,\theta_1 \,+\, i\, \sin\,\theta_1) \end{aligned} \tag{19}$$

Figure 2 shows how a complex number is represented in polar coordinates.

FIGURE 2. Graphical Representation of Complex Number in Polar Coordinates




7.8. **Complex Exponentials.** The exponential $e^x$ is a real number. We want to define $e^z$ when z is a complex number in such a way that the principle properties of the real exponential function will be preserved. The main properties of $e^x$, for x real, are the law of exponents, $e^{x_1}e^{x_2} = e^{x_1+x_2}$ and the equation $e^0 = 1$. If we want the law of exponents to hold for complex numbers, then it must be that

$$e^z = e^{x+iy} = e^x\, e^{iy} \tag{20}$$

We already know the meaning of $e^x$. We therefore need to define what we mean by $e^{iy}$. Specifically we define $e^{iy}$ in equation 21

**Definition 4.**

$$e^{iy} = \cos\, y + i\, \sin\, y \tag{21}$$

With this in mind we can then define $e^z = e^{x+iy}$ as follows

**Definition 5.**

$$e^z = e^x\, e^{iy} = e^x(\cos\, y + i\, \sin\, y) \tag{22}$$

Obviously if x = 0 so that z is a pure imaginary number this yields

$$e^{iy} = (\cos\, y + i\, \sin\, y) \tag{23}$$

It is easy to show that $e^0 = 1$. If z is real then y = 0. Equation 22 then becomes

$$\begin{aligned} e^z &= e^x\, e^{i\,0} = e^x(\cos\,(0) + i\, \sin\,(0)\,) \\ &= e^x\, e^{i\,0} = e^x(1 + 0\,) = e^x \end{aligned} \tag{24}$$

So $e^0$ obviously is equal to 1.

To show that $e^x e^{iy} = e^{x\,+\,iy}$ or $e^{z_1}e^{z_2} = e^{z_1\,+\,z_2}$ we will need to remember some trigonometric formulas.

**Theorem 1.**

$$\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta \tag{25a}$$

$$\sin(\phi - \theta) = \sin\phi\cos\theta - \cos\phi\sin\theta \tag{25b}$$

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta \tag{25c}$$

$$\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta \tag{25d}$$

Now to the theorem showing that $e^{z_1}e^{z_2} = e^{z_1 + z_2}$

**Theorem 2.** *If $z_1 = x_1 + iy_t$ and $z_2 = x_2 + iy_2$ are two complex numbers, then we have $e^{z_1}e^{z_2} = e^{z_1+z_2}$.*

*Proof.*

$$\begin{aligned} e^{z_1} &= e^{x_1}(cos\ y_1 + i\ sin\ y_1),\ e^{z_2} = e^{x_2}(cos\ y_1 + i\ sin\ y_2), \\ e^{z_1}e^{z_2} &= e^{x_1}e^{x_2}[cos\ y_1\ cos\ y_2 - sin\ y_1\ sin\ y_2 \\ &\quad + i(cos\ y_1\ sin\ y_2 + sin\ y_1\ cos\ y_2)]. \end{aligned} \tag{26}$$

Now $e^{x_1}e^{x_2} = e^{x_1+x_2}$, since $x_1$ and $x_2$ are both real. Also,

$$cos\ y_1\ cos\ y_2 - sin\ y_1\ sin\ y_2 = cos\,(y_1 + y_2) \tag{27}$$

and

$$cos\ y_1\ sin\ y_2 + sin\ y_1\ cos\ y_2 = sin\,(y_1 + y_2), \tag{28}$$

and hence

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[cos\,(y_1 + y_2) + i\ sin\,(y_1 + y_2) = e^{z_1+z_2}. \tag{29}$$

□

Now combine the results in equations 19, 22 and 23 to obtain the polar representation of a complex number

$$\begin{aligned} z_1 &= x_1 + i\,y_1 \\ &= r_1\,(\cos\theta_1 + i\sin\theta_1) \\ &= r_1\,e^{i\,\theta_1}, \qquad \text{(by equation 21)} \end{aligned} \tag{30}$$

The usual rules for multiplication and division hold so that

$$\begin{aligned} z_1\,z_2 &= r_1\,e^{i\theta_1} r_2\,e^{i\theta_2} = (r_1\,r_2)\,e^{i(\theta_1+\,\theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1\,e^{i\theta_1}}{r_2\,e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right)\,e^{i(\theta_1-\,\theta_2)} \end{aligned} \tag{31}$$

## 8. Appendix A – Proof that $\sqrt{2}$ is not a Rational Number

We can show that $\sqrt{2} \notin Q\ (rationals)$ as follows.

*Proof.* The proof is by contradiction. Suppose that $\sqrt{2} \in Q$. Then there exist integers p and q such that $p/q = \sqrt{2}$, where we choose the smallest such p and q. Now $p^2/q^2 = 2,\ or\ p^2 = 2q^2$, so that $p^2$ must be an even number. Since the square of an odd number is always odd, p is even so we may write p = 2r, where r is also an integer. We then have

$$p^2 = 4r^2 = 2q^2$$

and so $q^2 = 2r^2$. Thus $q^2$, and hence q, must be even. But if both p and q are even, they have a common factor of 2. This result contradicts the assumption that p and q were the smallest numbers to give $p/q = \sqrt{2}$. Thus, the statement $\sqrt{2} \in Q$ must be false.

□

## 9. Appendix B – Proofs of Theorems on Properties of Real Numbers and Equations

**Theorem 3.**

**a:** *If z and a are elements of* **R** *such that z + a = a, then z = 0.*
**b:** *If u and b* $\neq$ *0 are elements of* **R** *such that u* $\cdot$ *b = b, then u = 1.*

*Proof of a.* The hypothesis is that z + a = a. We add the element -a, whose existence is given in (A4), to both sides to get

$$(z + a) + (-a) = a + (-a).$$

If we use (A2), (A4), and (A3) in succession on the left side, we obtain

$$(z + a) + (-a) = z + (a + (-a)) = z + 0 = z;$$

If we use (A4) on the right side, we obtain

$$a + (-a) = 0.$$

Hence, we conclude that z = 0.

□

The proof of (b) is left as an exercise. Note that the hypothesis b $\neq$ 0 is crucial.

**Theorem 4.**

**a:** *If a and b are elements of* **R** *such that a + b = 0, then b = -a.*
**b:** *If a* $\neq$ *0 and b are elements of* **R** *such that a* $\cdot$ *b = 1, then b = 1/a.*

*Proof of a.* If a + b = 0, then we add -a to both sides to get

$$(-a) + (a + b) = (-a) + 0.$$

If we use (A2), (A4), and (A3) on the left side, we obtain

$$(-a) + (a + b) = ((-a) + a) + b = 0 + b = b;$$

if we use (A3) on the right side, we obtain

$$(-a) + 0 = -a.$$

Hence, we conclude that b = -a.

□

The proof of (b) is left as an exercise. Note that the hypothesis a $\neq$ 0 is used.

If we view the preceding properties in terms of solving equations, we note that (A4) and (M4) enable us to solve the equations a + x = 0 and a · x = 1 (when a $\neq$ 0) for x, and theorem 4 implies that the solutions are unique.

**Theorem 5.**
*Let a, b be arbitrary elements of* **R**. *Then:*

**a:** *the equation a + x = b has the unique solution x = (-a) + b;*
**b:** *if a $\neq$ 0, the equation a · x = b has the unique solution x = (1/a) · b.*

*Proof of a.* Using properties (A2), (A4), and (A3), we obtain

$$a + [(-a) + b] = [a + (-a)] + b = 0 + b = b.$$

which implies that x = (-a) + b is a solution of the equation a + x = b. To show that it is the only solution, suppose that $x_1$ is any solution of the equation, then a + $x_1$ = b, and if we add -a to both sides, we get

$$(-a) + (a + x_1) = (-a) + b.$$

If we now use (A2), (A4), and (A3) on the left side, we obtain

$$(-a) + (a + x_1) = (-a + a) + x_1 = 0 + x_1 = x_1.$$

Hence we conclude that $x_1$ = (-a) + b.

□

The proof of part (b) is left as an exercise.

**Theorem 6.** *If a and b are any elements of* **R**, *then:*

**a:** *a · 0 = 0,*
**b:** *(-1) · a = -a,*
**c:** *-(-a) = a,*
**d:** *(-1) · (-1) = 1,*
**e:** *-(a+b) = (-a) + (-b).*

*Proof of theorem 6a.* From (M3) we know that a · 1 = a. Then adding a · 0 and using (D) and (A3) we obtain

$$\begin{aligned} a + a \cdot 0 &= a \cdot 1 + a \cdot 0 \\ &= a \cdot (1 + 0) = a \cdot 1 = a. \end{aligned}$$

Thus, by theorem 3a we conclude that a · 0 = 0.

□

*Proof of theorem 6b .* We use (D), in conjunction with (M3), (A4), and part (a), to obtain

$$\begin{aligned} a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\ &= (1 + (-1)) \cdot a = 0 \cdot a = 0 \end{aligned}$$

Thus, from theorem 4a, we conclude that (-1) · a = - a.

□

*Proof of theorem 6c.* By (A4) we have (-a) + a = 0. Thus from theorem 4a, it follows that a = -(-a). □

*Proof of theorem 6d.* In part (b), substitute a = -1. Then

$$(-1) \cdot (-1) = -(-1).$$

Hence, the assertion follows from part (c) with a = 1. □

*Proof of theorem 6e.* To prove part e we use the following two lemmas.

**Lemma 1.** *If a, b, and c are any numbers, then*

$$a + b + c = a + c + b = b + a + c = b + c + a = c + a + b = c + b + a.$$

Proof left to the reader.

**Lemma 2.** *If a, b, c, and d are numbers, then*

$$(a + c) + (b + d) = (a + b) + (c + d).$$

□

*Proof.* Using lemma 1 and the axioms, we have

$$\begin{aligned}(a + c) + (b + d) &= [(a + c) + b] + d \\ &= (a + c + b) + d = (a + b + c) + d \\ &= [(a + b) + c] + d = (a + b) + (c + d).\end{aligned}$$

□

Now to prove part e of the theorem.

*Proof.* We know from the definition of negative that

$$(a + b) + [-(a + b)] = 0.$$

Furthermore, using lemma 2 we have

$$(a + b) + [(-a) + (-b)] = a + (-a)] + [b + (-b)] = 0 + 0 = 0.$$

The result follows from theorem 4a. □

**Theorem 7.** *Let x, y, z be arbitrary elements of R. If x + y = x + z then y = z.*

*Proof.* If x + y = x + z, the axioms of addition give

$$\begin{aligned}y = 0 + y &= (-x + x) + y = -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z.\end{aligned}$$

□

**Theorem 8.** *For any a and b ε R, we have*

**a:** *ab = 0 if and only if a = 0 or b = 0 or both.*
**b:** *a ≠ 0 and b ≠ 0 if and only if ab ≠ 0.*

*Proof.* We must prove two statements in each of parts (a) and (b). To prove (a), observe that if a = 0 or b = 0, or both, then it follows from theorem 6a that ab = 0. Going the other way in (a), suppose that ab = 0. Then there are two cases, either a = 0 or a $\neq$ 0. If a = 0, the result follows. If a $\neq$ 0, then we see that

$$b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0.$$

Hence b = 0 and (a) is established. To prove (b), first suppose a $\neq$ 0 and b $\neq$ 0. Then ab $\neq$ 0, because a $\neq$ 0 and b $\neq$ 0 is the negation of the statement "a = 0 or b = 0 or both." Thus (a) applies. For the second part of (b), suppose ab $\neq$ 0. Then a $\neq$ 0 and b $\neq$ 0 for, if one of them were zero, theorem 6a would apply to give ab = 0.

□

**Theorem 9.** *Let a, b, c, be elements of* **R**.

**a:** *If $a \neq 0$, then $1/a \neq 0$ and $1/(1/a) = a$.*
**b:** *If $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.*
**c:** *If $a \neq 0$ and $b \neq 0$, then $(a \cdot b)^{-1} = (a^{-1}) \cdot (b^{-1})$.*

*Proof of a.* We are given a $\neq$ 0, so that 1/a exists. If 1/a = 0, then 1 = a · (1/a) = a · 0 = 0, contrary to (M3). Thus 1/a $\neq$ 0 and since (1/a) · a = 1, theorem 4b implies that 1/(1/a) = a.

□

*Proof of b.* If we multiply both sides of the equation a · b = a · c by 1/a and apply the associative property (M2), we get

$$((1/a) \cdot a) \cdot b = ((1/a) \cdot a) \cdot c.$$

Thus, 1 · b = 1 · c which is the same as b = c.

□

*Proof of c.* The proof of this theorem is like the proof of theorem 6 with addition replaced by multiplication, 0 replaced by 1, and (-a), (-b) replaced by $a^{-1}$, $b^{-1}$. The details are left to the reader. Note that if a $\neq$ 0, then $a^{-1} \neq 0$ because $aa^{-1} = 1$ and $1 \neq 0$. Then theorem 8b may be used with $b = a^{-1}$.

□

**Theorem 10.** *If $b \neq 0$ and $d \neq 0$, then $b \cdot d \neq 0$ and $\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{a \cdot c}{b \cdot d}$ .*

In order to prove the theorem we will use the following lemmas, similar to lemma 1 and lemma 2. They can be proven by the reader. For the lemmas, we define abc as the common value of (ab)c and a(bc).

**Lemma 3.** *If a, b, and c are numbers, then*

$$abc = acb = bac = bca = cab = cba.$$

**Lemma 4.** *If a, b, c and d are numbers, then*

$$(ac) \cdot (bd) = (ab) \cdot (cd).$$

*Proof.* That b · d $\neq$ 0 follows from theorem 8. Using the notation for fractions, lemma 4 and theorem 9c, we find

$$\begin{aligned}\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) &= (a \cdot b^{-1}) \cdot (cd^{-1}) \\ &= (a \cdot c) \cdot (b^{-1}d^{-1}) = (a \cdot c)(bd)^{-1} = \frac{a \cdot c}{b \cdot d}.\end{aligned}$$

□

**Theorem 11.**

**a:** *If* $b \neq 0$ *and* $c \neq 0$*, then* $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$.

**b:** *If* $c \neq 0$*, then* $\frac{a}{c} + \frac{b}{c} = \frac{(a+b)}{c}$.

**c:** *If* $b \neq 0$*, then* $-b \neq 0$ *and* $\frac{(-a)}{b} = \frac{a}{-b} = -\left(\frac{a}{b}\right)$.

**d:** *If* $b \neq 0$*,* $c \neq 0$*, and* $d \neq 0$*, then* $(c/d) \neq 0$ *and* $\frac{(a/b)}{(c/d)} = \frac{a \cdot d}{b \cdot c} = \left(\frac{a}{b}\right) \cdot \left(\frac{d}{c}\right)$.

**e:** *If* $b \neq 0$ *and* $d \neq 0$*, then* $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

*Proof.* Left to the reader. □