# THE IMPLICIT FUNCTION THEOREM

## 1. A SIMPLE VERSION OF THE IMPLICIT FUNCTION THEOREM

### 1.1. Statement of the theorem.

**Theorem 1** (Simple Implicit Function Theorem). *Suppose that $\phi$ is a real-valued functions defined on a domain D and continuously differentiable on an open set $D^1 \subset D \subset R^n$, $\left(x_1^0,\, x_2^0,\, \ldots,\, x_n^0\right) \in D^1$, and*

$$\phi\left(x_1^0,\, x_2^0,\, \ldots,\, x_n^0\right) \;=\; 0 \tag{1}$$

*Further suppose that*

$$\frac{\partial \phi_(x_1^0,\, x_2^0,\, \ldots,\, x_n^0)}{\partial x_1} \;\neq\; 0 \tag{2}$$

*Then there exists a neighborhood $V_\delta(x_2^0,\, x_3^0,\, \ldots,\, x_n^0) \subset D^1$, an open set $W \subset R^1$ containing $x_1^0$ and a real valued function $\psi_1 : V \rightarrow W$, continuously differentiable on V, such that*

$$\begin{aligned} x_1^0 \;&=\; \psi_1\left(x_2^0,\, x_3^0,\, \ldots,\, x_n^0\right) \\ \phi\left(\psi_1(x_2^0,\, x_3^0,\, \ldots,\, x_n^0), x_2^0,\, x_3^0,\, \ldots,\, x_n^0\right) \;&\equiv\; 0 \end{aligned} \tag{3}$$

*Furthermore for k = 2, . . . , n,*

$$\frac{\partial \psi_1(x_2^0,\, x_3^0,\, \ldots,\, x_n^0)}{\partial x_k} \;=\; -\frac{\frac{\partial \phi(\psi_1(x_2^0,\, x_3^0,\, \ldots,\, x_n^0), x_2^0,\, x_3^0,\, \ldots,\, x_n^0)}{\partial x_k}}{\frac{\partial \phi(\psi_1(x_2^0,\, x_3^0,\, \ldots,\, x_n^0), x_2^0,\, x_3^0,\, \ldots,\, x_n^0)}{\partial x_1}} \;=\; -\frac{\frac{\partial \phi(x^0)}{\partial x_k}}{\frac{\partial \phi(x^0)}{\partial x_1}} \tag{4}$$

We can prove this last statement as follows.
Define g:V $\rightarrow$ R$^n$ as

$$g\left(x_2^0,\, x_3^0,\, \ldots,\, x_n^0\right) \;=\; \begin{bmatrix} \psi_1(x_2^0,\, \ldots,\, x_n^0) \\ x_2^0 \\ x_3^0 \\ \vdots \\ x_n^0 \end{bmatrix} \tag{5}$$

Then $(\phi \circ g)(x_2^0,\, \ldots,\, x_n^0) \;\equiv\; 0$ for all $(x_2^0,\, \ldots,\, x_n^0) \in V$. Thus by the chain rule

*Date*: October 10, 2005.

$$
\begin{aligned}
D\,(\phi \circ g)(x_2^0,\, \ldots,\, x_n^0) \;&=\; D\,\phi\left(g(x_2^0,\, \ldots,\, x_n^0)\right) D\,g(x_2^0,\, \ldots\, x_n^0) \;= 0 \\
&=\; D\,\phi\left(x_1^0, x_2^0,\, \ldots,\, x_n^0\right) D\,g(x_2^0,\, \ldots\, x_n^0) \;= 0 \\
&=\; D\,\phi \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ \vdots \\ x_n^0 \end{pmatrix} D \begin{bmatrix} \psi_1(x_2^0,\, \ldots,\, x_n^0) \\ x_2^0 \\ x_3^0 \\ \vdots \\ x_n^0 \end{bmatrix} = 0 \\
\Rightarrow \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n}\right] & \begin{bmatrix} \frac{\partial \psi_1}{\partial x_2} & \frac{\partial \psi_1}{\partial x_3} & \ldots & \frac{\partial \psi_1}{\partial x_n} \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}
\end{aligned} \tag{6}
$$

For any k = 2, 3, . . . , n, we then have

$$
\begin{aligned}
\frac{\partial \phi(x_1^0, x_2^0, \ldots, x_n^0)}{\partial x_1} \frac{\partial \psi_1(x_2^0, \ldots, x_n^0)}{\partial x_k} + \frac{\partial \phi(x_1^0, x_2^0, \ldots, x_n^0)}{\partial x_k} &= 0 \\
\Rightarrow \frac{\partial \psi_1(x_2^0, \ldots, x_n^0)}{\partial x_k} &= -\frac{\frac{\partial \phi(x_1^0, x_2^0, \ldots, x_n^0)}{\partial x_k}}{\frac{\partial \phi(x_1^0, x_2^0, \ldots, x_n^0)}{\partial x_1}}
\end{aligned} \tag{7}
$$

We can, of course, solve for any other $x_k = \psi_k\left(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_{k+1}^0, \ldots, x_n^0\right)$, as a function of the other x's rather than $x_1$ as a function of $\left(x_2^0, x_3^0 \ldots, x_n^0\right)$.

## 1.2. **Examples.**

### 1.2.1. *Production function.* Consider a production function given by

$$
y = f(x_1,\, x_2\, \ldots,\, x_n) \tag{8}
$$

Write this equation implicitly as

$$
\phi(x_1,\, x_2,\, \ldots,\, x_n,\, y) \;=\; f(x_1,\, x_2\, \ldots,\, x_n) \;-\; y \;=\; 0 \tag{9}
$$

where $\phi$ is now a function of n+1 variables instead of n variables. Assume that $\phi$ is continuously differentiable and the Jacobian matrix has rank 1. i.e.,

$$
\frac{\partial \phi}{\partial x_j} \;=\; \frac{\partial f}{\partial x_j} \;\neq\; 0, \quad j = 1, 2, \ldots, n \tag{10}
$$

Given that the implicit function theorem holds, we can solve equation 9 for $x_k$ as a function of y and the other x's i.e.

$$
x_k^* = \;\psi_k(x_1,\; x_2,\; \ldots,\; x_{k-1},\; x_{k+1},\; \ldots,\; y\,) \tag{11}
$$

Thus it will be true that

$$
\phi(y,\; x_1,\; x_2,\; \ldots,\; x_{k-1},\; x_k^*,\; x_{k+1},\; \ldots,\; x_n\,) \;\equiv\; 0 \tag{12}
$$

or that

$$y \equiv f(x_1,\, x_2,\, \ldots,\, x_{k-1},\, x_k^*,\, x_{k+1},\, \ldots,\, x_n\,) \tag{13}$$

Differentiating the identity in equation 13 with respect to $x_j$ will give

$$0 = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_k}\,\frac{\partial x_k}{\partial x_j} \tag{14}$$

or

$$\frac{\partial x_k}{\partial x_j} = \frac{-\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_k}} = RTS = MRS \tag{15}$$

Or we can obtain this directly as

$$\begin{aligned}\frac{\partial\phi(y,\, x_1,\, x_2,\, \ldots,\, x_{k-1},\, \psi_k,\, x_{k+1},\, \ldots,\, x_n)}{\partial x_k}\frac{\partial\psi_k}{\partial x_j} &= \frac{-\partial\phi(y,\, x_1,\, x_2,\, \ldots,\, x_{k-1},\, \psi_k,\, x_{k+1},\, \ldots,\, x_n)}{\partial x_j}\\ \Rightarrow \frac{\partial\phi}{\partial x_k}\,\frac{\partial x_k}{\partial x_j} &= -\,\frac{\partial\phi}{\partial x_j}\\ \Rightarrow \frac{\partial x_k}{\partial x_j} &= \frac{-\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_k}} = MRS\end{aligned} \tag{16}$$

Consider the specific example

$$y^0 = f\,(x_1,\, x_2\,) = x_1^2 - 2x_1\,x_2 + x_1\,x_2^3$$

To find the partial derivative of $x_2$ with respect to $x_1$ we find the two partials of f as follows

$$\begin{aligned}\frac{\partial x_2}{\partial x_1} &= -\,\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}\\ &= -\,\frac{2\,x_1 - 2\,x_2 + x_2^3}{3\,x_1\,x_2^2 - 2\,x_1}\\ &= \frac{2\,x_2 - 2\,x_1 - x_2^3}{3\,x_1\,x_2^2 - 2\,x_1}\end{aligned}$$

1.2.2. *Utility function.*

$$u^0 = u\,(x_1,\, x_2\,)$$

To find the partial derivative of $x_2$ with respect to $x_1$ we find the two partials of U as follows

$$\frac{\partial x_2}{\partial x_1} = -\,\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$$

1.2.3. *Another utility function.* Consider a utility function given by

$$u = x_1^{\frac{1}{4}}\,x_2^{\frac{1}{3}}\,x_3^{\frac{1}{6}}$$

This can be written implicitly as

$$f\,(u,\, x_1,\, x_2,\, x_3\,) = u - x_1^{\frac{1}{4}}\,x_2^{\frac{1}{3}}\,x_3^{\frac{1}{6}} = 0$$

Now consider some of the partial derivatives arising from this implicit function. First consider the marginal rate of substitution of $x_1$ for $x_2$

$$
\begin{aligned}
\frac{\partial x_1}{\partial x_2} &= \frac{\frac{-\partial f}{\partial x_2}}{\frac{\partial f}{\partial x_1}} \\
&= \frac{-\left(-\frac{1}{3}\right) x_1^{\frac{1}{4}}\, x_2^{-\frac{2}{3}}\, x_3^{\frac{1}{6}}}{\left(-\frac{1}{4}\right) x_1^{-\frac{3}{4}}\, x_2^{\frac{1}{3}}\, x_3^{\frac{1}{6}}} \\
&= -\,\frac{4}{3}\,\frac{x_1}{x_2}
\end{aligned}
$$

Now consider the marginal rate of substitution of $x_2$ for $x_3$

$$
\begin{aligned}
\frac{\partial x_2}{\partial x_3} &= \frac{\frac{-\partial f}{\partial x_3}}{\frac{\partial f}{\partial x_2}} \\
&= \frac{-\left(-\frac{1}{6}\right) x_1^{\frac{1}{4}}\, x_2^{\frac{1}{3}}\, x_3^{-\frac{5}{6}}}{\left(-\frac{1}{3}\right) x_1^{\frac{1}{4}}\, x_2^{-\frac{2}{3}}\, x_3^{\frac{1}{6}}} \\
&= -\,\frac{1}{2}\,\frac{x_2}{x_3}
\end{aligned}
$$

Now consider the marginal utility of $x_2$

$$
\begin{aligned}
\frac{\partial u}{\partial x_2} &= \frac{\frac{-\partial f}{\partial x_2}}{\frac{\partial f}{\partial u}} \\
&= \frac{-\left(-\frac{1}{3}\right) x_1^{\frac{1}{4}}\, x_2^{-\frac{2}{3}}\, x_3^{\frac{1}{6}}}{1} \\
&= \frac{1}{3}\, x_1^{\frac{1}{4}}\, x_2^{-\frac{2}{3}}\, x_3^{\frac{1}{6}}
\end{aligned}
$$

## 2. The general implicit function theorem with p independent variables and m implicit equations

**2.1. Motivation for implicit function theorem.** We are often interested in solving implicit systems of equations for m variables, say $x_1, x_2, \ldots, x_m$ in terms of m+p variables where there are a minimum of m equations in the system. We typically label the variables $x_{m+1}, x_{m+2}, \ldots, x_{m+p}, y_1, y_2, \ldots, y_p$. We are frequently interested in the derivatives $\frac{\partial x_i}{\partial x_j}$ where it is implicit that all other $x_k$ and all $y_\ell$ are held constant. The conditions guaranteeing that we can solve for m of the variables in terms of p variables along with a formula for computing derivatives are given by the implicit function theorem.

One motivation for the implicit function theorem is that we can eliminate m variables from a constrained optimization problem using the constraint equations. In this way the constrained problem is converted to an unconstrained problem and we can use the results on unconstrained problems to determine a solution.

**2.2. Description of a system of equations.** Suppose that we have m equations depending on m + p variables (parameters) written in implicit form as follows

$$\begin{aligned}
\phi_1 \left( x_1,\, x_2,\, \cdots,\, x_m,\, y_1,\, y_2,\, \cdots,\, y_p \right) \quad &= \; 0 \\
\phi_2 \left( x_1,\, x_2,\, \cdots,\, x_m,\, y_1,\, y_2,\, \cdots,\, y_p \right) \quad &= \; 0 \\
\vdots \qquad \vdots \qquad\qquad\qquad &= \; 0 \\
\phi_m \left( x_1,\, x_2,\, \cdots,\, x_m,\, y_1,\, y_2,\, \cdots,\, y_p \right) \quad &= \; 0
\end{aligned} \tag{17}$$

For example with m = 2 and p = 3, we might have

$$\begin{aligned}
\phi_1 \left( x_1,\; x_2,\; p,\; w_1,\; w_2 \right) \; &= \; (0.4)\; p\, x_1^{-0.6}\, x_2^{0.2} \; - \; w_1 \; = 0 \\
\phi_2 \left( x_1,\; x_2,\; p,\; w_1,\; w_2 \right) \; &= \; (0.2)\; p\, x_1^{0.4}\, x_2^{-\,0.8} \; - \; w_2 \; = 0
\end{aligned} \tag{18}$$

where p, $w_1$ and $w_2$ are the "independent" variables $y_1$, $y_2$, and $y_3$.

2.3. **Jacobian matrix of the system.** The Jacobian matrix of the system in 17 is defined as matrix of first partials as follows

$$J \; = \; \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_m} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \frac{\partial \phi_m}{\partial x_2} & \cdots & \frac{\partial \phi_m}{\partial x_m} \end{bmatrix} \tag{19}$$

This matrix will be of rank m if its determinant is not zero.

2.4. **Statement of implicit function theorem.**

**Theorem 2** (Implicit Function Theorem)**.** *Suppose that $\phi_i$ are real-valued functions defined on a domain D and continuously differentiable on an open set $D^1 \subset D \subset R^{m+p}$, where $p > 0$ and*

$$\begin{aligned}
\phi_1(x_1^0,\, x_2^0,\, \ldots,\, x_m^0,\, y_1^0,\, y_2^0,\, \ldots,\, y_p^0) \; &= 0 \\
\phi_2(x_1^0,\, x_2^0,\, \ldots,\, x_m^0,\, y_1^0,\, y_2^0,\, \ldots,\, y_p^0) \; &= 0 \\
\vdots \; &= 0 \\
\phi_m(x_1^0,\, x_2^0,\, \ldots,\, x_m^0,\, y_1^0,\, y_2^0,\, \ldots,\, y_p^0) \; &= 0 \\
(x^0,\, y^0) \; &\in D^1.
\end{aligned} \tag{20}$$

*We often write equation 20 as follows*

$$\phi_i(x^0,\, y^0) \; = \; 0, \quad i \; = \; 1,\, 2,\, \ldots,\, m, \; \textit{and} \; (x^0,\, y^0) \in D^1. \tag{21}$$

*Assume the Jacobian matrix $\left[ \frac{\partial \phi_i(x^0,\, y^0)}{\partial x_j} \right]$ has rank m. Then there exists a neighborhood $N_\delta(x^0, y^0) \subset D^1$, an open set $D^2 \subset R^p$ containing $y^0$ and real valued functions $\psi_k$, k = 1, 2, ..., m, continuously differentiable on $D^2$, such that the following conditions are satisfied:*

$$
\begin{aligned}
x_1^0 &= \psi_1(y^0) \\
x_2^0 &= \psi_2(y^0) \\
&\vdots \\
x_m^0 &= \psi_m(y^0)
\end{aligned}
\tag{22}
$$

*For every $y \in D^2$, we have*

$$
\begin{aligned}
&\phi_i(\psi_1(y),\, \psi_2(y),\, \ldots,\, \psi_m(y),\, y_1,\, y_2,\, \ldots,\, y_p\,) \;\equiv\; 0, \quad i \,=\, 1,\, 2,\, \ldots,\, m. \\
&\quad \textit{or} \\
&\phi_i(\psi(y), y) \;\equiv\; 0, \quad i \,=\, 1,\, 2,\, \ldots,\, m.
\end{aligned}
\tag{23}
$$

*We also have that for all (x,y) $\in N_\delta(x^0,\, y^0)$, the Jacobian matrix $\left[\frac{\partial \phi_i(x,y)}{\partial x_j}\right]$ has rank m. Furthermore for $y \in D^2$, the partial derivatives of $\psi(y)$ are the solutions of the set of linear equations*

$$
\begin{aligned}
\sum_{k=1}^{m} \frac{\partial \phi_1(\psi(y), y)}{\partial x_k}\, \frac{\partial \psi_k(y)}{\partial y_j} &= \frac{-\partial \phi_1(\psi(y), y)}{\partial y_j} \\
\sum_{k=1}^{m} \frac{\partial \phi_2(\psi(y), y)}{\partial x_k}\, \frac{\partial \psi_k(y)}{\partial y_j} &= \frac{-\partial \phi_2(\psi(y), y)}{\partial y_j} \\
&\vdots \\
\sum_{k=1}^{m} \frac{\partial \phi_m(\psi(y), y)}{\partial x_k}\, \frac{\partial \psi_k(y)}{\partial y_j} &= \frac{-\partial \phi_m(\psi(y), y)}{\partial y_j}
\end{aligned}
\tag{24}
$$

*We can write equation 24 in the following alternative manner*

$$
\sum_{k=1}^{m} \frac{\partial \phi_i(\psi(y), y)}{\partial x_k}\, \frac{\partial \psi_k(y)}{\partial y_j} \;=\; \frac{-\partial \phi_i(\psi(y), y)}{\partial y_j} \qquad i = 1,\, 2,\, \ldots,\, m \tag{25}
$$

*or perhaps most usefully as the following matrix equation*

$$
\begin{bmatrix}
\frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_m} \\
\frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_m} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial \phi_m}{\partial x_1} & \frac{\partial \phi_m}{\partial x_2} & \cdots & \frac{\partial \phi_m}{\partial x_m}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \psi_1(y)}{\partial y_j} \\
\frac{\partial \psi_2(y)}{\partial y_j} \\
\vdots \\
\frac{\partial \psi_m(y)}{\partial y_j}
\end{bmatrix}
=
\begin{bmatrix}
\frac{-\partial \phi_1(\psi(y), y)}{\partial y_j} \\
\frac{-\partial \phi_2(\psi(y), y)}{\partial y_j} \\
\vdots \\
\frac{-\partial \phi_m(\psi(y), y)}{\partial y_j}
\end{bmatrix}
\tag{26}
$$

*This of course implies*

$$
\begin{bmatrix}
\frac{\partial \psi_1(y)}{\partial y_j} \\
\frac{\partial \psi_2(y)}{\partial y_j} \\
\vdots \\
\frac{\partial \psi_m(y)}{\partial y_j}
\end{bmatrix}
= -
\begin{bmatrix}
\frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_m} \\
\frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_m} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial \phi_m}{\partial x_1} & \frac{\partial \phi_m}{\partial x_2} & \cdots & \frac{\partial \phi_m}{\partial x_m}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\partial \phi_1(\psi(y), y)}{\partial y_j} \\
\frac{\partial \phi_2(\psi(y), y)}{\partial y_j} \\
\vdots \\
\frac{\partial \phi_m(\psi(y), y)}{\partial y_j}
\end{bmatrix}
\tag{27}
$$

*We can expand equation 27 to cover the derivatives with respect to the other* $y_\ell$ *by expanding the matrix equation*

$$\begin{bmatrix} \frac{\partial\psi_1(y)}{\partial y_1} & \frac{\partial\psi_1(y)}{\partial y_2} & \cdots & \frac{\partial\psi_1(y)}{\partial y_p} \\ \frac{\partial\psi_2(y)}{\partial y_1} & \frac{\partial\psi_2(y)}{\partial y_2} & \cdots & \frac{\partial\psi_2(y)}{\partial y_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\psi_m(y)}{\partial y_1} & \frac{\partial\psi_m(y)}{\partial y_2} & \cdots & \frac{\partial\psi_m(y)}{\partial y_p} \end{bmatrix} = - \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_m} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\phi_m}{\partial x_1} & \frac{\partial\phi_m}{\partial x_2} & \cdots & \frac{\partial\phi_m}{\partial x_m} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial\phi_1(\psi(y),y)}{\partial y_1} & \frac{\partial\phi_1(\psi(y),y)}{\partial y_2} & \cdots & \frac{\partial\phi_1(\psi(y),y)}{\partial y_p} \\ \frac{\partial\phi_2(\psi(y),y)}{\partial y_1} & \frac{\partial\phi_2(\psi(y),y)}{\partial y_2} & \cdots & \frac{\partial\phi_2(\psi(y),y)}{\partial y_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\phi_m(\psi(y),y)}{\partial y_1} & \frac{\partial\phi_m(\psi(y),y)}{\partial y_2} & \cdots & \frac{\partial\phi_m(\psi(y),y)}{\partial y_p} \end{bmatrix} \tag{28}$$

2.5. **Some intuition for derivatives computed using the implicit function theorem.** To see the intuition of equation 24, take the total derivative of $\phi_i$ in equation 23 with respect to $y_j$ as follows

$$\begin{gathered} \phi_i\left(\psi_1(y),\ \psi_2(y),\ \cdots,\ \psi_m(y),\ y\right) = 0 \\ \frac{\partial\phi_i}{\partial\psi_1}\frac{\partial\psi_1}{\partial y_j} + \frac{\partial\phi_i}{\partial\psi_2}\frac{\partial\psi_2}{\partial y_j} + \cdots + \frac{\partial\phi_i}{\partial\psi_m}\frac{\partial\psi_m}{\partial y_j} + \frac{\partial\phi_i}{\partial y_j} = 0 \end{gathered} \tag{29}$$

and then move $\frac{\partial\phi_i}{\partial y_j}$ to the right hand side of the equation. Then perform a similar task for the other equations to obtain m equations in the m partial derivatives, $\frac{\partial x_i}{\partial y_j} = \frac{\partial\psi_i}{\partial y_j}$

For the case of only one implicit equation 24 reduces to

$$\sum_{k=1}^{m} \frac{\partial\phi(\psi(y),\ y)}{\partial x_k}\frac{\partial\psi_k(y)}{\partial y_j} = -\frac{\partial\phi(\psi(y),\ y)}{\partial y_j} \tag{30}$$

If there is only one equation, we can solve for only one of the x variables in terms of the other x variables and the p y variables and obtain only one implicit derivative.

$$\frac{\partial\phi(\psi(y),\ y)}{\partial x_k}\frac{\partial\psi_k(y)}{\partial y_j} = -\frac{\partial\phi(\psi(y),\ y)}{\partial y_j} \tag{31}$$

which can be rewritten as

$$\frac{\partial\psi_k(y)}{\partial y_j} = \frac{-\frac{\partial\phi(\psi(y),\ y)}{\partial y_j}}{\frac{\partial\phi(\psi(y),\ y)}{\partial x_k}} \tag{32}$$

This is more or less the same as equation 4.

If there are only two variables, $x_1$ and $x_2$ where $x_2$ is now like $y_1$, we obtain

$$\begin{gathered} \frac{\partial\phi(\psi_1(x_2),x_2)}{\partial x_1}\frac{\partial\psi_1(x_2)}{\partial x_2} = -\frac{\partial\phi(\psi_1(x_2),x_2)}{\partial x_2} \\ \Rightarrow \frac{\partial x_1}{\partial x_2} = \frac{\partial\psi_1(x_2)}{\partial x_2} = \frac{-\frac{\partial\phi(\psi_1(x_2),x_2)}{\partial x_2}}{\frac{\partial\phi(\psi_1(x_2),x_2)}{\partial x_1}} \end{gathered} \tag{33}$$

which is like the example in section 1.2.1 where $\phi$ takes the place of f.

2.6. **Examples.**

2.6.1. *One implicit equation with three variables.*

$$\phi(x_1^0,\, x_2^0,\, y^0) \;=\; 0 \tag{34}$$

The implicit function theorem says that we can solve equation 34 for $x_1^0$ as a function of $x_2^0$ and $y^0$, i.e.,

$$x_1^0 \;=\; \psi_1(x_2^0,\, y^0) \tag{35}$$

and that

$$\phi(\psi_1(x_2,\, y),\, x_2,\, y) \;=\; 0 \tag{36}$$

The theorem then says that

$$\begin{aligned} \frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1}\frac{\partial \psi_1}{\partial x_2} &= \frac{-\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2} \\ \Rightarrow\; \frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1}\;\frac{\partial x_1(x_2,\, y)}{\partial x_2} &= -\;\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2} \\ \Rightarrow\; \frac{\partial x_1(x_2,\, y)}{\partial x_2} &= \frac{-\;\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2}}{\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1}} \end{aligned} \tag{37}$$

2.6.2. *Production function example.*

$$\begin{aligned} \phi(x_1^0,\, x_2^0,\, y^0) \;&=\; 0 \\ y^0 \;-\; f(x_1^0,\, x_2^0) \;&=\; 0 \end{aligned} \tag{38}$$

The theorem says that we can solve the equation for $x_1^0$.

$$x_1^0 \;=\; \psi_1(x_2^0,\, y^0) \tag{39}$$

It is also true that

$$\begin{aligned} \phi(\psi_1(x_2,\, y),\, x_2,\, y) \;&= 0 \\ y \;-\; f(\psi_1(x_2,\, y), x_2) \;&= 0 \end{aligned} \tag{40}$$

Now compute the relevant derivatives

$$\begin{aligned} \frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1} \;&=\; -\;\frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_1} \\ \frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2} \;&=\; -\;\frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_2} \end{aligned} \tag{41}$$

The theorem then says that

$$\begin{aligned} \frac{\partial x_1(x_2,\, y)}{\partial x_2} \;&=\; -\;\left[\frac{\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2}}{\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1}}\right] \\ &=\; -\;\left[\frac{-\;\frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_2}}{-\;\frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_1}}\right] \\ &=\; -\;\frac{\frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_2}}{\frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_1}} \end{aligned} \tag{42}$$

2.7. **General example with two equations and three variables.** Consider the following system of equations

$$\begin{aligned} \phi_1(x_1,\, x_2,\, y) &= 3x_1 \,+\, 2x_2 \,+\, 4y \,=\, 0 \\ \phi_2\,(x_1,\, x_2,\, y) &= 4x_1 \,+\, \ x_2 \,+\, \ y \,=\, 0 \end{aligned} \tag{43}$$

The Jacobian is given by

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \tag{44}$$

We can solve system 43 for $x_1$ and $x_2$ as functions of y. Move y to the right hand side in each equation.

$$3x_1 \,+\, 2x_2 \,=\, -4y \tag{45a}$$

$$4x_1 \,+\, x_2 \,=\, -y \tag{45b}$$

Now solve equation 45b for $x_2$

$$x_2 \,=\, -y \,-4x_1 \tag{46}$$

Substitute the solution to equation 46 into equation 45a and simplify

$$\begin{aligned} 3x_1 \,+\, 2(-y \,-4x_1) \,&=\, -4y \\ \Rightarrow\, 3x_1 \,-\, 2y \,-\, 8x_1 \,&=\, -4y \\ \Rightarrow\, -5x_1 \,&=\, -2y \\ \Rightarrow\, x_1 \,=\, \frac{2}{5}y \,&=\, \psi_1(y) \end{aligned} \tag{47}$$

Substitute the solution to equation 47 into equation 46 and simplify

$$\begin{aligned} x_2 \,=\, -y \,-4\left[\frac{2}{5}y\right] & \\ \Rightarrow\, x_2 \,=\, -\,\frac{5}{5}y \,-\, \frac{8}{5}y & \\ =\, -\,\frac{13}{5}y \,=\, \psi_2(y) & \end{aligned} \tag{48}$$

If we substitute these expressions for $x_1$ and $x_2$ into equation 43 we obtain

$$\begin{aligned} \phi_1\left(\frac{2}{5}y\,,\, -\frac{13}{5}y,\, y\right)) &= 3\left[\frac{2}{5}y\right] \,+\, 2\left[-\,\frac{13}{5}y\right] \,+\, 4y \\ &= \frac{6}{5}y \,-\, \frac{26}{5}y \,+\, \frac{20}{5}y \\ &= -\,\frac{20}{5}y \,+\, \frac{20}{5}y \,=\, 0 \end{aligned} \tag{49}$$

and

$$\begin{aligned}\phi_2\left(\frac{2}{5}y\,,\,-\,\frac{13}{5}y,\,y\right)) &= 4\left[\frac{2}{5}y\right] \,+\, \left[-\,\frac{13}{5}y\right] \,+\, y \\ &= \frac{8}{5}y \,-\, \frac{13}{5}y \,+\, \frac{5}{5}y \\ &= \frac{13}{5}y \,-\, \frac{13}{5}y \,=\, 0\end{aligned} \tag{50}$$

Furthermore

$$\begin{aligned}\frac{\partial\psi_1}{\partial y} &= \frac{2}{5} \\ \frac{\partial\psi_2}{\partial y} &= -\,\frac{13}{5}\end{aligned} \tag{51}$$

We can solve for these partial derivatives using equation 24 as follows

$$\frac{\partial\phi_1}{\partial x_1}\,\frac{\partial\psi_1}{\partial y} \,+\, \frac{\partial\phi_1}{\partial x_2}\,\frac{\partial\psi_2}{\partial y} \,=\frac{-\partial\phi_1}{\partial y} \tag{52a}$$

$$\frac{\partial\phi_2}{\partial x_1}\,\frac{\partial\psi_1}{\partial y} \,+\, \frac{\partial\phi_2}{\partial x_2}\,\frac{\partial\psi_2}{\partial y} \,=\frac{-\partial\phi_2}{\partial y} \tag{52b}$$

Now substitute in the derivatives of $\phi_1$ and $\phi_2$ with respect to $x_1$, $x_2$, and y.

$$3\,\frac{\partial\psi_1}{\partial y} \,+\, 2\,\frac{\partial\psi_2}{\partial y} \,=\, -\,4 \tag{53a}$$

$$4\,\frac{\partial\psi_1}{\partial y} \,+\, 1\,\frac{\partial\psi_2}{\partial y} \,=\, -\,1 \tag{53b}$$

Solve equation 53b for $\frac{\partial\psi_2}{\partial y}$

$$\frac{\partial\psi_2}{\partial y} \,=\, -1 \,-\, 4\,\frac{\partial\psi_1}{\partial y} \tag{54}$$

Now substitute the answer from equation 54 into equation 53a

$$\begin{aligned}3\,\frac{\partial\psi_1}{\partial y} \,+\, 2\left(-\,1 \,-\, 4\,\frac{\partial\psi_1}{\partial y}\right) &= -\,4 \\ \Rightarrow\; 3\,\frac{\partial\psi_1}{\partial y} \,-\, 2 \,-\, 8\,\frac{\partial\psi_1}{\partial y} &= -\,4 \\ \Rightarrow\; -\,5\,\frac{\partial\psi_1}{\partial y} &= -\,2 \\ \Rightarrow\; \frac{\partial\psi_1}{\partial y} &= \frac{2}{5}\end{aligned} \tag{55}$$

If we substitute equation 55 into equation 54 we obtain

$$
\begin{aligned}
\frac{\partial \psi_2}{\partial y} &= -\,1 \,-\, 4\,\frac{\partial \psi_1}{\partial y} \\
\Rightarrow\ \frac{\partial \psi_2}{\partial y} &= -\,1 \,-\, 4\left(\frac{2}{5}\right) \\
&= \frac{-5}{5} \,-\, \frac{8}{5} \,=\, -\frac{13}{5}
\end{aligned} \tag{56}
$$

We could also do this by inverting the matrix.

2.7.1. *Profit maximization example 1.* Consider the system in equation 18. We can solve this system of m implicit equations for all m of the x variables as functions of the p independent (y) variables. Specifically, we can solve the two equations for $x_1$ and $x_2$ as functions of p, $w_1$, and $w_2$, i.e., $x_1 = \psi_1$(p, $w_1$, $w_2$ ) and $x_2 = \psi_2$(p, $w_1$, $w_2$ ). We can also find $\frac{\partial x_1}{\partial p}$and$\frac{\partial x_2}{\partial p}$ , that is $\left(\frac{\partial \psi_i}{\partial p}\right)$ , from the following two equations derived from equation 18 which is repeated here.

$$
\begin{aligned}
\phi_1\,(x_1,\ x_2,\ p,\ w_1,\ w_2) &= (0.4)\ p\,x_1^{-0.6}\ x_2^{0.2} \,-\, w_1 \,=0 \\
\phi_2\,(x_1,\ x_2,\ p,\ w_1,\ w_2\,) &= (0.2)\ p\,x_1^{0.4}\ x_2^{-\,0.8} \,-\, w_2 \,=0
\end{aligned}
$$

$$
\begin{aligned}
\left[\,(-\,0.24)\,p\,x_1^{-1.6}\,x_2^{0.2}\,\right]\ \frac{\partial\,x_1}{\partial\,p} \,+\, \left[\,(\,0.08)\,p\,x_1^{-0.6}\,x_2^{-\,0.8}\,\right]\ \frac{\partial\,x_2}{\partial\,p} \,&=\, -\ (0.4)\ x_1^{-0.6}\ x_2^{0.2} \\
\left[\,(\,0.08)\,p\,x_1^{-0.6}\,x_2^{-0.8}\,\right]\ \frac{\partial\,x_1}{\partial\,p} \,+\, \left[\,(\,-\,0.16)\,p\,x_1^{0.4}\,x_2^{-\,1.8}\,\right]\ \frac{\partial\,x_2}{\partial\,p} \,&=\, -\ (0.2)\ x_1^{0.4}\ x_2^{-\,0.8}
\end{aligned} \tag{57}
$$

We can write this in matrix form as

$$
\begin{bmatrix} (-\,0.24)\,p\,x_1^{-1.6}\,x_2^{0.2} & (\,0.08)\,p\,x_1^{-0.6}\,x_2^{-\,0.8} \\ (\,0.08)\,p\,x_1^{-0.6}\,x_2^{-0.8} & (\,-0.16)\,p\,x_1^{0.4}\,x_2^{-\,1.8} \end{bmatrix} \begin{bmatrix} \frac{\partial\,x_1}{\partial\,p} \\ \frac{\partial\,x_2}{\partial\,p} \end{bmatrix} \,=\, \begin{bmatrix} -\ (0.4)\ x_1^{-0.6}\ x_2^{0.2} \\ -\ (0.2)\ x_1^{0.4}\ x_2^{-\,0.8} \end{bmatrix} \tag{58}
$$

2.7.2. *Profit maximization example 2.* Let the production function for a firm by given by

$$
y \,=\, 14x_1 \,+\, 11x_2 \,-\, x_1^2 \,-\, x_2^2 \tag{59}
$$

Profit for the firm is given by

$$
\begin{aligned}
\pi\ &= py \,-\, w_1\,x_1 \,-\, w_2\,x_2 \\
&= p\,(14x_1 \,+\, 11x_2 \,-\, x_1^2 \,-\, x_2^2) \,-\, w_1\,x_1 \,-\, w_2\,x_2
\end{aligned} \tag{60}
$$

The first order conditions for profit maximization imply that

$$
\begin{aligned}
\pi \,&=\, 14\,p\,x_1 \,+\, 11\,p\,x_2 \,-\, p\,x_1^2 \,-\, p\,x_2^2 \,-\, w_1\,x_1 \,-\, w_2\,x_2 \\
\frac{\partial\,\pi}{\partial\,x_1} \,&=\, \phi_1 \,=14\,p \,-\, 2\,p\,x_1 \,-\, w_1 \,=\, 0 \\
\frac{\partial\,\pi}{\partial\,x_2} \,&=\, \phi_1 \,=11\,p \,-\, 2\,p\,x_2 \,-\, w_2 \,=\, 0
\end{aligned} \tag{61}
$$

We can solve the first equation for $x_1$ as follows

$$\begin{aligned}\frac{\partial\pi}{\partial x_1} &= 14p - 2px_1 - w_1 = 0\\ \Rightarrow\ 2px_1 &= 14p - w_1\\ \Rightarrow\ x_1 &= \frac{14p - w_1}{2p}\\ &= 7 - \frac{w_1}{2p}\end{aligned}\tag{62}$$

In a similar manner we can find $x_2$ from the second equation

$$\begin{aligned}\frac{\partial\pi}{\partial x_2} &= 11p - 2px_2 - w_2 = 0\\ \Rightarrow\ 2px_2 &= 11p - w_2\\ \Rightarrow\ x_2 &= \frac{11p - w_2}{2p}\\ &= 5.5 - \frac{w_2}{2p}\end{aligned}\tag{63}$$

We can find the derivatives of $x_1$ and $x_2$ with respect to p, $w_1$ and $w_2$ directly as follows:

$$\begin{aligned}x_1 &= 7 - \frac{1}{2}w_1p^{-1}\\ x_2 &= 5.5 - \frac{1}{2}w_2p^{-1}\\ \frac{\partial x_1}{\partial p} &= \frac{1}{2}w_1p^{-2}\\ \frac{\partial x_1}{\partial w_1} &= -\frac{1}{2}p^{-1}\\ \frac{\partial x_1}{\partial w_2} &= 0\\ \frac{\partial x_2}{\partial p} &= \frac{1}{2}w_2p^{-2}\\ \frac{\partial x_2}{\partial w_2} &= -\frac{1}{2}p^{-1}\\ \frac{\partial x_2}{\partial w_2} &= 0\end{aligned}\tag{64}$$

We can also find these derivatives using the implicit function theorem. The two implicit equations are

$$\begin{aligned}\phi_1(x_1,\ x_2,\ p,\ w_1,\ w_2) &= 14p - 2px_1 - w_1 = 0\\ \phi_2(x_1,\ x_2\ p,\ w_1,\ w_2) &= 11p - 2px_2 - w_2 = 0\end{aligned}\tag{65}$$

First we check the Jacobian of the system. It is obtained by differentiating $\phi_1$ and $\phi_2$ with respect to $x_1$ and $x_2$ as follows

$$J = \begin{bmatrix}\frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2}\\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix}-2p & 0\\ 0 & -2p\end{bmatrix}\tag{66}$$

The determinant is $4p^2$ which is positive. Now we have two equations we can solve using the implicit function theorem. The theorem says

$$\sum_{k=1}^{m} \frac{\partial \phi_i(g\,(y),\; y)}{\partial x_k}\, \frac{\partial g_k(y)}{\partial y_j} \;=\; -\,\frac{\partial \phi_i(\psi(y),\; y)}{\partial y_j}\;,\; i \;=\; 1,\, 2,\, \cdots \tag{67}$$

For the case of two equations we obtain

$$\begin{aligned} \frac{\partial\,\phi_1}{\partial\,x_1}\,\frac{\partial\,x_1}{\partial\,p} \;+\; \frac{\partial\,\phi_1}{\partial\,x_2}\,\frac{\partial\,x_2}{\partial\,p} \;&=\; -\;\frac{\partial\,\phi_1}{\partial\,p} \\ \frac{\partial\,\phi_2}{\partial\,x_1}\,\frac{\partial\,x_1}{\partial\,p} \;+\; \frac{\partial\,\phi_2}{\partial\,x_2}\,\frac{\partial\,x_2}{\partial\,p} \;&=\; -\;\frac{\partial\,\phi_2}{\partial\,p} \end{aligned} \tag{68}$$

We can write this in matrix form as follows

$$\begin{bmatrix} \frac{\partial\,\phi_1}{\partial\,x_1} & \frac{\partial\,\phi_1}{\partial\,x_2} \\ \frac{\partial\,\phi_2}{\partial\,x_1} & \frac{\partial\,\phi_2}{\partial\,x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial\,x_1}{\partial\,p} \\ \frac{\partial\,x_2}{\partial\,p} \end{bmatrix} \;=\; \begin{bmatrix} -\,\frac{\partial\,\phi_1}{\partial\,p} \\ -\,\frac{\partial\,\phi_2}{\partial\,p} \end{bmatrix} \tag{69}$$

Solving for $\frac{\partial\,x_1}{\partial\,p}$ and $\frac{\partial\,x_2}{\partial\,p}$ we obtain

$$\begin{bmatrix} \frac{\partial\,x_1}{\partial\,p} \\ \frac{\partial\,x_2}{\partial\,p} \end{bmatrix} \;=\; \begin{bmatrix} \frac{\partial\,\phi_1}{\partial\,x_1} & \frac{\partial\,\phi_1}{\partial\,x_2} \\ \frac{\partial\,\phi_2}{\partial\,x_1} & \frac{\partial\,\phi_2}{\partial\,x_2} \end{bmatrix}^{-1} \begin{bmatrix} -\,\frac{\partial\,\phi_1}{\partial\,p} \\ -\,\frac{\partial\,\phi_2}{\partial\,p} \end{bmatrix} \tag{70}$$

We can compute the various partial derivatives of $\phi$ as follows

$$\begin{aligned} \phi_1(x_1,\; x_2,\; p,\, w_1,\, w_2) \;&= 14\,p \;-\; 2\,p\,x_1 \;-\; w_1 \;=\; 0 \\ \frac{\partial\,\phi_1}{\partial\,x_1} \;&=\; -\;2\,p \\ \frac{\partial\,\phi_1}{\partial\,x_2} \;&= 0 \\ \frac{\partial\,\phi_1}{\partial\,p} \;&= 14 \;-\; 2\,x_1 \\ \phi_2(x_1,\; x_2\;p,\, w_1,\, w_2) \;&= 11\,p \;-\; 2\,p\,x_2 \;-\; w_2 \;=\; 0 \\ \frac{\partial\,\phi_2}{\partial\,x_1} \;&=\; -\;2\,p \\ \frac{\partial\,\phi_2}{\partial\,x_2} \;&= 0 \\ \frac{\partial\,\phi_1}{\partial\,p} \;&= 11 \;-\; 2\,x_2 \end{aligned} \tag{71}$$

Now writing out the system we obtain for the case at hand we obtain

$$\begin{bmatrix} \frac{\partial\,x_1}{\partial\,p} \\ \frac{\partial\,x_2}{\partial\,p} \end{bmatrix} \;=\; \begin{bmatrix} \frac{\partial\,\phi_1}{\partial\,x_1} & \frac{\partial\,\phi_1}{\partial\,x_2} \\ \frac{\partial\,\phi_2}{\partial\,x_1} & \frac{\partial\,\phi_2}{\partial\,x_2} \end{bmatrix}^{-1} \begin{bmatrix} -\,\frac{\partial\,\phi_1}{\partial\,p} \\ -\,\frac{\partial\,\phi_2}{\partial\,p} \end{bmatrix} \tag{72}$$

Now substitute in the elements of the inverse matrix

$$\begin{bmatrix} \frac{\partial\,x_1}{\partial\,p} \\ \frac{\partial\,x_2}{\partial\,p} \end{bmatrix} \;=\; \begin{bmatrix} -2p & 0 \\ 0 & -2p \end{bmatrix}^{-1} \begin{bmatrix} 2\,x_1 \;-\; 14 \\ 2\,x_2 \;-\; 11 \end{bmatrix} \tag{73}$$

If we then invert the matrix we obtain

$$\begin{bmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2p} & 0 \\ 0 & -\frac{1}{2p} \end{bmatrix} \begin{bmatrix} 2\,x_1 \;-\; 14 \\ 2\,x_2 \;-\; 11 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7 \;-\; x_1}{p} \\ \frac{5.5 \;-\; x_2}{p} \end{bmatrix} \tag{74}$$

Given that

$$\begin{aligned} x_1 &= 7 \;-\; \frac{1}{2}\, w_1\, p^{-1} \\ x_2 &= 5.5 \;-\; \frac{1}{2}\, w_2\, p^{-1} \end{aligned} \tag{75}$$

we obtain

$$\begin{aligned} \frac{\partial\, x_1}{\partial\, p} &= \frac{7 \;-\; x_1}{p} \\ &= \frac{7 \;-\; (7 \;-\; \frac{1}{2}\, w_1\, p^{-1})}{p} \\ &= \frac{1}{2}\, w_1\, p^{-2} \\ \frac{\partial\, x_2}{\partial\, p} &= \frac{5.5 \;-\; x_2}{p} \\ &= \frac{5.5 \;-\; (5.5 \;-\; \frac{1}{2}\, w_2\, p^{-1})}{p} \\ &= \frac{1}{2}\, w_2\, p^{-2} \end{aligned} \tag{76}$$

which is the same as before.

2.7.3. *Profit maximization example 3.* Verify the implicit function theorem derivatives $\frac{\partial\, x_1}{\partial\, w_1}$, $\frac{\partial\, x_1}{\partial\, w_2}$, $\frac{\partial\, x_2}{\partial\, w_1}$, $\frac{\partial\, x_2}{\partial\, w_2}$ for the profit maximization example 2.

2.7.4. *Profit maximization example 4.* A firm sells its output into a perfectly competitive market and faces a fixed price p. It hires labor in a competitive labor market at a wage w, and rents capital in a competitive capital market at rental rate r. The production is f(L, K). The production function is strictly concave. The firm seeks to maximize its profits which are

$$\pi \;=\; pf(L,\; K) \;-\; wL \;-\; rK \tag{77}$$

The first-order conditions for profit maximization are

$$\begin{aligned} \pi_L &= p\, f_L(L^*,\; K^*) \;-\; w \;=\; 0 \\ \pi_K &= p\, f_K(L^*,\; K^*) \;-\; r \;=\; 0 \end{aligned} \tag{78}$$

This gives two implicit equations for K and L. The second order conditions are

$$\frac{\partial^2\, \pi}{\partial L^2}\,(L^*,\, K^*) \;<\; 0, \qquad \frac{\partial^2\, \pi}{\partial L^2}\, \frac{\partial^2\, \pi}{\partial K^2} \;-\; \left[ \frac{\partial^2\, \pi}{\partial L\, \partial K} \right]^2 \;>\; 0\; at\; (L^*,\;\; K^*) \tag{79}$$

We can compute these derivatives as

$$
\begin{aligned}
\frac{\partial^2 \pi}{\partial L^2} &= \frac{\partial \left( p\, f_L\,(L^*,\ K^*)\ -\ w \right)}{\partial\, L} \ = \ p\, F_{LL} \\
\frac{\partial^2 \pi}{\partial K^2} &= \frac{\partial \left( p\, f_K\,(L^*,\ K^*)\ -\ r \right)}{\partial\, L} \ = \ p\, F_{KK} \\
\frac{\partial^2 \pi}{\partial\, L\, \partial\, K} &= \frac{\partial \left( p\, f_L\,(L^*,\ K^*)\ -\ w \right)}{\partial\, K} \ = \ p\, F_{KL}
\end{aligned}
\tag{80}
$$

The first of the second order conditions is satisfied by concavity of f. We then write the second condition as

$$
\begin{aligned}
D \ &= \begin{vmatrix} p\, f_{LL} & p\, f_{L\,K} \\ p\, f_{K\,L} & p\, f_{K\,K} \end{vmatrix} \ > \ 0 \\
&\Rightarrow\ p^2 \begin{vmatrix} f_{LL} & f_{L\,K} \\ f_{K\,L} & f_{K\,K} \end{vmatrix} \ > \ 0 \\
&\Rightarrow\ p^2 (f_{LL}\ f_{KK}\ -\ f_{KL}\ f_{LK})\ >\ 0
\end{aligned}
\tag{81}
$$

The expression is positive or at least non-negative because f is assumed to be strictly concave.

Now we wish to determine the effects on input demands, $L^*$ and $K^*$, of changes in the input prices. Using the implicit function theorem in finding the partial derivatives with respect to w, we obtain for the first equation

$$
\begin{aligned}
\pi_{LL}\, \frac{\partial\, L^*}{\partial\, w} \ + \ \pi_{LK}\, \frac{\partial\, K^*}{\partial\, w} \ + \ \pi_{Lw} \ &= 0 \\
\Rightarrow\ p\, f_{LL}\, \frac{\partial\, L^*}{\partial\, w} \ + \ p\, f_{KL}\, \frac{\partial\, K^*}{\partial\, w} \ - \ 1 \ &= 0
\end{aligned}
\tag{82}
$$

For the second equation we obtain

$$
\begin{aligned}
\pi_{KL}\, \frac{\partial\, L^*}{\partial\, w} \ + \ \pi_{KK}\, \frac{\partial\, K^*}{\partial\, w} \ + \ \pi_{Kw} \ &= 0 \\
\Rightarrow\ p\, f_{KL}\, \frac{\partial\, L^*}{\partial\, w} \ + \ p\, f_{KK}\, \frac{\partial\, K^*}{\partial\, w} \ &= 0
\end{aligned}
\tag{83}
$$

We can write this in matrix form as

$$
\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{KL} & pf_{KK} \end{bmatrix} \begin{bmatrix} \partial L^*/\partial w \\ \partial K^*/\partial w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\tag{84}
$$

Using Cramer's rule, we then have the comparative-statics results

$$
\begin{aligned}
\frac{\partial L^*}{\partial w} &= \frac{\begin{vmatrix} 1 & pf_{LK} \\ 0 & pf_{KK} \end{vmatrix}}{D} = \frac{p\, f_{KK}}{D} \\
\frac{\partial K^*}{\partial w} &= \frac{\begin{vmatrix} pf_{LL} & 1 \\ pf_{KL} & 0 \end{vmatrix}}{D} = -\frac{p\, f_{KL}}{D}
\end{aligned}
\tag{85}
$$

The sign of the first is negative because $f_{KK} < 0$ and $D > 0$ from the second-order conditions. Thus, the demand curve for labor has a negative slope. However, in order to sign the effect of a change in the wage rate on the demand for capital, we need to know the sign of $f_{KL}$, the effect of a change in the labor input on the marginal product of capital. We can derive the effects of a change in the rental rate of capital in a similar way.

2.7.5. *Profit maximization example 5.* Let the production function be Cobb-Douglas of the form $y = L^{\alpha} K^{\beta}$. Find the comparative-static effects $\partial L^*/\partial w$ and $\partial K^*/\partial w$. Profits are given by

$$\begin{aligned} \pi &= p\, f\,(L,\ K)\ -\ wL\ -\ rK \\ &= p\, L^{\alpha}\, K^{\beta}\ -\ w\, L\ -\ r\, K \end{aligned} \tag{86}$$

The first-order conditions are for profit maximization are

$$\begin{aligned} \pi_L &= \alpha\, p\, L^{\alpha\,-\,1}\, K^{\beta}\ -\ w\ =\ 0 \\ \pi_K &= \beta\, p\, L^{\alpha}\, K^{\beta\,-\,1}\ -\ r\ =\ 0 \end{aligned} \tag{87}$$

This gives two implicit equations for K and L. The second order conditions are

$$\frac{\partial^2\,\pi}{\partial L^2}\,(L^*,\ K^*)\ <\ 0, \qquad \frac{\partial^2\,\pi}{\partial L^2}\,\frac{\partial^2\,\pi}{\partial K^2}\ -\ \left[\frac{\partial^2\,\pi}{\partial L\,\partial K}\right]^2\ >\ 0\ at\ (L^*,\ K^*) \tag{88}$$

We can compute these derivatives as

$$\begin{aligned} \frac{\partial^2\,\pi}{\partial L^2} &= \alpha\,(\alpha\ -\ 1)\,p\ L^{\alpha\,-\,2}\,K^{\beta} \\ \frac{\partial^2\,\pi}{\partial K^2} &\equiv \beta\,(\beta\ -\ 1)\,p\ L^{\alpha}\ K^{\beta\,-\,2} \\ \frac{\partial^2\,\pi}{\partial\,L\,\partial\,K} &= \alpha\,\beta\,p\ L^{\alpha\,-\,1}\ K^{\beta\,-\,1} \end{aligned} \tag{89}$$

The first of the second order conditions is satisfied as long as $\alpha, \beta < 1$. We can write the second condition as

$$D\ =\ \begin{vmatrix} \alpha\,(\alpha\ -\ 1)\,p\ L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\,p\ L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\,p\ L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\ -\ 1)\,p\ L^{\alpha}\,K^{\beta\,-\,2} \end{vmatrix}\ >\ 0 \tag{90}$$

We can simplify D as follows

$$\begin{aligned} D\ &=\ \begin{vmatrix} \alpha\,(\alpha\ -\ 1)\,p\ L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\,p\ L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\,p\ L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\ -\ 1)\,p\ L^{\alpha}\,K^{\beta\,-\,2} \end{vmatrix} \\ &= p^2\ \begin{vmatrix} \alpha\,(\alpha\ -\ 1)\ L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\ L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\ L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\ -\ 1)\ L^{\alpha}\,K^{\beta\,-\,2} \end{vmatrix} \\ &= p^2\ \left[\alpha\ \beta\,(\alpha\ -\ 1)\,(\beta\ -\ 1)\ L^{\alpha\,-\,2}\ K^{\beta}\ L^{\alpha}\ K^{\beta\,-\,2}\ -\ \alpha^2\,\beta^2\ L^{2\alpha\,-\,2}\,K^{2\beta\,-\,2}\right] \\ &= p^2\ \left[\alpha\ \beta\,L^{\alpha\,-\,2}\ K^{\beta}\ L^{\alpha}\ K^{\beta\,-\,2}\ \left(\,(\alpha\ -\ 1)\,(\beta\ -\ 1)\ -\ \alpha\,\beta\,\right)\right] \\ &= p^2\ \left[\alpha\ \beta\,L^{\alpha\,-\,2}\ K^{\beta}\ L^{\alpha}\ K^{\beta\,-\,2}\ \left(\,\alpha\beta\ -\ \beta\ -\ \alpha\ +\ 1\ -\ \alpha\,\beta\,\right)\right] \\ &= p^2\ \left[\alpha\ \beta\,L^{\alpha\,-\,2}\ K^{\beta}\ L^{\alpha}\ K^{\beta\,-\,2}\ \left(\,1\ -\ \alpha\ -\ \beta\,\right)\right] \end{aligned} \tag{91}$$

The condition is then that

$$D\ =\ p^2\,\alpha\,\beta\,L^{2\alpha-2}\ K^{2\beta-2}\,(1\ -\ \alpha\ -\ \beta)\ >\ 0 \tag{92}$$

This will be true if $\alpha+\beta < 1$.

Now we wish to determine the effects on input demands, L* and K*, of changes in the input prices. Using the implicit function theorem in finding the partial derivatives with respect to w, we obtain for the first equation

$$\begin{aligned} \pi_{LL}\,\frac{\partial L^*}{\partial w} \;+\; \pi_{LK}\,\frac{\partial K^*}{\partial w} \;+\; \pi_{Lw} &= 0 \\ \Rightarrow\; \alpha\,(\alpha\,-\,1)\,p\,L^{\alpha\,-\,2}\,K^{\beta}\,\frac{\partial L^*}{\partial w} \;+\; \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1}\,\frac{\partial K^*}{\partial w} \;-\; 1 &= 0 \end{aligned} \tag{93}$$

For the second equation we obtain

$$\begin{aligned} \pi_{KL}\,\frac{\partial L^*}{\partial w} \;+\; \pi_{KK}\,\frac{\partial K^*}{\partial w} \;+\; \pi_{Kw} &= 0 \\ \Rightarrow\; \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1}\,\frac{\partial L^*}{\partial w} \;+\; \beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2} &= 0 \end{aligned} \tag{94}$$

We can write this in matrix form as

$$\begin{bmatrix} \alpha\,(\alpha\,-\,1)\,p\,L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2} \end{bmatrix} \begin{bmatrix} \partial L^*/\partial w \\ \partial K^*/\partial w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{95}$$

Using Cramer's rule, we then have the comparative-statics results

$$\begin{aligned} \frac{\partial L^*}{\partial w} &= \frac{\begin{vmatrix} 1 & \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ 0 & \beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2} \end{vmatrix}}{D} = \frac{\beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2}}{D} \\ \frac{\partial K^*}{\partial w} &= \frac{\begin{vmatrix} \alpha\,(\alpha\,-\,1)\,p\,L^{\alpha\,-\,2}\,K^{\beta} & 1 \\ \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & 0 \end{vmatrix}}{D} = \frac{-\,\alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1}}{D} \end{aligned} \tag{96}$$

The sign of the first of is negative because $p\,\beta\,(\beta\,-\,1)\,L^{\alpha}\,K^{\beta\,-\,2}\;<\;0\;(\beta < 1)$, and D $>$ 0 by the second order conditions. The cross partial derivative $\frac{\partial K^*}{\partial w}$ is also less than zero because $p\,\alpha\,\beta\,L^{\alpha\,-\,1}K^{\beta\,-\,1}\;>\;0$ and D $>$ 0. So, for the case of a two-input Cobb-Douglas production function, an increase in the wage unambiguously reduces the demand for capital.

## 3. Tangents to $\phi(x_1, x_2, \ldots, ) = c$ and Properties of the Gradient

### 3.1. **Direction numbers.**

A direction in $\Re^2$ is determined by an ordered pair of two numbers (a,b), not both zero, called direction numbers. The direction corresponds to all lines parallel to the line through the origin (0,0) and the point (a,b). The direction numbers (a,b) and the direction numbers (ra,rb) determine the same direction, for any nonzero r.

A direction in $\Re^n$ is determined by an ordered n-tuple of numbers $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$, not all zero, called direction numbers. The direction corresponds to all lines parallel to the line through the origin (0,0, . . . ,0) and the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$. The direction numbers $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ and the direction numbers $\left(rx_1^0, rx_2^0, \ldots, rx_n^0\right)$ determine the same direction, for any nonzero r. If pick an r in the following way.

$$|r| \;=\; \frac{1}{\sqrt{(x_1^0)^2 \;+\; (x_2^0)^2 \;+\; \ldots \;+\; (x_n^0)^2}} \;=\; \frac{1}{|x^0|} \tag{97}$$

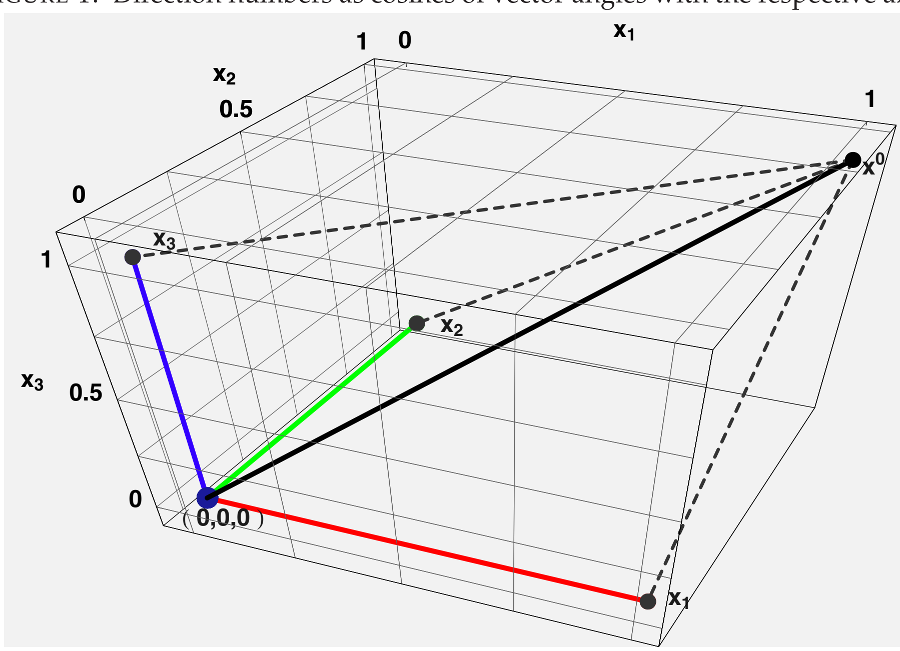
then

$$
\begin{aligned}
\cos\gamma_1 &= \frac{x_1^0}{\sqrt{(x_1^0)^2 \,+\, (x_2^0)^2 \,+\, \ldots \,+\, (x_n^0)^2}} \\
\cos\gamma_2 &= \frac{x_2^0}{\sqrt{(x_1^0)^2 \,+\, (x_2^0)^2 \,+\, \ldots \,+\, (x_n^0)^2}} \\
&\vdots \\
\cos\gamma_n &= \frac{x_n^0}{\sqrt{(x_1^0)^2 \,+\, (x_2^0)^2 \,+\, \ldots \,+\, (x_n^0)^2}}
\end{aligned} \tag{98}
$$

where $\gamma_j$ is the angle of the vector running through the origin and the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ with the $x_j$ axis. With this choice of r, the direction numbers (ra,rb) are just the cosines of the angles that the line makes with the positive $x_1$,$x_2$, $\ldots$ and $x_n$ axes. These angles $\gamma_1$, $\gamma_2$ , $\ldots$ are called the direction angles, and their cosines $(\cos[\gamma_1], \cos[\gamma_2], \ldots)$ are called the direction cosines of that direction. See figure 1. They satisfy

$$
\cos^2[\gamma_1] \,+\, \cos^2[\gamma_2] \,+\, \ldots \,+\, \cos^2[\gamma_n] \,=\, 1
$$

FIGURE 1. Direction numbers as cosines of vector angles with the respective axes




From equation 98, we can also write

$$
\begin{aligned}
x^0 \;&=\; \left(x_1^0, x_2^0, \ldots, x_n^0\right) \;=\; \left(|x^0| \cos[\gamma_1],\; |x^0| \cos[\gamma_2],\; \ldots\;, |x^0| \cos[\gamma_n]\right) \\
&= |x^0| \left(\cos[\gamma_1], \cos[\gamma_2], \ldots, \cos[\gamma_n]\right)
\end{aligned} \tag{99}
$$

Therefore

$$\frac{1}{|x^0|}x^0 \;=\; (\cos[\gamma_1], \cos[\gamma_2], \ldots, \cos[\gamma_n]) \tag{100}$$

which says that the direction cosines of $x^0$ are the components of the unit vector in the direction of $x^0$.

**Theorem 3.** *If$\theta$ is the angle between the vectors $\vec{a}$ abd $\vec{b}$, then*

$$a \cdot b \;=\; |a|\,|b|\cos\theta \tag{101}$$

3.2. **Planes in $\Re^n$.**

3.2.1. *Planes through the origin.* A plane through the origin in $\Re^n$ is given implicitly by the equation

$$\begin{aligned} p'x \;&= 0 \\ \rightarrow p_1x_1 \,+\, p_2x_2 \,+\, \ldots \,+\, p_nx_n \;&= 0 \end{aligned} \tag{102}$$

3.2.2. *More general planes in $\Re^n$.* A plane in $\Re^n$ through the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ is given implicitly by the equation

$$\begin{aligned} p'(x \,-\, x^0) \;&= 0 \\ \rightarrow p_1(x_1 - x_1^0) \,+\, p_2(x_2 - x_2^0) \,+\, \ldots \,+\, p_n(x_n - x_n^0) \;&= 0 \end{aligned} \tag{103}$$

We say that the vector p is orthogonal to the vector $\left(x \,-\, x^0\right)$ or

$$\begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix} \begin{bmatrix} x_1 \,-\, x_1^0 \\ x_2 \,-\, x_2^0 \\ \vdots \\ x_n \,-\, x_n^0 \end{bmatrix} \;=\; 0 \tag{104}$$

The coefficients [$p_1$,$p_2$, ... , $p_n$] are the direction numbers of a line L passing through the point $x^0$ or alternatively they are the coordinates of a point on a line through the origin that is parallel to L. If $p_1$ is equal to -1, we can write equation 104 as follows

$$x_1 \,-\, x_1^0 \;=\; p_2(x_2 - x_2^0) \,+\, p_3(x_2 - x_3^0) \,+\, \ldots \,+\, p_n(x_n - x_n^0) \tag{105}$$

3.3. **The general equation for a tangent hyperplane.**

**Theorem 4.** *Suppose $D^1 \subset \Re^n$ is open, $\left(x_1^0, x_2^0, \ldots, x_n^0\right) \in D^1$, $\phi : D^1 \rightarrow \Re^1$ is continuously differentiable, and $D\phi\left(x_1^0, x_2^0, \ldots, x_n^0\right) \neq 0$. Suppose $\phi\left(x_1^0, x_2^0, \ldots, x_n^0\right) = c$ Then consider the level surface of the function $\phi\left(x_1, x_2, \ldots, x_n\right) = \phi\left(x_1^0, x_2^0, \ldots, x_n^0\right) = c$. Denote this level service by M = $\phi^{-1}(\{c\})$ which consists all all values of $(x_1, x_2, \ldots, x_n)$ such that $\phi(x_1, x_2, \ldots, x_n) = c$. Then the tangent hyperplane at $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ of M is given by*

$$T_{x^0}M \;=\; \{x \in \Re^n : D\phi(x^0)(x - x^0) = 0\} \tag{106}$$

*that is $\nabla\phi(x^0)$ is nomal to the tangent hyperplane.*

*Proof.* Because $D\phi(x^0) \neq 0$, we may assume without loss of generality that $\frac{\partial \phi}{\partial x_1}(x^0) \neq 0$. Apply the implicit function theorem (Theorem 1) to the function $\phi - c$. We then know that the level set can be expressed as a graph of the function $x_1 = \psi_1(x_2, x_3, \ldots, x_n)$ for some continuous function $\psi_1$. Now the tangent plane to the graph $x_1 = \psi_1(x_2, x_3, \ldots, x_n)$ at the point $x^0 = [\psi_1(x_2^0, x_3^0, \ldots, x_n^0), x_2^0, x_3^0, \ldots x_n^0]$ is the graph of the gradient $D\psi_1(x_2^0, x_3^0, \ldots, x_n^0)$ translated so that it passes through $x^0$. Using equation 105 where $\frac{\partial \psi_1}{\partial x_k}\left(x_2^0, x_3^0, \ldots, x_n^0\right)$ are the direction numbers we obtain

$$x_1 - x_1^0 = \sum_{k=2}^{n} \frac{\partial \psi_1}{\partial x_k}\left(x_2^0, x_3^0, \ldots, x_n^0\right)(x_k - x_k^0) \tag{107}$$

Now make the substitution from equation 4 in equation 107 to obtain

$$\begin{aligned} x_1 - x_1^0 &= \sum_{k=2}^{n} - \left( \frac{\frac{\partial \phi(\psi_1(x_2^0, x_3^0, \ldots, x_n^0), x_2^0, x_3^0, \ldots, x_n^0)}{\partial x_k}}{\frac{\partial \phi(\psi_1(x_2^0, x_3^0, \ldots, x_n^0), x_2^0, x_3^0, \ldots, x_n^0)}{\partial x_1}} \right)(x_k - x_k^0) \\ &= \sum_{k=2}^{n} - \left( \frac{\frac{\partial \phi(x^0)}{\partial x_k}}{\frac{\partial \phi(x^0)}{\partial x_1}} \right)(x_k - x_k^0) \end{aligned} \tag{108}$$

Rearrange equation 108 to obtain

$$\sum_{k=2}^{n} \frac{\partial \phi(x^0)}{\partial x_k}(x_k - x_k^0) + \frac{\partial \phi(x^0)}{\partial x_1}(x_1 - x_1^0) = D\phi(x^0)(x - x^0) = 0 \tag{109}$$

□

3.4. **Example.** Consider a function

$$y = f\,(x_1, x_2, \cdots, x_n) \tag{110}$$

evaluated at the point $(\, x_1^0,\; x_2^0,\; \cdots,\; x_n^0 \,)$

$$y^0 = f\,(x_1^0, x_2^0, \cdots, x_n^0) \tag{111}$$

The equation of the hyperplane tangent to the surface at this point is given by

$$\begin{aligned} (y - y^0) &= \frac{\partial f}{\partial x_1}(x_1 - x_1^0) + \frac{\partial f}{\partial x_2}(x_2 - x_2^0) + \cdots + \frac{\partial f}{\partial x_n}(x_n - x_n^0) \\ \Rightarrow \frac{\partial f}{\partial x_1}(x_1 - x_1^0) &+ \frac{\partial f}{\partial x_2}(x_2 - x_2^0) + \cdots + \frac{\partial f}{\partial x_n}(x_n - x_n^0) - (y - y^0) = 0 \\ \Rightarrow f(x_1, x_2, \cdots, x_n) &= f(x_1^0, x_2^0, \cdots, x_n^0) + \frac{\partial f}{\partial x_1}(x_1 - x_1^0) + \frac{\partial f}{\partial x_2}(x_2 - x_2^0) + \cdots + \frac{\partial f}{\partial x_n}(x_n - x_n^0) \end{aligned} \tag{112}$$

where the partial derivatives $\frac{\partial f}{\partial x_i}$ are evaluated at $(\, x_1^0,\; x_2^0,\; \cdots,\; x_n^0 \,)$. We can also write this as

$$f\,(x) - f\,(x^0) = \nabla f\,(x^0)\,(x - x^0) \tag{113}$$

Compare this to the tangent equation with one variable

$$\begin{aligned} y - f\,(x^0) &= f'(x^0)\,(x - x^0) \\ \Rightarrow y &= f\,(x^0) + f'(x^0)\,(x - x^0) \end{aligned} \tag{114}$$

and the tangent plane with two variables

$$\begin{aligned} y \;-\; f(x_1^0, x_2^0) \;&= \frac{\partial f}{\partial x_1}(x_1 - x_1^0) \;+\; \frac{\partial f}{\partial x_2}(x_2 - x_2^0) \\ \Rightarrow\; y \;&= f(x_1^0, x_2^0) \;+\; \frac{\partial f}{\partial x_1}(x_1 - x_1^0) \;+\; \frac{\partial f}{\partial x_2}(x_2 - x_2^0) \end{aligned} \tag{115}$$

3.5. **Vector functions of a real variable.** Let $x_1, x_2, \ldots, x_n$ be functions of a variable t defined on an interval I and write

$$x(t) \;=\; (x_1(t), x_2(t), \ldots, x_n(t)) \tag{116}$$

The function t → x(t) is a transformation from R to $R^n$ and is called a vector function of a real variable. As x runs through I, x(t) traces out a set of points in n-space called a curve. In particular, if we put

$$x_1\,(t) \;=\; x_1^0 \;+\; t\,a_1\;,\; x_2\,(t) \;=\; x_2^0 \;+\; t\,a_2\;,\; \cdots\;,\; x_n\,(t) \;=\; x_n^0 \;+\; t\,a_n \tag{117}$$

the resulting curve is a straight line in n-space. It passes through the point $x^0 \;=\; (\,x_1^0,\; x_2^0,\; \cdots\;,\; x_n^0\,)$ at t = 0, and it is in the direction of the vector $a \;=\; (\,a_1,\; a_2,\; \cdots\;,\; a_n\,)$. We can define the derivative of x(t) as

$$\frac{d\,x}{d\,t} \;= \dot{x}\,(t) \;=\; \left( \frac{d\,x_1(t)}{d\,t},\, \frac{d\,x_2(t)}{d\,t}\,,\, \cdots\,,\, \frac{d\,x_n(t)}{d\,t} \right) \tag{118}$$

If K is a curve in n-space traced out by x(t), then $\dot{x}\,(t)$ can be interpreted as a vector tangent to K at the point t.

3.6. **Tangent vectors.**

Consider the function $r(t) \;=\; (x_1(t),\, x_2(t), \cdots, x_n(t)\,)$ and its derivative $\dot{r}\,(t) \;=\; (\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t))$. This function traces out a curve in $R^n$. We can call the curve K. For fixed t,

$$\dot{r}\,(t) \;=\; \lim_{h\,\rightarrow\,0} \frac{r\,(t\;+\;h)\;-\;r\,(t)}{h} \tag{119}$$

If $\dot{r}\,(t) \neq 0$, then for t + h close enough to t, the vector r(t+h) - r(t) will not be zero. As h tends to 0, the quantity r(t+h) - r(t) will come closer and closer to serving as a direction vector for the tangent to the curve at the point P. This can be seen in figure 2

It may be tempting to take this difference as approximation of the direction of the tangent and then take the limit

$$\lim_{h\,\rightarrow\,0}\;[\,r\,(t\;+\;h)\;-\;r\,(t)\,] \tag{120}$$

and call the limit the direction vector for the tangent. But the limit is zero and the zero vector has no direction. Instead we use the a vector that for small h has a greater length, that is we use

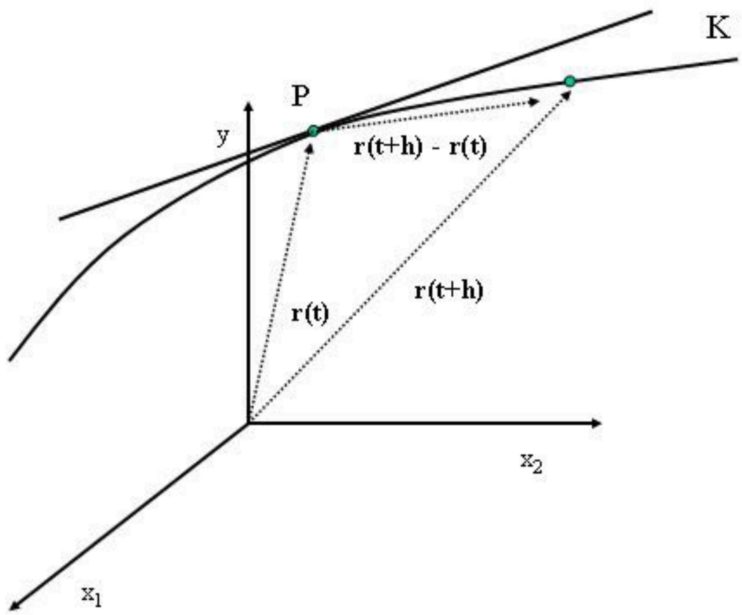
$$\frac{r\,(t\;+\;h)\;-\;r\,(t)}{h} \tag{121}$$

For any real number h, this vector is parallel to r(t+h) - r(t). Therefore its limit

$$\dot{r}\,(t) \;=\; \lim_{h\,\rightarrow\,0} \frac{r\,(t\;+\;h)\;-\;r\,(t)}{h} \tag{122}$$

can be taken as a direction vector for the tangent line.

FIGURE 2. Tangent to a Curve




3.7. **Tangent lines, level curves and gradients.** Consider the equation

$$f\,(x) \;=\; f\;(x_1,\,x_2,\,\cdots,\,x_n) \;=\; c \tag{123}$$

which defines a level curve for the function f. Let $x^0 \;=\; (\,x_1^0,\,x_2^0,\;\cdots\,,\;x_n^0\,)$ be a point on the surface and let $x\,(t) \;=\; (\,x_1(t),\,x_2(t)\,,\,\cdots\,,\,x_n(t)\,)$ represent a differentiable curve K lying on the surface and passing through $x^0$ at t = $t^0$. Because K lies on the surface, f [x(t)] = f [$x_1$(t), $x_2$(t), ,,, $x_n$(t)] = c for all t. Now differentiate equation 123 with respect to t

$$\begin{aligned} \frac{\partial\,f\,(x)}{\partial\,x_1}\,\frac{\partial\,x_1}{\partial\,t} \;+\; \frac{\partial\,f\,(x)}{\partial\,x_2}\,\frac{\partial\,x_2}{\partial\,t} \;+\;\cdots\;+\; \frac{\partial\,f\,(x)}{\partial\,x_2}\,\frac{\partial\,x_2}{\partial\,t} \;&=0 \\ \Rightarrow\;\nabla\,f\,(x^0)\;\cdot\;\dot{x}\,(t^0)\;&=0 \end{aligned} \tag{124}$$

Because the vector $\dot{x}\,(t) \;=\; (\,\dot{x}_1(t),\,\dot{x}_2(t)\,,\,\cdots\,,\,\dot{x}_n(t)\,)$ has the same direction as the tangent to the curve K at $x^0$, the gradient of f is orthogonal to the curve K at the point $x^0$.

**Theorem 5.** *Suppose f($x_1$, $x_2$, . . . , $x_n$) is continuous and differentiable in a domain A and suppose that x=($x_1$, $x_2$, . . . , $x_n$) $\in$ A. The gradient*

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

*has the following properties at points x where $\nabla f(x) \neq 0$.*

(i) *$\nabla f(x)$ is orthogonal to the level surfaces $f(x_1, x_2, \ldots, x_n) = c$.*
(ii) *$\nabla f(x)$ points in the direction of the steepest increase on f.*
(iii) *$||\nabla f(x)||$ is the value of the directional derivative in the direction of steepest increase.*

## References

[1] Hadley, G. *Linear Algebra*. Reading, MA: Addison-Wesley, 1961.
[2] Sydsaeter, Knut. *Topics in Mathematical Analysis for Economists*. New York: Academic Press, 1981.