MATRIX ALGEBRA AND SYSTEMS OF EQUATIONS

1. Systems of Equations and Matrices

1.1. **Representation of a linear system.** The general system of m equations in n unknowns can be written

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
\vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
\tag{1}
$$

In this system, the a_{ij}'s and b_i's are given real numbers; a_{ij} is the coefficient for the unknown x_j in the i^{th} equation. We call the set of all a_{ij}'s arranged in a rectangular array the coefficient matrix of the system. Using matrix notation we can write the system as

$$
Ax = b
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ . \\ . \\ x_n
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 \\ . \\ . \\ b_m
\end{pmatrix}
\tag{2}
$$

We define the augmented coefficient matrix for the system as

$$
\hat{A} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\
\vdots & \vdots & \vdots & \vdots & \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{pmatrix}
\tag{3}
$$

Consider the following matrix A and vector b

$$
A = \begin{pmatrix}
1 & 2 & 1 \\
2 & 5 & 2 \\
-3 & -4 & -2
\end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix}
\tag{4}
$$

We can then write

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$$\begin{aligned} Ax &= b \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix} \end{aligned} \tag{5}$$

for the linear equation system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + 5x_2 + 2x_3 &= 8 \\ -3x_1 - 4x_2 - 2x_3 &= -4 \end{aligned} \tag{6}$$

1.2. **Row-echelon form of a matrix.**

1.2.1. *Leading zeroes in the row of a matrix.* A row of a matrix is said to have k leading zeroes if the first k elements of the row are all zeroes and the (k+1)th element of the row is not zero.

1.2.2. *Row echelon form of a matrix.* A matrix is in row echelon form if each row has more leading zeroes than the row preceding it.

1.2.3. *Examples of row echelon matrices.* The following matrices are all in row echelon form

$$\begin{aligned} A &= \begin{pmatrix} 3 & 4 & 7 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{pmatrix} \end{aligned} \tag{7}$$

The matrix C would imply the equation system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= b_1 \\ 4x_2 + x_3 &= b_2 \\ 3x_3 &= b_3 \end{aligned} \tag{8}$$

which could be easily solved for the various variable after dividing the third equation by 3.

1.2.4. *Pivots.* The first non-zero element in each row of a matrix in row-echelon form is called a **pivot**. For the matrix A above the pivots are 3,5,4. For the matrix B they are 1,2 and for C they are 1,4,3. For the matrix B there is no pivot in the last row.

1.2.5. *Reduced row echelon form.* A row echelon matrix in which each pivot is a 1 and in which each column containing a pivot contains no other nonzero entries, is said to be in reduced row echelon form. This implies that columns containing pivots are columns of an identity matrix. The matrices D and E below are in reduced row echelon form.

$$\begin{aligned} D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ E &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{9}$$

The matrix F is in row echelon form but not in reduced row echelon form.

$$F = \begin{pmatrix} 0 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

1.2.6. *Reducing a matrix to row echelon or reduced row echelon form.* Any system of equations can be reduced to a system with the A portion of the augmented matrix $\tilde{A}$ in echelon or reduced echelon form by performing what are called elementary row operations on the matrix $\tilde{A}$. These operations are the some ones that we used when solving a linear system using the method of Gaussian elimination.

1: Changing the order in which the equations or rows are listed produces an equivalent system.

2: Multiplying both sides of a single equation of the system (or a single row of $\tilde{A}$) by a nonzero number (leaving the other equations unchanged) results in a system of equations that is equivalent to the original system.

3: Adding a multiple of one equation or row to another equation or row (leaving the other equations unchanged) results in a system of equations that is equivalent to the original system.

1.2.7. *Rank.*

Definition 1 (Rank)**.** The number of non-zero rows in the row echelon form of an m $\times$ n matrix A produced by elementary operations on A is called the rank of A. Alternatively, the rank of an m $\times$ n matrix A is the number of pivots we obtain by reducing a matrix to row echelon form.

Matrix D in equation 9 has rank 3, matrix E has rank 2, while matrix F in 10 has rank 3.

Definition 2 (Full column rank)**.** We say that an m $\times$ n matrix A has full column rank if r = n. If r = n, all columns of A are pivot columns.

Definition 3 (Full row rank)**.** We say that an m $\times$ n matrix A has full row rank if r = m. If r = m, all rows of A have pivots and the reduced row echelon form has no zero rows.

1.2.8. *Solutions to equations (stated without proof).*

a: A system of linear equations with coefficient matrix A which is m $\times$ n, a right hand side vector b which is m $\times$ 1, and augmented matrix $\hat{A}$ has a solution if and only if

$$rank\,(A) \;=\; rank\,(\hat{A})$$

b: A linear system of equations must have either no solution, one solution, or infinitely many solutions.
c: If the rank of A = r = n < m, the linear system Ax = b has no solution or exactly one solution. In other words, if the matrix A is of full column rank and has less columns than rows, the system will be inconsistent, or will have exactly one solution.
d: If the rank of A = r = m < n, the linear system Ax = b will always have a solution and there will be an infinite number of solutions. In other words, if the matrix A is of full row rank and has more columns than rows, the system will always have an infinite number of solutions.
e: If the rank of A = r where r< m and r < n, the linear system Ax = b will either have no solutions or there will be an infinite number of solutions. In other words, if the matrix A has neither full row nor column rank, the system will be inconsistent or will have an infinite number of solutions.
f: If the rank of A = r = m = n, the linear system Ax = b will exactly one solution. In other words, if the matrix A is square and has full rank, the system will have a unique solution.

A coefficient matrix is said to be **nonsingular**, that is, the corresponding linear system has one and only one solution for every choice of right hand side b_1, b_2, ... , b_m, if and only if

$$number\ of\ rows\ of\ A\ =\ number\ of\ columns\ of\ A\ =\ rank\,(A)$$

1.3. **Solving systems of linear equations by finding the reduced echelon form of a matrix and back substitution.** To solve a system of linear equations represented by a matrix equation, we first add the right hand side vector to the coefficient matrix to form the augmented coefficient matrix. We then perform elementary row and column operations on the augmented coefficient matrix until it is in row echelon form or reduced row echelon form. If the matrix is in row echelon we can solve it by back substitution, if it is in reduced row echelon form we can read the coefficients off directly from the matrix. Consider matrix equation 5 with its augmented matrix $\tilde{A}$.

$$\begin{aligned} Ax &= b \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix} \\ \tilde{A} &= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix} \end{aligned} \tag{11}$$

Step 1. Consider the first row and the first column of the matrix. Element a_{11} is a non-zero number and so is appropriate for the first row of the row echelon form. If element a_{11} were zero, we would exchange row 1 with a row of the matrix with a non-zero element in column 1.

Step 2. Use row operations to "knock out" a_{21} by turning it into a zero. This would then make the second row have one more leading zero than the first row. We can turn the a_{21} element into a zero by adding negative 2 times the first row to the second row.

$$\begin{aligned} -2 \times &\begin{pmatrix} 1 & 2 & 1 & 3 \end{pmatrix} \\ \Rightarrow &\begin{pmatrix} -2 & -4 & -2 & -6 \end{pmatrix} \end{aligned}$$

$$\begin{array}{cccc} -2 & -4 & -2 & -6 \\ 2 & 5 & 2 & 8 \\ -- & -- & -- & -- \\ 0 & 1 & 0 & 2 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_1$.

$$\tilde{A}_1 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -3 & -4 & -2 & -4 \end{pmatrix} \tag{12}$$

We can obtain $\tilde{A}_1$ by premultiplying $\tilde{A}$ by what is called the a_{21} elimination matrix denoted by E_{21} where E_{21} is given by

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

The first and third rows of E_{21} are just rows of the identity matrix because the first and third rows of $\tilde{A}$ and $\tilde{A}_1$ are the same. The second row of $\tilde{A}_1$ is a linear combination of the rows of $\tilde{A}$ where we add negative 2 times the first row of $\tilde{A}$ to 1 times the second row of $\tilde{A}$. To convince yourself of this fact remember what it means to premultiply a matrix by a row vector. For example

$$\begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} -5 & -6 \end{pmatrix}$$

For the case at hand we have

$$\tilde{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -3 & -4 & -2 & -4 \end{pmatrix} \tag{14}$$

Step 3. Use row operations to "knock out" a_{31} by turning it into a zero. This would then make the third row have one more leading zero than the first row. We can turn the a_{31} element into a zero by adding 3 times the first row to the third row.

$$3\times \begin{pmatrix} 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 6 & 3 & 9 \end{pmatrix}$$

$$\begin{array}{cccc} 3 & 6 & 3 & 9 \\ -3 & -4 & -2 & -4 \\ -- & -- & -- & -- \\ 0 & 2 & 1 & 5 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_2$.

$$\tilde{A}_2 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{pmatrix} \tag{15}$$

We can obtain $\tilde{A}_2$ by premultiplying $\tilde{A}_1$ by what is called the a_{31} elimination matrix denoted by E_{31} where E_{31} is given by

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \tag{16}$$

The first and second rows of E_{31} are just rows of the identity matrix because the first and second rows of $\tilde{A}_1$ and $\tilde{A}_2$ are the same. The third row of $\tilde{A}_2$ is a linear combination of the rows of $\tilde{A}_1$ where we add 3 times the first row of $\tilde{A}_1$ to 1 times the third row of $\tilde{A}_1$. For the case at hand we have

$$\tilde{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -3 & -4 & -2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{pmatrix} \tag{17}$$

Step 4. Use row operations to "knock out" a_{32} by turning it into a zero. This would then make the third row have one more leading zero than the second row. We can turn the a_{32} element into a zero by adding negative 2 times the second row of $\tilde{A}_2$ to the third row of $\tilde{A}_2$.

$$\begin{gathered} -2\times \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 0 & -2 & 0 & -4 \end{pmatrix} \\ \begin{array}{cccc} 0 & -2 & 0 & -4 \\ 0 & 2 & 1 & 5 \\ -- & -- & -- & -- \\ 0 & 0 & 1 & 1 \end{array} \end{gathered}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_3$.

$$\tilde{A}_3 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{18}$$

We can obtain $\tilde{A}_3$ by premultiplying $\tilde{A}_3$ by what is called the a_{32} elimination matrix denoted by E_{32} where E_{32} is given by

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \tag{19}$$

The first and second rows of E_{32} are just rows of the identity matrix because the first and second rows of $\tilde{A}_2$ and $\tilde{A}_3$ are the same. The third row of $\tilde{A}_3$ is a linear combination of the rows of $\tilde{A}_2$ where we add negative 2 times the first second of $\tilde{A}_2$ to 1 times the third row of $\tilde{A}_2$. For the case at hand we have

$$\tilde{A}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{20}$$

We can then read off the solution for x_3 because the third equation says that $x_3 = 1$. The second equation says that $x_2 = 2$. We then solve for x_1 by substituting in the first equation as follows.

$$\begin{aligned} x_1 \ + \ 2x_2 \ + \ x_3 \ &= 3 \\ \Rightarrow \ x_1 \ + \ 2(2) \ + \ (1) \ &= 3 \\ \Rightarrow \ x_1 \ + \ 4 + \ 1 \ &= 3 \\ \Rightarrow \ x_1 \ &= -2 \end{aligned} \tag{21}$$

1.4. **Solving systems of linear equations by finding the reduced row echelon form of a matrix.** To solve a system of linear equations represented by a matrix equation, we first add the right hand side vector to the coefficient matrix to form the augmented coefficient matrix. We then perform elementary row and column operations on the augmented coefficient matrix until it is in reduced row echelon form. We can read the coefficients off directly from the matrix. Consider the system of equations in section 1.3

$$\begin{aligned} Ax \ &= b \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ &= \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix} \\ \tilde{A} \ &= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix} \end{aligned} \tag{22}$$

After performing suitable row operations we obtained the matrix $\tilde{A}_3$ in equation 20

$$\tilde{A}_3 \ = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{23}$$

¿From this equation system we obtained the solutions by back substitution. To find the reduced row echelon form we need to eliminate all non-zero entries in the pivot columns. We begin with column 3. The a_{23} element is already zero, so we can proceed to the a_{13} element. We denote this in step 5.

Step 5. Use row operations to ”knock out” a_{13} by turning it into a zero. We can turn the a_{13} element into a zero by adding negative 1 times the third row to the first row.

$$\begin{gathered} -1 \times \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} \\ \begin{array}{cccc} 0 & 0 & -1 & -1 \\ 1 & 2 & 1 & 3 \\ -- & -- & -- & -- \\ 1 & 2 & 0 & 2 \end{array} \end{gathered}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_4$.

$$\tilde{A}_4 \ = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{24}$$

We can obtain $\tilde{A}_4$ by premultiplying $\tilde{A}_3$ by what is called the a_{13} elimination matrix denoted by E_{13} where E_{13} is given by

$$E_{13} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

For the case at hand we have

$$\tilde{A}_4 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{26}$$

The third column of the matrix has a pivot in the third row and zeroes in the first and second rows.

Step 6. Use row operations to "knock out" a_{12} by turning it into a zero. We can turn the a_{12} element into a zero by adding negative 2 times the second row to the first row.

$$\begin{aligned} -2 \times & \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix} \\ \Rightarrow & \begin{pmatrix} 0 & -2 & 0 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{array}{cccc} 0 & -2 & 0 & -4 \\ 1 & 2 & 0 & 2 \\ -- & -- & -- & -- \\ 1 & 0 & 0 & -2 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_5$.

$$\tilde{A}_5 = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{27}$$

We can obtain $\tilde{A}_5$ by premultiplying $\tilde{A}_4$ by what is called the a_{12} elimination matrix denoted by E_{12} where E_{12} is given by

$$E_{12} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{28}$$

For the case at hand we have

$$\tilde{A}_5 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{29}$$

It is obvious that x_1 = -2, x_2 = 2, and x_3 = 1.

1.5. **Systems of linear equations and determinants.**

1.5.1. *Solving a general 2x2 equation system using elementary row operations.* Consider the following simple 2x2 system of linear equations where the A matrix is written in a completely general form.

$$\begin{aligned} a_{11}\, x_1 \;+\; a_{12} x_2 \;&= b_1 \\ a_{21}\, x_1 \;+\; a_{22}\, x_2 \;&= b_2 \end{aligned} \tag{30}$$

We can write this in matrix form as

$$\begin{aligned} Ax \;&= b \\ A \;&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\; x \;=\; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\; b \;=\; \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \end{aligned} \tag{31}$$

If we append the column vector b to the matrix A, we obtain the augmented matrix for the system. This is written as

$$\tilde{A} \;=\; \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} \tag{32}$$

We can perform row operations on this matrix to reduce it to reduced row echelon form. We will do this in steps. The first step is to divide each element of the first row by a_{11}. This will make the first pivot a one. This will give

$$\tilde{A}_1 \;=\; \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & b_2 \end{bmatrix} \tag{33}$$

Now multiply the first row by a_{21} to yield

$$\begin{bmatrix} a_{21} & \dfrac{a_{21}\, a_{12}}{a_{11}} & \dfrac{a_{21}\, b_1}{a_{11}} \end{bmatrix} \tag{34}$$

and subtract it from the second row

$$\begin{array}{lll} a_{21} & a_{22} & b_2 \\ a_{21} & \dfrac{a_{21}\, a_{12}}{a_{11}} & \dfrac{a_{21}\, b_1}{a_{11}} \\ -\,-\,- & -\,-\,-\,- & -\,-\,-\,- \\ 0 & a_{22} \;-\; \dfrac{a_{21}\, a_{12}}{a_{11}} & b_2 \;-\; \dfrac{a_{21}\, b_1}{a_{11}} \end{array} \tag{35}$$

This will give a new matrix on which to operate.

$$\tilde{A}_2 \;=\; \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{a_{11}\, a_{22} \,-\, a_{21}\, a_{12}}{a_{11}} & \frac{a_{11}\, b_2 \,-\, a_{21}\, b_1}{a_{11}} \end{bmatrix} \tag{36}$$

Now multiply the second row by $\frac{a_{11}}{a_{11}\, a_{22} \,-\, a_{21}\, a_{12}}$ to obtain

$$\tilde{A}_3 \;=\; \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{a_{11}\, b_2 \,-\, a_{21}\, b_1}{a_{11}\, a_{22} \,-\, a_{21}\, a_{12}} \end{bmatrix} \tag{37}$$

Now multiply the second row by $\frac{a_{12}}{a_{11}}$ and subtract it from the first row. First multiply the second row by $\frac{a_{12}}{a_{11}}$ to yield.

$$\begin{bmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{12}\,(\,a_{11}\, b_2 \,-\, a_{21}\, b_1\,)}{a_{11}\,(\,a_{11}\, a_{22} \,-\, a_{21}\, a_{12}\,)} \end{bmatrix} \tag{38}$$

Now subtract the expression in equation 38 from the first row of $\tilde{A}_3$ to obtain the following row.

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \end{bmatrix} - \begin{bmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{12}\,(a_{11}\,b_2 - a_{21}\,b_1)}{a_{11}\,(a_{11}\,a_{22} - a_{21}\,a_{12})} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{b_1}{a_{11}} - \frac{a_{12}\,(a_{11}\,b_2 - a_{21}\,b_1)}{a_{11}\,(a_{11}\,a_{22} - a_{21}\,a_{12})} \end{bmatrix} \tag{39}$$

Now replace the first row in $\tilde{A}_3$ with the expression in equation 39 to obtain $\tilde{A}_4$

$$\tilde{A}_4 = \begin{bmatrix} 1 & 0 & \frac{b_1}{a_{11}} - \frac{a_{12}\,(a_{11}\,b_2 - a_{21}\,b_1)}{a_{11}\,(a_{11}\,a_{22} - a_{21}\,a_{12})} \\ 0 & 1 & \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} \end{bmatrix} \tag{40}$$

This can be simplified by putting the upper right hand term over a common denominator, and canceling like terms as follows

$$\begin{aligned} \tilde{A}_4 &= \begin{bmatrix} 1 & 0 & \frac{b_1\,a_{11}^2\,a_{22} - b_1\,a_{11}\,a_{21}\,a_{12} - a_{12}\,a_{11}^2\,b_2 + a_{11}\,a_{12}\,a_{21}\,b_1}{a_{11}^2\,(a_{11}\,a_{22} - a_{21}\,a_{12})} \\ 0 & 1 & \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \frac{b_1\,a_{11}^2\,a_{22} - a_{12}\,a_{11}^2\,b_2}{a_{11}^2\,(a_{11}\,a_{22} - a_{21}\,a_{12})} \\ 0 & 1 & \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \frac{b_1\,a_{22} - a_{12}\,b_2}{a_{11}\,a_{22} - a_{21}\,a_{12}} \\ 0 & 1 & \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} \end{bmatrix} \end{aligned} \tag{41}$$

We can now read off the solutions for x_1 and x_2. They are

$$\begin{aligned} x_1 &= \frac{b_1\,a_{22} - a_{12}\,b_2}{a_{11}\,a_{22} - a_{21}\,a_{12}} \\ x_2 &= \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} \end{aligned} \tag{42}$$

Each of the fractions has a common denominator. It is called the **determinant** of the matrix A. Note that this part of the solution of the equation Ax=b does not depend on b; it will be the same no matter what right hand side is chosen. That is why it is called the determinant of the matrix A and not of the system Ax = b. The **determinant** of an arbitrary 2x2 matrix A is given by

$$\det(A) = |A| = a_{11}\,a_{22} - a_{21}\,a_{12} \tag{43}$$

The determinant of the 2x2 matrix is given by difference in the product of the elements along the main diagonal and the reverse main diagonal. The determinant of the matrix A = $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is 2-6 = -4.

If we look closely at the solutions in equation 42, we can see that we can also write the numerator of each expression as a determinant. In particular

$$\begin{aligned} x_1 &= \frac{b_1\,a_{22} - a_{12}\,b_2}{a_{11}\,a_{22} - a_{21}\,a_{12}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|} \\ x_2 &= \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|A|} \end{aligned}$$

The matrix of which we compute the determinant in the numerator of the first expression is the matrix A, where the first column has been replaced by the b vector. The matrix of which we compute the determinant in the numerator of the second expression is the matrix A where the second column has been replaced by the b vector. This procedure for solving systems of equations is called **Cramer's rule** and will be discussed in more detail later. For those interested, here is a simple example.

1.5.2. *An example problem with Cramer's rule.* Consider the system of equations

$$\begin{aligned} 3x_1 \;+\; 5x_2 \;&= 11 \\ 8x_1 \;-\; 3x_2 \;&= 13 \end{aligned} \tag{44}$$

Using Cramer's rule

$$\begin{aligned} x_1 \;&= \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|\,A\,|} \;=\; \frac{\begin{vmatrix} 11 & 5 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 8 & -3 \end{vmatrix}} \;=\; \frac{(-33) \;-\; (65)}{(-9) \;-\; (40)} \;=\; \frac{-98}{-49} \;=\; 2 \\ x_2 \;&= \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|\,A\,|} \;=\; \frac{\begin{vmatrix} 3 & 11 \\ 8 & 13 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 8 & -3 \end{vmatrix}} \;=\; \frac{(39) \;-\; (88)}{(-9) \;-\; (40)} \;=\; \frac{-\,49}{-49} \;=\; 1 \end{aligned} \tag{45}$$

2. Determinants

2.1. **Introduction.** Associated with every square matrix is a number called the determinant of the matrix. The determinant provides a useful summary "measurement" of that matrix. Of particular importance is whether the determinant of a matrix is zero. As we saw in section 1.5, elements of the solution of the equation Ax = b for the 2x2 case can be written in terms of the determinant of the matrix A. For a square matrix A, the equation Ax = 0 has a non-trivial solution (x $\neq$ 0) if and only if the determinant of A is zero.

2.2. **Definition and analytical computation.**

2.2.1. *Definition of a determinant.* The determinant of an n $\times$ n matrix A = $\|\; a_{ij} \;\|$, written |A|, is defined to be the number computed from the following sum where each element of the sum is the product of n elements:

$$|A| \;=\; \sum (\pm)\; a_{1i}\, a_{2\,j} ... a_{n\,r}, \tag{46}$$

the sum being taken over all n! permutations of the second subscripts. A term is assigned a plus sign if (i,j,...,r) is an even permutation of (1,2,...,n) and a minus sign if it is an odd permutation. An even permutation is defined as making an even number of switches of the indices in (1,2,...,n), similarly for an odd permutation. Thus each term in the summation is the product of n terms, one term from each column of the matrix.

When there are no switches in the indices from the natural order, the product will be a_{11} a_{22} a_{33} . . . a_{nn} or the product of the terms on the main diagonal. When the only switch is between the subscripts 1 and 2 of the first two terms we obtain the product a_{12} a_{21} a_{33} . . . a_{nn}. This product will be assigned a minus sign. If we switch subscripts 1 and 3 of the first and third terms we obtain the product $a_{13}\, a_{22}\, a_{31} \ldots a_{nn}$ which also has a minus sign.

2.2.2. *Example 1.* As a first example consider the following 2x2 case

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \tag{47}$$

Here a_{11} = 1, a_{12} = 2, a_{21} = 2, and a_{22} = 5. There are two permutations of (1,2).

1: (1,2) is an even permutation with 0 switches of the indices (1,2): + $a_{11}\, a_{22}$
2: (2,1) is odd permutation with 1 switch of the indices: - $a_{12}\, a_{21}$

Notice that the permutations are over the second subscripts. The determinant of A is then

$$|A| \;=\; +a_{11}\, a_{22} \;-\; a_{12}\, a_{21} \;=\; 1 * 5 \;-\; 2 * 2 \;=\; 1 \tag{48}$$

The 2 x 2 case is easy to remember because there are just 2 rows and 2 columns. Each term in the summation has two elements and there are two terms. The first term is the product of the diagonal elements and the second term is the product of the off-diagonal elements. The first term has a plus sign because the second subscripts are 1 and 2 while the second term has a negative sign because the second subscripts are 2 and 1 (one permutation of 1 and 2).

2.2.3. *Example 2.* Now consider another 2x2 example

$$A = \begin{pmatrix} 3 & 5 \\ 8 & -3 \end{pmatrix} \tag{49}$$

Here a_{11} = 3, a_{12} = 5, a_{21} = 8, and a_{22} = -3. The determinant of A is given by

$$|A| \;=\; +a_{11}\, a_{22} \;-\; a_{12}\, a_{21} \;=\; (3)\,(-3) \;-\; (8)(5) \;=\; -49 \tag{50}$$

2.2.4. *Example 3.* Consider the following 3x3 case

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix} \tag{51}$$

The indices are the numbers 1, 2 and 3. There are 6 (3!) permutations of (1, 2, 3). They are as follows: (1, 2, 3), **(2, 1, 3), (1, 3, 2), (3, 2, 1)**, (2, 3, 1) and (3, 1, 2). The three in bold come from switching one pair of indices in the natural order (1, 2, 3). The final two come from making a switch in one of the bold sets. Specifically

1: (1, 2, 3) is even (0 switches): + $a_{11}\, a_{22}\, a_{33}$
2: (2, 1, 3) is odd (1 switches): - $a_{12}\, a_{21}\, a_{33}$
3: (1, 3, 2) is odd (1 switches): - $a_{11}\, a_{23}\, a_{32}$
4: (3, 2, 1) is odd (1 switches): - $a_{13}\, a_{22}\, a_{31}$
5: (2, 3, 1) is even (2 switches): + $a_{12}\, a_{23}\, a_{31}$
6: (3, 1, 2) is even (2 switches): + $a_{13}\, a_{21}\, a_{32}$

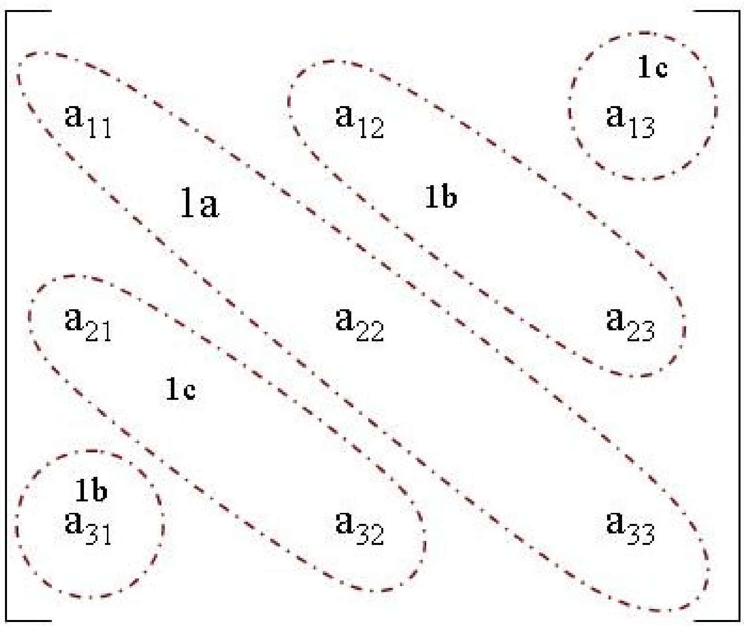
Notice that the permutations are over the second subscripts, that is, every term is of the form $a_{1\,.}\; a_{2\,.}\; a_{3\,.}$ with the second subscript varying. The determinant of A is then

$$
\begin{aligned}
|A| \;=\; & + a_{11}\, a_{22}\, a_{33} \\
& - a_{12}\, a_{21}\, a_{33} \\
& - a_{11}\, a_{23},\;\, a_{32} \\
& - a_{13}\, a_{22}\, a_{31} \\
& + a_{12}\, a_{23}\, a_{31} \\
& + a_{13}\, a_{21}\, a_{32} \\
& = 20 - 0 - 0 - 15 + 4 + 0 \;=\; 9
\end{aligned} \tag{52}
$$

The first subscripts in each term do not change but the second subscripts range over all permutations of the numbers 1, 2, 3, ... , n. The first term is the product of the diagonal elements and has a plus sign because the second subscripts are (1, 2, 3). The last term also has a plus sign because the second subscripts are (3, 1, 2). This permutation come from 2 switches [(1, 2, 3) $\rightarrow$ (3,2,1) $\rightarrow$ (3,1,2)].

A simple way to remember this formula for a 3x3 matrix is to use diagram in figure 1. The elements labeled 1j are multiplied together and receive a plus sign in the summation.

FIGURE 1. 3 x 3 Determinant – Plus Signs

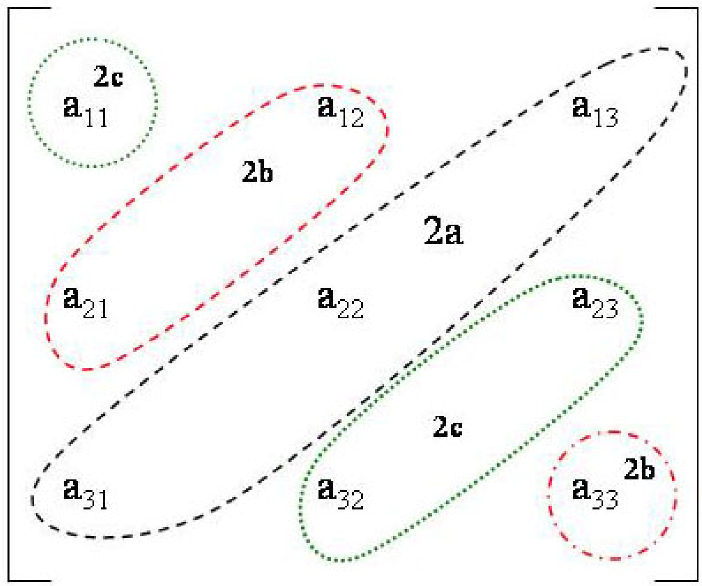


The elements labeled 2j are multiplied together and receive a minus sign in the summation as in figure 2.

2.2.5. *Example 4.* Consider the following 3x3 case

$$
A = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 1 & -3 \\ -1 & 2 & 0 \end{pmatrix} \tag{53}
$$

FIGURE 2. 3 x 3 Determinant – Minus Signs



The determinant of A is

$$\begin{aligned}|A| \;=\; & +a_{11}\,a_{22}\,a_{33}\\ & +a_{12}\,a_{23}\,a_{31}\\ & +a_{13}\,a_{21}\,a_{32}\\ & -a_{13}\,a_{22}\,a_{31}\\ & -a_{12}\,a_{21}\,a_{33}\\ & -a_{11}\,a_{23},\;a_{32}\\ & = 0+6+16-(-4)-0-6 \;=\; 20\end{aligned} \tag{54}$$

2.2.6. *Example 5.* Consider the following 4x4 case

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix} \tag{55}$$

The indices are the numbers 1, 2, 3, and 4. There are 24 (4!) permutations of (1, 2, 3, 4). They are contained in table 1. Notice that the permutations are over the second subscripts, that is, every term is of the form $a_{1\,.}\ a_{2\,.}\ a_{3\,.}\ a_{4\,.}$ with the second subscript varying. The general formula is rather difficult to remember or implement for large n and so another method based on ”cofactors” is normally used.

2.3. **Submatrices and partitions.** Before discussing the computation of determinants using cofactors a few definitions concerning matrices and submatrices will be useful.

TABLE 1. **Determinant of a 4x4 Matrix**

Permutation	Sign	element of sum	Permutation	Sign	element of sum
1234	+	$a_{11}a_{22}a_{33}a_{44}$	3142	-	$a_{13}a_{21}a_{34}a_{42}$
1243	-	$a_{11}a_{22}a_{34}a_{43}$	3124	+	$a_{13}a_{21}a_{32}a_{44}$
1324	-	$a_{11}a_{23}a_{32}a_{44}$	3214	-	$a_{13}a_{22}a_{31}a_{44}$
1342	+	$a_{11}a_{23}a_{34}a_{42}$	3241	+	$a_{13}a_{22}a_{34}a_{41}$
1432	-	$a_{11}a_{24}a_{33}a_{42}$	3421	-	$a_{13}a_{24}a_{32}a_{41}$
1423	+	$a_{11}a_{24}a_{32}a_{43}$	3412	+	$a_{13}a_{24}a_{31}a_{42}$
2134	-	$a_{12}a_{21}a_{33}a_{44}$	4123	-	$a_{14}a_{21}a_{32}a_{43}$
2143	+	$a_{12}a_{21}a_{34}a_{43}$	4132	+	$a_{14}a_{21}a_{33}a_{42}$
2341	-	$a_{12}a_{23}a_{34}a_{41}$	4231	-	$a_{14}a_{22}a_{33}a_{41}$
2314	+	$a_{12}a_{23}a_{31}a_{44}$	4213	+	$a_{14}a_{22}a_{31}a_{43}$
2413	-	$a_{12}a_{24}a_{31}a_{43}$	4312	-	$a_{14}a_{23}a_{31}a_{42}$
2431	+	$a_{12}a_{24}a_{33}a_{41}$	4321	+	$a_{14}a_{23}a_{32}a_{41}$

2.3.1. *Submatrix.* A submatrix is a matrix formed from a matrix A by taking a subset consisting of j rows with column elements from a set k of the columns. For example consider A({1,3},{2,3}) below

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \; A(\{1,3\},\{2,3\}) = \begin{pmatrix} 4 & 7 \\ 0 & 4 \end{pmatrix} \tag{56}$$

The notation A({1,3},{2,3}) means that we take the first and third rows of A and include the second and third elements of each row.

2.3.2. *Principal submatrix.* A principal submatrix is a matrix formed from a square matrix A by taking a subset consisting of n rows and column elements from the same numbered columns. For example consider A({1,3},{1,3}) below

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \qquad A(\{1,3\},\{1,3\}) = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix} \tag{57}$$

2.3.3. *Minor.* A minor is the determinant of a square submatrix of the matrix A. For example consider $\mid A(\{2,3\},\{1,3\}) \mid$.

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \qquad A(\{2,3\},\{1,3\}) = \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}, \qquad |A\left(\{2,3\},\{1,3\}\right)| \;=\; 6 \tag{58}$$

2.3.4. *Principal minor.* A principal minor is the determinant of a principal submatrix of A. For example consider $\mid A(\{1,2\},\{1,2\}) \mid$.

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \qquad A(\{1,2\},\{1,2\}) = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}, \qquad \mid A(\{1,2\},\{1,2\}) \mid \;=\; 7 \tag{59}$$

2.3.5. *Leading principal minor.* Let A =(a_{ij}) be any n $\times$ n . The **leading principle minors** of A are the n determinants:

$$D_k = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix}, \qquad k = 1, 2, \dots, n \tag{60}$$

D_k is obtained by crossing out the last n-k columns and n-k rows of the matrix. Thus for k = 1, 2, 3, ... n, the leading principle minors are, respectively

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \dots, \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \tag{61}$$

2.3.6. *Cofactor.* The cofactor (denoted A_{ij}) of the element a_{ij} of any square matrix A is $(-1)^{i+j}$ times the minor of A that is obtained by including all but the i^{th} row and the j^{th} column, or alternatively the minor that is obtained by deleting the i^{th} row and the j^{th} column. For example the cofactor of a_{12} below is found as

$$\begin{aligned} A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \begin{pmatrix} 3 & 4 & 7 \\ 2 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix} \\ A(\{2,3\},\{1,3\}) &= \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix} \\ A_{12} = (-1)^3 \, |A(\{2,3\},\{1,3\}\,| &= (-1)^3 \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} \\ &= (-1)^3 \, (6) = -6 \end{aligned} \tag{62}$$

2.4. **Computing determinants using cofactors.**

2.4.1. *Definition of a cofactor expansion.* The determinant of a square matrix A can be found inductively using the following formula

$$\det\ A = |\,A\,| = \sum_{j=1}^{n} a_{ij}\, A_{ij} \tag{63}$$

where i denotes the i^{th} row of the matrix A. This is called an expansion of |A| by column i of A. The result is the same for any other row. This can also be done for columns letting the sum range over i instead of j.

2.4.2. *Examples.*

1: Consider as an example the following 2x2 matrix and expand using the first row

$$B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

$$|B| = 4 * (-1)^2 * (2) + 3 * (-1)^3 * (1) = 8 - 3 = 5$$

where the cofactor of 4 is (-1)$^{(1+1)}$ times the submatrix that remains when we delete row and column 1 from the matrix B while the cofactor of 3 is (-1)$^{(1+2)}$ times the submatrix that remains when we delete row 1 and column 2 from the matrix B. In this case each the principle submatrices are just single numbers so there is no need to formally compute a determinant.

2: Now consider a 3x3 example computed using the first row of the matrix.

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} |B| &= 1 * (-1)^2 \begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} + 2 * (-1)^3 \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} + 3 * (-1)^4 \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} \\ &= (1) * (20) + (-2) * (-2) + 3 * (-5) = 9 \end{aligned}$$

We can also compute it using the third row of the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} |B| &= 1 * (-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} + 0 * (-1)^5 \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + 4 * (-1)^6 \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \\ &= (1) * (-11) + (0) * (2) + (4) * (5) = 9 \end{aligned}$$

or the first column

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} |B| &= 1 * (-1)^2 \begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} + 0 * (-1)^3 \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} + 1 * (-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} \\ &= (1) * (20) + (0) * (-8) + (1) * (-11) = 9 \end{aligned}$$

3: Consider the following 3x3 matrix

$$A = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 1 & -3 \\ -1 & 2 & 0 \end{pmatrix} \tag{64}$$

The determinant of A computed using the cofactor expansion along the third row is given by

$$|A| \;=\; -\,1*(-1)^4 \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} \;+\; 2*(-1)^5 \begin{vmatrix} -1 & 4 \\ 2 & -3 \end{vmatrix} \;+\; 0*(-1)^6 \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}$$
$$= (-1)*(-10) \;+\; (-2)*(-5) \;+\; (0)*(-5) \;=\; 20$$

2.5. **Expansion by alien cofactors.**

2.5.1. *Definition of expansion by alien cofactors.* When we expand a row of the matrix using the cofactors of a different row we call it expansion by alien cofactors.

$$\sum_j a_{kj}\, A_{ij} \;=\; 0\, k \;\neq\; i \tag{65}$$

2.5.2. *Some examples.*

1: example 1

$$\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}$$

Expand row 1 using the 2nd row cofactors to obtain (4) (-1)$^{(2+1)}$ (3) + (3) (-1)$^{(2+2}$ (4) = -12 + 12 = 0.

2: example 2

$$\begin{vmatrix} 4 & 3 & 1 \\ 6 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix}$$

Expand row 1 using the 2nd row cofactors. The second row cofactors are

$$(-1)^{(2+1)} \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}$$
$$and$$
$$(-1)^{(2+2)} \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix}$$
$$and$$
$$(-1)^{(2+3)} \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}$$

This then gives for the cofactor expansion(4) (-1) (0) + (3) (1) (3) + (1) (-1) (9) = 0.

2.5.3. *General principle.* If we expand the rows of a matrix by alien cofactors, the expansion will equal zero.

2.6. **Singular and nonsingular matrices.** The square matrix A is said to be singular if |A| = 0, and nonsingular if |A| $\neq$ 0.

2.6.1. *Determinants, Minors, and Rank.*

Theorem 1. *The rank of an $n \times n$ matrix A is k if and only if every minor in A of order $k+1$ vanishes, while there is at least one minor of order k which does not vanish.*

Proposition 1. Consider an $n \times n$ matrix A.

1: det A= 0 if every minor of order n - 1 vanishes.
2: If every minor of order n equals zero, then the same holds for the minors of higher order.
3 (restatement of theorem): The largest among the orders of the non-zero minors generated by a matrix is the rank of the matrix.

2.7. **Basic rules for determinants.**

1: The determinant of an n $\times$ n identity matrix is 1.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \qquad \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = 1$$

2: If two rows A are interchanged, then the determinant of A changes sign, but keeps its absolute value. If two columns of A are interchanged, then the determinant of A changes sign, but keeps its absolute value.
3: The determinant is a linear function of each row separately. If the first row is multiplied by α, the determinant is multiplied by α. If first rows are added, the determinants are added. This rule applies only when other rows do not change. Consider the following example matrices.

$$\begin{vmatrix} \alpha\, a_{11} & \alpha\, a_{12} & \alpha\, a_{13} & \dots & \alpha\, a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \dots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

Or consider the following numerical example

$$
\begin{aligned}
\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix} \\
&= 2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix} \\
&= (2)(4) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix} \\
&= 8 + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{vmatrix}
\end{aligned}
\tag{66}
$$

Property 3 can be used to show that if A is an x $times$ n matrix and α is a real number then $|\alpha A| = \alpha^n |A|$.

4: If two rows or columns of A are equal, then |A| = 0.

5: If a scalar multiple of one row (or column) of A is added to another row (or column) of A, then the value of the determinant does not change. This property can be used to show that if two rows or columns of A are proportional, then |A| = 0. If we multiply one of the rows by the factor of proportionality and add it to the other row, we will obtain a row of zeroes and a row of zeroes implies det A = 0.

6: A matrix with a row (or column) of zeros has $|A| = 0$. Given this property of determinants, the determinant in equation 66 is equal to 8 as the determinants of the two matrices on the right are both zero.

7: If A is triangular, then det A = $|A|$ = a_{11} a_{22} $\ldots$ a_{nn} = the product of the diagonal entries. The product of the diagonal entries in equation 66 is equal to 8. This property, of course, implies that the determinant of a diagonal matrix is the product of the diagonal elements.

8: If A is singular (to be defined later), then the determinant of A = 0. If A is invertible (to be defined later), then det A $\neq$ 0.

9: If A and B are both x $\times$ n then |AB| = |A| |B|.

10: The determinant of A' is equal to the determinant of A, i.e., $|A| = |A'|$.

11: The determinant of a sum is not necessarily the sum of the determinants.

2.8. **Some problems to solve.** Provide an example for each of items 1 -11 in section 2.7.

3. The inverse of a matrix A

3.1. **Definition of the inverse of a matrix.** Given a matrix A, if there exists a matrix B such that

$$AB = BA = I \tag{67}$$

then we say that B is the **inverse** of A. Moreover, A is said to be invertible in this case. Because BA = AB = I, the matrix A is also an inverse of B - that is, A and B are inverses of each other. Inverses are only defined for square matrices. We usually denote the inverse of A by A^{-1} and write

$$A^{-1}A \;=\; AA^{-1} \;=\; I \tag{68}$$

3.2. **Some examples of matrix inversion.** Show that the following pairs of matrices are inverses of each other.

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 1 \\ 6 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{16}{21} & \frac{3}{7} & \frac{10}{21} \end{bmatrix}$$

3.3. **Existence of the inverse.** A square matrix A has an inverse $\Leftrightarrow |A| \neq 0$. A square matrix is said to be **singular** if $|A| = 0$ and **nonsingular** if $|A| \neq 0$. Thus a matrix has an inverse if and only if it is nonsingular.

3.4. **Uniqueness of the inverse.** The inverse of a square matrix is unique if it exists.

3.5. **Some implications of inverse matrices.**

$$\begin{aligned} AX &= I \Rightarrow X = A^{-1} \\ BA &= I \Rightarrow B = A^{-1} \end{aligned} \tag{69}$$

3.6. **Properties of the inverse.** Let A and B be invertible n $\times$ n matrices.

1: A^{-1} is invertible and $(A^{-1})^{-1} = A$
2: AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3: A' is invertible and $(A^{-1})' = (A')^{-1}$
4: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, if A, B, and C have inverses
5: $(cA)^{-1} = c^{-1}\,A^{-1}$ whenever c is a number $\neq 0$
6: In general, $(A + B)^{-1} \neq A^{-1} + B^{-1}$

3.7. **Orthogonal matrices.** A matrix A is called orthogonal if its inverse is equal to its transpose, that is if

$$A^{-1} \;=\; A' \tag{70}$$

3.8. **Finding the inverse of a 2x2 matrix.** For a general 2x2 matrix A, we can find the inverse using the following matrix equation

$$\begin{aligned} A\,B &= I \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{71}$$

We can divide this into two parts as follows

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (72)$$
$$\text{and}$$
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can solve each system using Cramer's rule. For the first system we obtain

$$b_{11} = \frac{\begin{vmatrix} 1 & a_{12} \\ 0 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{22}}{a_{11}\,a_{22} - a_{21}\,a_{12}} \qquad (73)$$
$$\text{and}$$
$$b_{21} = \frac{\begin{vmatrix} a_{11} & 1 \\ a_{21} & 0 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{-a_{21}}{a_{11}\,a_{22} - a_{21}\,a_{12}}$$

In a similar fashion we can show that

$$b_{12} = \frac{-a_{12}}{a_{11}\,a_{22} - a_{21}\,a_{12}} \qquad (74)$$

$$b_{22} = \frac{a_{11}}{a_{11}\,a_{22} - a_{21}\,a_{12}}$$

Combining the expressions we see that

$$A^{-1} = \frac{1}{a_{11}\,a_{22} - a_{12}\,a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad (75)$$

This leads to the familiar rule that to compute the determinant of a 2x2 matrix we switch the diagonal elements, change the sign of the off-diagonal ones, and divide by the determinant.

3.9. **Finding the inverse of a general n $\times$ n matrix.** The most efficient way to find the inverse of an n $\times$ n matrix is to use elementary row operations or elimination on the augmented matrix $[\,A\ I\,]$. Or in other words solve the following systems of equations for $b_1, b_2, \ldots, b_n$.

$$
\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} \\
&\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{vmatrix} \\
&\qquad\qquad\vdots \\
&\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{vmatrix}
\end{aligned}
\tag{76}
$$

As discussed in section 6, this is done by performing elimination on the matrix

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & 1 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & 0 & 0 & 0 & \dots & 1 \end{vmatrix}
\tag{77}
$$

4. Solving equations by matrix inversion

4.1. **Procedure for solving equations using a matrix inverse.** Let A be an n × n matrix. Let B be an arbitrary matrix. Then we ask whether there are matrices C and D of suitable dimension such that

$$
\begin{aligned}
A\,C &= B \\
D\,A &= B
\end{aligned}
\tag{78}
$$

In the first case the matrix B must have n rows, while in the second B must have n columns. Then we have the following theorem.

Theorem 2. *If* $|A| \neq 0$*, then:*

$$
\begin{aligned}
A\,C &= B \;\Leftrightarrow\; C \;=\; A^{-1}\,B \\
D\,A &= B \;\Leftrightarrow\; D \;=\; B\,A^{-1}
\end{aligned}
\tag{79}
$$

As an example, solve the following system of equations using theorem 2.

$$
\begin{aligned}
3\,x_1 \;+\; 5\,x_2 &= 14 \\
8\,x_1 \;-\; 2\,x_2 &= 22
\end{aligned}
\tag{80}
$$

We can write the system as

$$
\begin{aligned}
A\,x &= b \\
A &= \begin{bmatrix} 3 & 5 \\ 8 & -2 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; b = \begin{bmatrix} 14 \\ 22 \end{bmatrix} \\
x &= A^{-1}\,b \;=\; A^{-1} \begin{bmatrix} 14 \\ 22 \end{bmatrix}
\end{aligned} \tag{81}
$$

We can compute A^{-1} from equation 75.

$$
\begin{aligned}
A^{-1} &= \frac{1}{(3)\,(-2)\;-\;(8)\,(5)} \begin{bmatrix} -2 & -5 \\ -8 & 3 \end{bmatrix} \\
&= \frac{-1}{46} \begin{bmatrix} -2 & -5 \\ -8 & 3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{46} & \frac{5}{46} \\ \frac{8}{46} & -\frac{3}{46} \end{bmatrix}
\end{aligned} \tag{82}
$$

We can now compute A^{-1} b as follows

$$
\begin{aligned}
A^{-1}\,b &= \begin{bmatrix} \frac{2}{46} & \frac{5}{46} \\ \frac{8}{46} & -\frac{3}{46} \end{bmatrix} \begin{bmatrix} 14 \\ 22 \end{bmatrix} = \begin{bmatrix} \frac{28}{46} + \frac{110}{46} \\ \frac{112}{46} - \frac{66}{46} \end{bmatrix} \\
&= \begin{bmatrix} \frac{138}{46} \\ \frac{46}{46} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\end{aligned} \tag{83}
$$

4.2. **Some example systems.** Solve each of the following using matrix inversion

$$
\begin{aligned}
3x_1 \;+\; x_2 &= 6 \\
5x_1 \;-\; x_2 &= 2
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
4x_1 \;+\; x_2 &= 10 \\
3x_1 \;+\; 2\,x_2 &= 10
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
4x_1 \;+\; 3\,x_2 &= 10 \\
x_1 \;+\; 2\,x_2 &= 5
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
4x_1 \;+\; 3\,x_2 &= 11 \\
x_1 \;+\; 2\,x_2 &= 4
\end{aligned} \tag{87}
$$

5. A GENERAL FORMULA FOR THE INVERSE OF A MATRIX USING ADJOINT MATRICES

5.1. **The adjoint of a matrix.** The **adjoint** of the matrix A denoted adj (A) or A^+ is the transpose of the matrix obtained from A by replacing each element a_{ij} by its cofactor A_{ij}. For example consider the matrix

$$
A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix} \tag{88}
$$

Now find the cofactor of each element. For example the cofactor of the 1 in the upper left hand corner is computed as

$$
A_{11} = (-1)^{(1+1)} \begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} \;=\; 20 \tag{89}
$$

Similarly the cofactor of $a_{23} = 2$ is given by

$$A_{23} = (-1)^{(2+3)} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2 \tag{90}$$

The entire matrix of cofactors is given by

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \cdot & & & \cdot \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} = \begin{pmatrix} 20 & 2 & -5 \\ -8 & 1 & 2 \\ -11 & -2 & 5 \end{pmatrix} \tag{91}$$

We can then compute the adjoint by taking the transpose

$$\begin{aligned} A^{+} &= \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \cdot & & & \cdot \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}' = \begin{pmatrix} 20 & 2 & -5 \\ -8 & 1 & 2 \\ -11 & -2 & 5 \end{pmatrix}' \\ &= \begin{pmatrix} 20 & -8 & -11 \\ 2 & 1 & -2 \\ -5 & 2 & 5 \end{pmatrix} \end{aligned} \tag{92}$$

5.2. **Finding an inverse matrix using adjoints.** For a square nonsingular matrix A, its inverse is given by

$$A^{-1} = \frac{1}{|A\,|} A^{+} \tag{93}$$

5.3. **An example of finding the inverse using the adjoint.** First find the cofactor matrix.

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix} \\ A_{ij} &= \begin{pmatrix} 20 & 2 & -5 \\ -8 & 1 & 2 \\ -11 & -2 & 5 \end{pmatrix} \end{aligned} \tag{94}$$

Now find the transpose of A_{ij}

$$A^{+} = \begin{pmatrix} 20 & -8 & -11 \\ 2 & 1 & -2 \\ -5 & 2 & 5 \end{pmatrix} \tag{95}$$

The determinant of A is

$$|A| = 9$$

Putting it all together we obtain

$$A^{-1} = \frac{1}{9} \cdot \begin{pmatrix} 20 & -8 & -11 \\ 2 & 1 & -2 \\ -5 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 20/9 & -8/9 & -11/9 \\ 2/9 & 1/9 & -2/9 \\ -5/9 & 2/9 & 5/9 \end{pmatrix} \tag{96}$$

We can check the answer as follows.

$$AA^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 2 \\ 1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 20/9 & -8/9 & -11/9 \\ 2/9 & 1/9 & -2/9 \\ -5/9 & 2/9 & 5/9 \end{pmatrix} = \begin{pmatrix} 9/9 & 0 & 0 \\ 0 & 9/9 & 0 \\ 0 & 0 & 9/9 \end{pmatrix} = I \tag{97}$$

6. Finding the inverse of a matrix using elementary row operations

The most effective computational way to find the inverse of a matrix is to use elementary row operations. This is done by forming the augmented matrix [A : I] similar to the augmented matrix we use when solving a system equation, except that we now add n columns of the identity matrix instead of the right hand side vector in the equations system. In effect we are writing the system

$$A\,X = I \tag{98}$$

and then solving for the matrix X, which will be A^{-1}. Consider the following matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{pmatrix} \tag{99}$$

Now append an identity matrix as follows

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \tag{100}$$

Add negative 1 times the first row to the last row to obtain

$$\begin{array}{cccccc} -1 & -2 & -2 & -1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ -- & -- & -- & -- & -- & -- \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}$$

$$\Rightarrow\ \tilde{A}_1 = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{101}$$

Now divide the second row by 4 to obtain

$$\tilde{A}_2 = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{102}$$

Now multiply the second row by 2 and add to the third row

$$\begin{array}{cccccc} 0 & 2 & 1/2 & 0 & 1/2 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ -- & -- & -- & -- & -- & -- \\ 0 & 0 & 1/2 & -1 & 1/2 & 1 \end{array}$$

$$\Rightarrow\ \tilde{A}_3 = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/2 & -1 & 1/2 & 1 \end{pmatrix} \tag{103}$$

Now multiply the third row by 2

$$\tilde{A}_4 = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{pmatrix} \tag{104}$$

Now multiply the second row by negative 2 and add to the first row

$$\begin{array}{cccccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1/2 & 0 & -1/2 & 0 \\ -- & -- & -- & -- & -- & -- \\ 1 & 0 & 3/2 & 1 & -1/2 & 0 \end{array}$$
$$\Rightarrow \tilde{A}_5 = \begin{pmatrix} 1 & 0 & 3/2 & 1 & -1/2 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{pmatrix} \tag{105}$$

Now multiply the third row by negative 3/2 and add to the first row

$$\begin{array}{cccccc} 1 & 0 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & -3/2 & 3 & -3/2 & -3 \\ -- & -- & -- & -- & -- & -- \\ 1 & 0 & 0 & 4 & -2 & -3 \end{array}$$
$$\Rightarrow \tilde{A}_6 = \begin{pmatrix} 1 & 0 & 0 & 4 & -2 & -3 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{pmatrix} \tag{106}$$

Finally multiply the third row by negative 1/4 and add to the second row

$$\begin{array}{cccccc} 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & -1/4 & 1/2 & -1/4 & -1/2 \\ -- & -- & -- & -- & -- & -- \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \end{array}$$
$$\Rightarrow \tilde{A}_7 = \begin{pmatrix} 1 & 0 & 0 & 4 & -2 & -3 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{pmatrix} \tag{107}$$

The inverse of A is then given by

$$A^{-1} = \begin{pmatrix} 4 & -2 & -3 \\ 1/2 & 0 & -1/2 \\ -2 & 1 & 2 \end{pmatrix} \tag{108}$$

We can check this as follows

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 & -3 \\ 1/2 & 0 & -1/2 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{109}$$

7. The general form of Cramer's rule

Consider the general equation system

$$A\,x \;=\; b \tag{110}$$

where A is n $\times$ n, x in nx1, and b is nx1.

Let D_j denote the determinant formed from $|A|$ by replacing the j^{th} column with the column vector b. Thus

$$D_j \;=\; \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1\,j-1} & b_1 & a_{1\,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2\,j-1} & b_2 & a_{2\,j+1} & \cdots & a_{2n} \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & \ldots & a_{n\,j-1} & b_n & a_{n\,j+1} & \cdots & a_{nn} \end{pmatrix} \tag{111}$$

Then the general system of n equations in n unknowns has unique solution if $|A| \neq 0$. The solution is

$$x_1 \;=\; \frac{D_1}{|\,A\,|}\,,\; x_2 \;=\; \frac{D_2}{|\,A\,|}\,,\;\cdots\;,\; x_n \;=\; \frac{D_n}{|\,A\,|} \tag{112}$$

8. A note on homogeneous systems of equations

The homogeneous system of n equations in n unknowns

$$Ax \;=\; 0 \tag{113}$$

or

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ \vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{114}$$

has a nontrivial solution (a trivial solution has $x_1 = 0$, $x_2 = 0$, ... $x_n = 0$ if and only if $|A| = 0$.

Lecture Questions

1: Provide two examples of linear systems that have no solution and explain why.

2: Provide two examples of linear systems that have exactly one solution and explain why.

3: Provide two examples of linear systems that have exactly infinitely many solutions and explain why. Provide at least one of these where the number of equations and unknowns is the same.

4: Explain why if a linear system has exactly one solution, the coefficient matrix A must have at least as many rows as columns. Give examples of why this is the case.

5: If a system of linear equations has more unknowns than equations, it must either have no solution or infinitely many solutions. Explain why. Provide at least two examples.