SIMPLE CONSTRAINED OPTIMIZATION

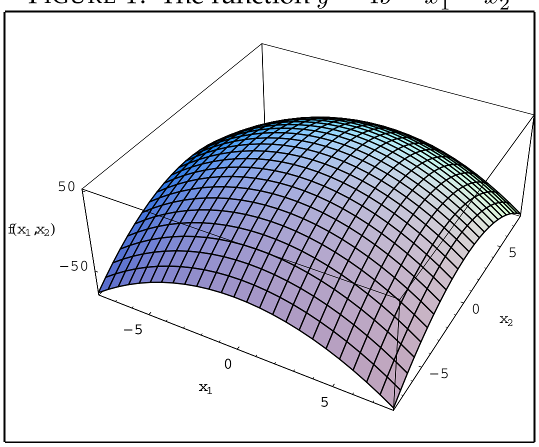
1. Intuitive Introduction to Constrained Optimization

Consider the following function which has a maximum at the origin.

$$y = f(x_1, x_2) = 49 - x_1^2 - x_2^2 \tag{1}$$

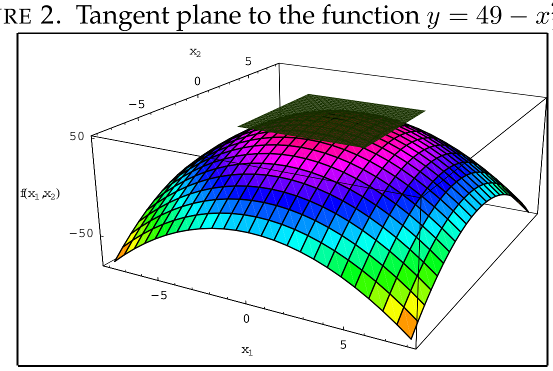
The graph is contained in figure 1.

FIGURE 1. The function $y = 49 - x_1^2 - x_2^2$



The tangent plane to the graph at the origin is shown in figure 2.

FIGURE 2. Tangent plane to the function $y = 49 - x_1^2 - x_2^2$



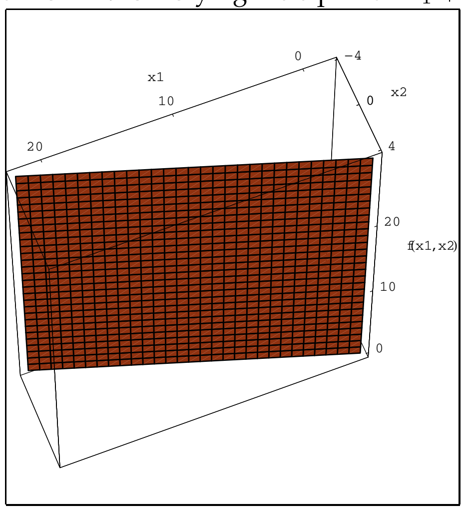
Date: September 3, 2004.

Now consider only values of x_1 and x_2 that satisfy the equation

$$x_1 + 3x_2 - 10 = 0 \tag{2}$$

Above this line in the x_1-x_2 plane are an infinity of points. We can construct a plane above this line in R^3. This plane is shown in figure 3.

FIGURE 3. Points in R^3 satisfying the equation $x_1 + 3x_2 - 10 = 0$



Now consider maximizing the function $y = 49 - x_1^2 - x_2^2$ subject to the condition that the values of x_1 and x_2 chosen lie on the line $x_1 + 3x_2 - 10 = 0$. Graphically we want to pick points on the surface that also lie on the plane above the line. Both the function and the plane on which we must pick the points are shown in figure 4.

From inspection of the graph, it is clear that the maximum of the function that also lies in the plane is less than the global maximum of the function. If we look at a closeup of the graph (Figure 5) in the vicinity of what seems to be the constrained maximum, we can visually guess at corresponding values of x_1 and x_2.

The constrained maximum looks to be somewhere around $x_1 = 1$ and $x_2 = 3$. This of course is quite a ways from the unconstrained maximum of (0, 0). As will be shown in section 2, the maximal value of the function is at $x_1 = 1$ and $x_2 = 3$ as can be seen in figure 6.

We can also graph the level curves of the function and the constraint as in the following diagram where level curves are indicated at values of 14, 21, 28, 35, 42 and 49. The constraint is the darker line in figure 7.

It seems clear that the constrained maximum must be between the level curves for 35 and 42. Adding a level curve for y = 39, we can see the optimum in figure 8.

Thus we have seemed to find a constrained maxima for the function using graphical methods. A few things seem to characterize the extreme point.

FIGURE 4. The function $y = 49 - x_1^2 - x_2^2$ and the constraint $x_1 + 3x_2 - 10 = 0$

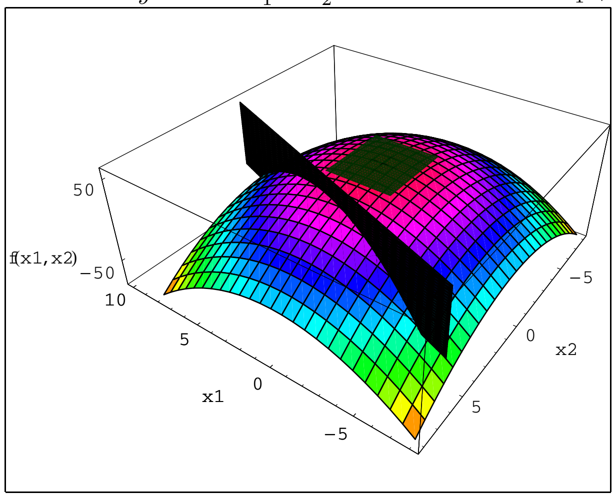
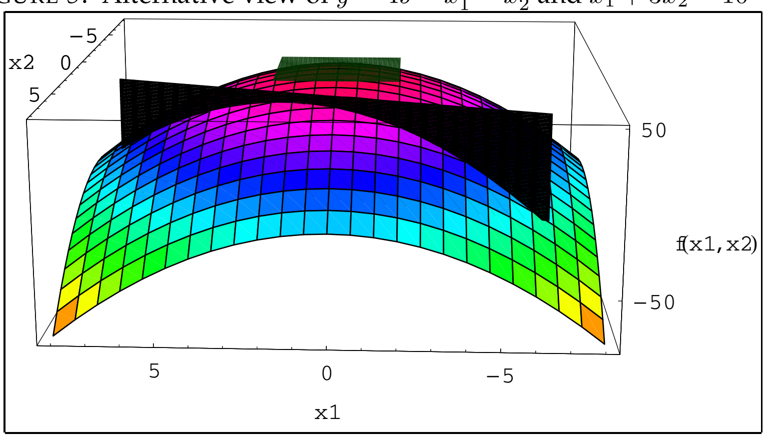


FIGURE 5. Alternative view of $y = 49 - x_1^2 - x_2^2$ and $x_1 + 3x_2 - 10 = 0$



1: The extreme point lies on the surface but above a point in x_1-x_2 space that satisfies the constraint.

2: At the constrained extreme point, the constraint and the level surfaces of the function are tangent.

FIGURE 6. Maximum of $y = 49 - x_1^2 - x_2^2$ subject to $x_1 + 3x_2 - 10 = 0$

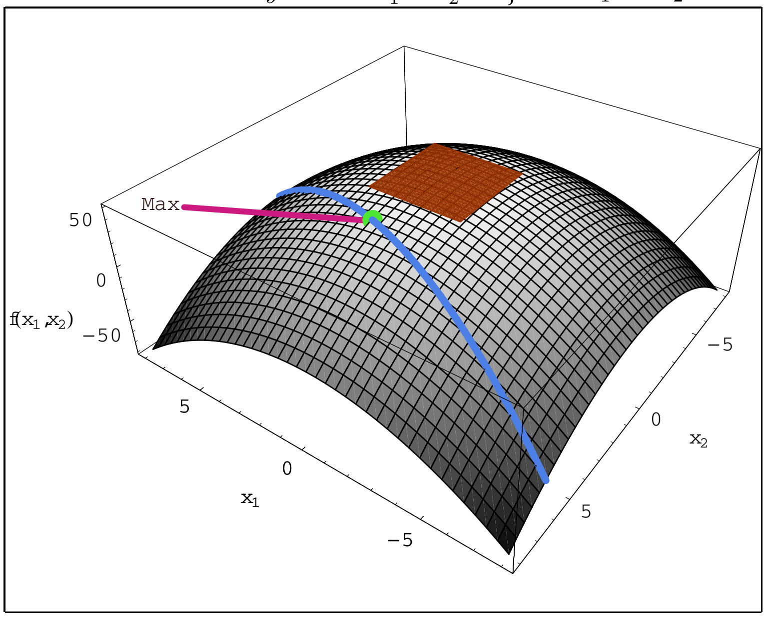
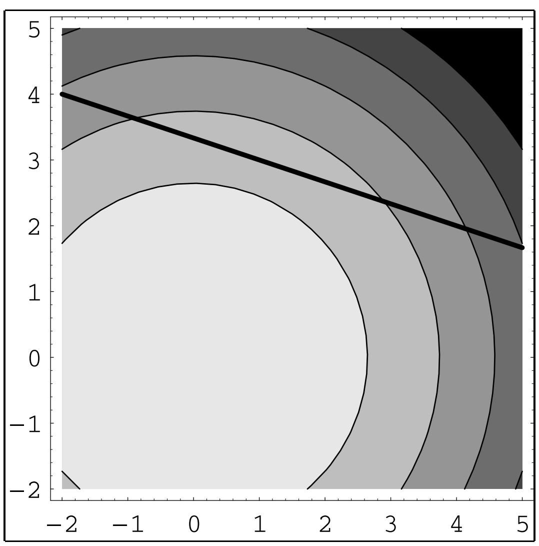


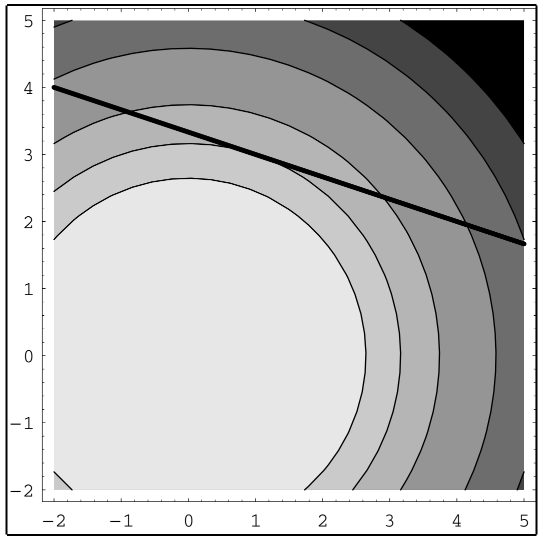
FIGURE 7. Level curves of the function $y = 49 - x_1^2 - x_2^2$ and constraint $x_1 + 3x_2 - 10 = 0$



2. FORMAL ANALYSIS OF CONSTRAINED OPTIMIZATION PROBLEMS

2.1. Formal setup of the constrained optimization problem. Consider the problem defined by

FIGURE 8. More level curves of the function $y = 49 - x_1^2 - x_2^2$ and constraint $x_1 + 3x_2 - 10 = 0$



$$\max_{x_1,\, x_2} \; f(x_1,\, x_2) \; subject\ to \; g(x_1,\, x_2) \; = \; 0. \tag{3}$$

where $g(x_1, x_2) = 0$ denotes a constraint on the values of x_1 and x_2.

2.2. **Solution by substitution.** One method that sometimes works is to solve the constraint equation for x_1 in terms of x_2 and then substitute in $f(x_1, x_2)$. For the problem at hand this would yield

$$\begin{aligned} x_1 \; + \; 3x_2 \; - \; 10 \; &= 0 \\ \Rightarrow \; x_1 \; &= 10 \; - \; 3x_2 \end{aligned} \tag{4}$$

If we rewrite the function f using this substitution we obtain

$$\begin{aligned} y \; = \; f(x_1,\, x_2) \; &= 49 \; - \; x_1^2 \; - \; x_2^2 \\ &= 49 \; - \; (10 \; - \; 3x_2)^2 \; - \; x_2^2 \\ &= 49 \; - \; (100 \; - \; 60x_2 + 9x_2^2) \; - \; x_2^2 \\ &= \; - \; 51 \; + \; 60x_2 - 10x_2^2 \end{aligned} \tag{5}$$

Now we can maximize this function by taking the derivative with respect to x_2, setting this equation equal to zero, and then solving for x_2 as follows

$$\begin{aligned} y \; &= - \; 51 \; + \; 60x_2 \; - \; 10x_2^2 \\ \frac{dy}{dx_2} \; &= 60 \; - \; 20x_2 \; = \; 0 \\ \Rightarrow \; 20x_2 \; &= 60 \\ \Rightarrow \; x_2 \; &= 3 \end{aligned} \tag{6}$$

Substituting in equation 4 we obtain

$$
\begin{aligned}
x_1 &= 10 \, - \, 3x_2 \\
&= 10 \, - \, (3)(3) \\
&= 10 \, - \, 9 \, = \, 1
\end{aligned} \tag{7}
$$

We can check to see that this is maximum by looking at the second derivative of y in equation 6. This will give

$$
\begin{aligned}
\frac{dy}{dx_2} \, &= 60 \, - \, 20x_2 \\
\frac{d^2y}{dx_2^2} \, &= \, - \, 40 \, x_2
\end{aligned} \tag{8}
$$

At the $x_2 = 3$ this is negative, and so we have a maximum.

While the method of substitution will work in many cases it breaks down in a number of situations. For example if there is no explicit way to solve the constraint for one of the variables, there is no easy way to make the substitution in f. In some situations, if we choose the wrong variable to solve out of the constraint we may end up with a point that maximizes or minimizes f but does not satisfy the constraint. And extending the method of substitution multiple variables is often difficult. We thus turn to another method due to Lagrange and make use of the fact that the level curves of the function and the constraint are tangent to one another.

2.3. **The method of Lagrange.**

2.3.1. *The Langrangian.* The solution to a constrained optimization problem is obtained by finding the critical values of the Langrangian function

$$
L(x_1, \, x_2, \, \lambda) \; = \; f(x_1, \, x_2) \; - \; \lambda \, g(x_1, \, x_2) \tag{9}
$$

Notice that the gradient of L with respect to x_1 and x_2 will involve a set of derivatives that looks like this

$$
\nabla L(x_1, \, x_2; \; \lambda \,) \; = \; \begin{pmatrix} \frac{\partial f}{\partial x_1} \; - \; \lambda \, \frac{\partial \, g}{\partial \, x_1} \\ \\ \frac{\partial f}{\partial x_2} \; - \; \lambda \, \frac{\partial \, g}{\partial \, x_2} \end{pmatrix} \tag{10}
$$

2.3.2. *Necessary first order conditions for an extreme point.* The necessary conditions for an extremum of $f(x_1, x_2)$ with the equality constraints $g(x_1, x_2) = 0$ are

$$
\nabla L(x_1^*, \, x_2^*, \, \lambda^*) \; = \; 0 \tag{11}
$$

This, of course implies that

$$
\nabla \, L(x_1, \, x_2 \, \lambda) \; = \; \begin{pmatrix} \frac{\partial f}{\partial x_1} \; - \; \lambda \, \frac{\partial \, g}{\partial \, x_1} \\ \\ \frac{\partial f}{\partial x_2} \; - \; \lambda \, \frac{\partial \, g}{\partial \, x_2} \\ \\ - \, g \, (x_1, \, x_2) \end{pmatrix} \; = \; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}
$$

2.3.3. *Simple example.* For the example problem the Langrangian is a follows

$$\begin{aligned} L(x_1,\, x_2,\, \lambda) &= f(x_1,\, x_2) \;-\; \lambda\, g(x_1,\, x_2) \\ &= 49 \;-\; x_1^2 \;-\; x_2^2 \;-\; \lambda\,(x_1 \;+\; 3\,x_2 \;-\; 10) \end{aligned} \tag{13}$$

Taking the partial derivatives with respect to x_1, x_2, and λ we obtain

$$\begin{aligned} L(\,x_1,\, x_2,\, \lambda) &= 49 \;-\; x_1^2 \;-\; x_2^2 \;-\; \lambda(x_1 \;+\; 3\,x_2 \;-\; 10) \\ \frac{\partial\, L}{\partial\, x_1} &= \;-\; 2x_1 \;-\; \lambda \;=\; 0 \\ \frac{\partial\, L}{\partial\, x_2} &= \;-\; 2x_2 \;-\; 3\,\lambda \;=\; 0 \\ \frac{\partial\, L}{\partial\, \lambda} &= \;-\; x_1 \;-\; 3x_2 \;+\; 10 \;=\; 0 \end{aligned} \tag{14}$$

The first equation can be solved for λ yielding

$$\begin{aligned} \frac{\partial\, L}{\partial\, x_1} &= \;-\; 2\,x_1 \;-\; \lambda \;=\; 0 \\ \Rightarrow\; 2\,x_1 &= \;-\; \lambda \\ \Rightarrow\; x_1 &= \frac{-\lambda}{2} \end{aligned} \tag{15}$$

Similarly the second equation implies that

$$\begin{aligned} \frac{\partial\, L}{\partial\, x_2} &= \;-\; 2x_2 \;-\; 3\,\lambda \;=\; 0 \\ \Rightarrow\; 2x_2 &= \;-\; 3\,\lambda \\ \Rightarrow\; x_2 &= \frac{-3\,\lambda}{2} \end{aligned} \tag{16}$$

Substituting both of these in the third equation of 14 will allow us to solve for λ as follows

$$\frac{\partial L}{\partial \lambda} = -x_1 - 3x_2 + 10 = 0$$

$$\Rightarrow -\left(\frac{-\lambda}{2}\right) - (3)\left(\frac{-3\lambda}{2}\right) = -10$$

$$\Rightarrow \left(\frac{\lambda}{2}\right) + \left(\frac{9\lambda}{2}\right) = -10 \tag{17}$$

$$\Rightarrow \frac{10}{2}\lambda = -10$$

$$\Rightarrow 5\lambda = -10$$

$$\Rightarrow \lambda = -2$$

Now substituting in the equations 15 and 16 for x_1 and x_2 we obtain

$$\begin{aligned} x_1 &= \frac{-\lambda}{2} = \frac{-(-2)}{2} = 1 \\ x_2 &= \frac{-3\lambda}{2} = \frac{(-3)(-2)}{2} = \frac{6}{2} = 3 \end{aligned} \tag{18}$$

Thus we obtain the same answer as before.

2.3.4. *Sufficient conditions for a constrained extremum problem.* The sufficient conditions will not be stated at this time.

2.4. **Example problems.**

2.4.1. *Production function.* Consider a production function given by

$$y = 20x_1 - x_1^2 + 15x_2 - x_2^2 \tag{19}$$

Let the prices of x_1 and x_2 be 10 and 5 respectively. Then minimize the cost of producing 100 units of output given these prices. The objective function is $10x_1 + 5x_2$. The constraint is $20x_1 - x_1^2 + 15x_2 - x_2^2 = 100$. The Lagrangian is given by

$$L = 10x_1 + 5x_2 - \lambda(20x_1 - x_1^2 + 15x_2 - x_2^2 - 100)$$

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 10 - \lambda(20 - 2x_1) = 0 \\ \frac{\partial L}{\partial x_2} &= 5 - \lambda(15 - 2x_2) = 0 \\ \frac{\partial L}{\partial \lambda} &= (-1)(20x_1 - x_1^2 + 15x_2 - x_2^2 - 100) = 0 \end{aligned} \tag{20}$$

If we take the ratio of the first two first order conditions in equation 20 we obtain

$$\begin{aligned} \frac{10}{5} &= 2 = \frac{20 - 2x_1}{15 - 2x_2} \\ \Rightarrow 30 - 4x_2 &= 20 - 2x_1 \\ \Rightarrow 10 - 4x_2 &= -2x_1 \\ \Rightarrow x_1 &= 2x_2 - 5 \end{aligned} \tag{21}$$

Now plug this result into the negative of the last first order condition in equation 20 to obtain

$$20(2x_2 - 5) - (2x_2 - 5)^2 + 15x_2 - x_2^2 - 100 = 0 \tag{22}$$

Multiplying out and solving for x_2 will give

$$\begin{aligned} 40x_2 - 100 - (4x_2^2 - 20x_2 + 25) + 15x_2 - x_2^2 - 100 &= 0 \\ \Rightarrow 40x_2 - 100 - 4x_2^2 + 20x_2 - 25 + 15x_2 - x_2^2 - 100 &= 0 \\ \Rightarrow -5x_2^2 + 75x_2 - 225 &= 0 \\ \Rightarrow 5x_2^2 - 75x_2 + 225 &= 0 \\ \Rightarrow x_2^2 - 15x_2 + 45 &= 0 \end{aligned} \tag{23}$$

Now solve this quadratic equation for x_2 as follows

$$\begin{aligned} x_2 &= \frac{15 \pm \sqrt{225 - 4(45)}}{2} \\ &= \frac{15 \pm \sqrt{45}}{2} \\ &= 4.15 \textit{ or } 10.85[10pt] \end{aligned} \tag{24}$$

Given that we know the objective is to minimize cost we choose 4.15. Therefore,

$$\begin{aligned} x_1 &= 2x_2 - 5 \\ &= 3.29 \end{aligned} \tag{25}$$

The minimum cost is obtained by substituting into the cost expression to obtain

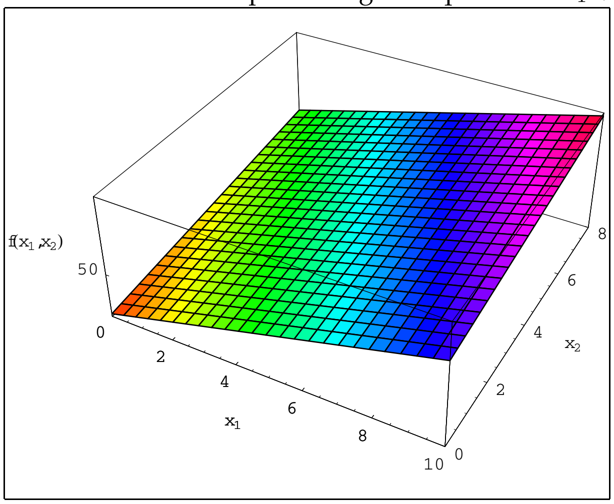
$$C = 10(3.29) + 5(4.15) = 53.65 \tag{26}$$

The Langrangian multiplier λ can be obtained by solving the first first order condition in equation 20 for λ

$$\begin{aligned} 10 - \lambda(20 - 2(3.29)) &= 0 \\ \rightarrow \lambda &= .75 \end{aligned} \tag{27}$$

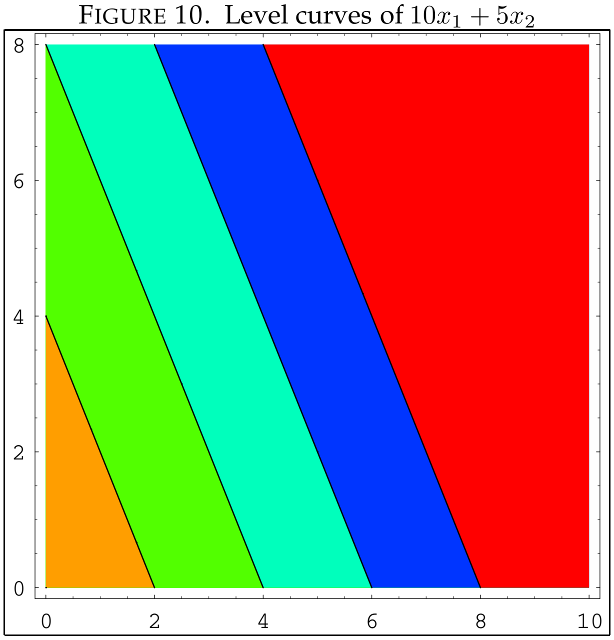
The objective function is a plane as shown in figure 9.

FIGURE 9. Plane in R^3 representing the equation $10x_1 + 5x_2$



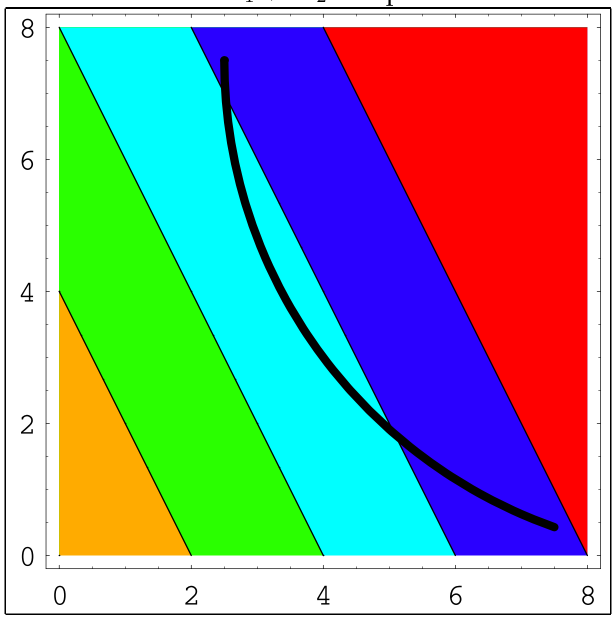
It is clearly minimized in the positive orthant at the point (0, 0). The level curves look as follows

FIGURE 10. Level curves of $10x_1 + 5x_2$



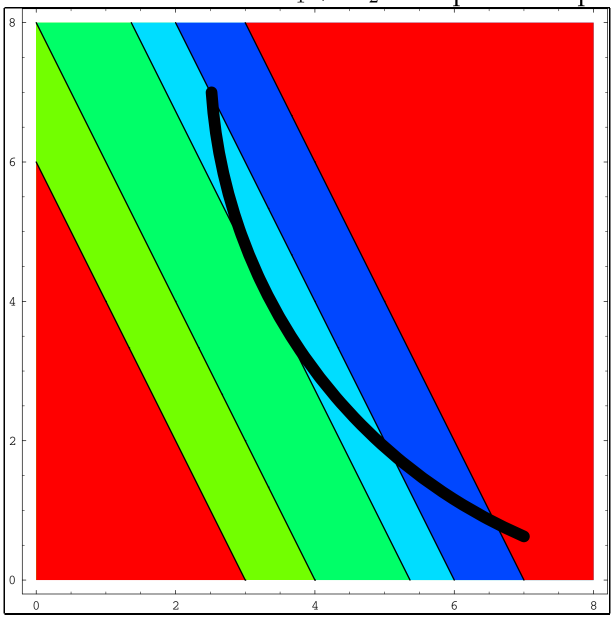
If we plot these level curves along with the value of the constraint we obtain a representation as in figure 11.

FIGURE 11. Level curves of $10x_1 + 5x_2$ and production function constraint



The optimum looks to be somewhere around 3.5 and 4. We can then refine the diagram as in figure 12.

FIGURE 12. Level curves of $10x_1 + 5x_2$ and optimum input levels



2.4.2. *Utility function.* Consider a utility function given by

$$u = x_1^{\alpha_1} x_2^{\alpha_2} \tag{28}$$

Now maximize this function subject to the constraint that

$$w_1 x_1 + w_2 x_2 = c_0 \tag{29}$$

where w_1 is the price of the first good, w_2 is the price of the second good, and c_0 is income.

Set up the Langrangian problem

$$L = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda \left[w_1 x_1 + w_2 x_2 - c_0 \right] \tag{30}$$

The first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \alpha_1 \, x_1^{\alpha_1 - 1} \, x_2^{\alpha_2} - \lambda \, w_1 = 0 \\ \frac{\partial L}{\partial x_2} &= \alpha_2 \, x_1^{\alpha_1} \, x_2^{\alpha_2 - 1} - \lambda \, w_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= - \, w_1 x_1 - w_2 x_2 + c_0 = 0 \end{aligned} \tag{31}$$

Taking the ratio of the first and second equations in equation 40 we obtain

$$\frac{w_1}{w_2} = \frac{\alpha_1 x_2}{\alpha_2 x_1}. \tag{32}$$

We can now solve this equation for x_2 as a function of x_1 and prices. Doing so we obtain

$$x_2 = \frac{\alpha_2 x_1 w_1}{\alpha_1 w_2}. \tag{33}$$

Substituting in the cost equation we obtain

$$\begin{aligned} w_1 x_1 + w_2 x_2 &= c_0 \\ \Rightarrow w_1 x_1 + w_2 \left[\frac{\alpha_2 x_1 w_1}{\alpha_1 w_2} \right] &= c_0 \\ \Rightarrow w_1 x_1 + \left[\frac{\alpha_2 w_1 w_2}{\alpha_1 w_2} \right] x_1 &= c_0 \\ \Rightarrow w_1 x_1 + \left[\frac{\alpha_2 w_1}{\alpha_1} \right] x_1 &= c_0 \\ \Rightarrow x_1 \left[w_1 + \frac{\alpha_2 w_1}{\alpha_1} \right] &= c_0 \\ \Rightarrow x_1 w_1 \left[1 + \frac{\alpha_2}{\alpha_1} \right] &= c_0 \\ \Rightarrow x_1 w_1 \left[\frac{\alpha_1 + \alpha_2}{\alpha_1} \right] &= c_0 \\ \Rightarrow x_1 &= \frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 + \alpha_2} \right] \end{aligned} \tag{34}$$

We can now obtain x_2 by substitution

$$\begin{aligned} x_2 &= x_1 \left[\frac{\alpha_2 w_1}{\alpha_1 w_2} \right] \\ &= \frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 + \alpha_2} \right] \left[\frac{\alpha_2 w_1}{\alpha_1 w_2} \right] \\ &= \frac{c_0}{w_2} \left[\frac{\alpha_2}{\alpha_1 + \alpha_2} \right] \end{aligned} \tag{35}$$

The utility level is obtained by substituting x_1 and x_2 in the utility function

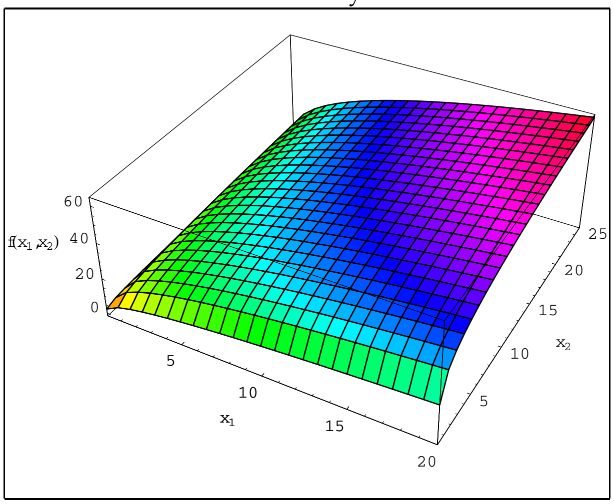
$$\begin{aligned} u &= x_1^{\alpha_1} x_2^{\alpha_2} \\ &= \left[\frac{c_0}{w_1} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \right) \right]^{\alpha_1} \left[\frac{c_0}{w_2} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \right) \right]^{\alpha_2} \\ &= \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \right)^{\alpha_2} \left(\frac{c_0}{w_1} \right)^{\alpha_1} \left(\frac{c_0}{w_2} \right)^{\alpha_2} \end{aligned} \tag{36}$$

2.4.3. *Numerical utility function example.* Consider a utility function given by

$$u = 10x_1^{0.4}\, x_2^{0.2} \tag{37}$$

The utility function is shown in figure 13.

FIGURE 13. Utility Function



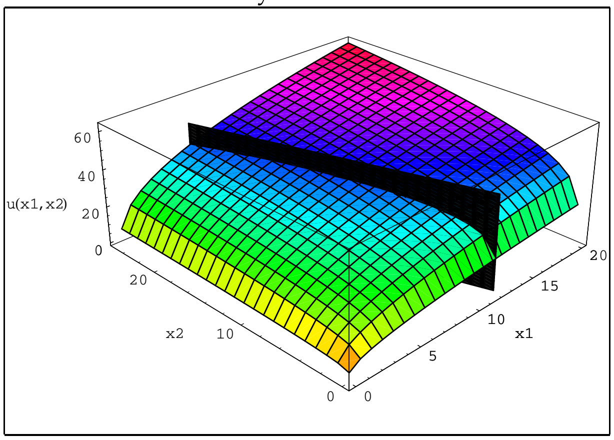
Now maximize this function subject to the constraint that

$$8\,x_1 \; + \; 2\,x_2 \; = \; 96 \tag{38}$$

where 8 is the price of the first good, 2 is the price of the second good, and 40 is the level of income.

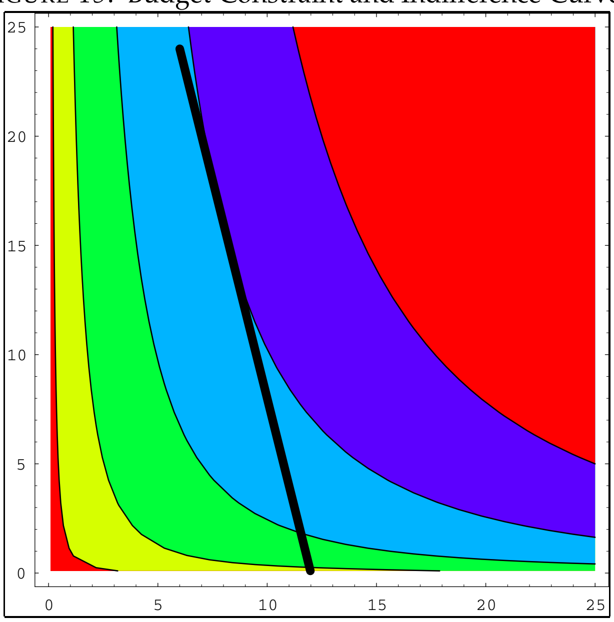
The utility function and the constraint are shown in figure 14.

FIGURE 14. Utility Function and Constraint



A set of level sets (indifference curves) and the budget constraint are shown in figure 15.

FIGURE 15. Budget Constraint and Indifference Curves



Set up the Langrangian problem

$$L = 10x_1^{0.4}\, x_2^{0.2} - \lambda\,[\,8\,x_1 + 2\,x_2 - 96] \tag{39}$$

The first order conditions are

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= 0.4\, x_1^{-0.6}\, x_2^{0.2} \;-\; 8\lambda \;=\; 0 \\ \frac{\partial L}{\partial x_2} &= 0.2\, x_1^{0.4}\, x_2^{-0.8} \;-\; 2\lambda \;=\; 0 \\ \frac{\partial L}{\partial \lambda} &= \;-\, 8\, x_1 \;-\; 2\, x_2 \;+\; 96 \;=\; 0\end{aligned} \tag{40}$$

Taking the ratio of the first and second equations in equation 40 we obtain

$$4 \;=\; 2\, \left(\frac{x_2}{x_1} \right). \tag{41}$$

We can now solve this equation for x_2 as a function of x_1 and prices. Doing so we obtain

$$x_2 \;=\; 2\, x_1 \tag{42}$$

Substituting in the cost equation we obtain

$$\begin{aligned} 8\, x_1 \;+\; 2\, x_2 \; &= 96 \\ \Rightarrow\; 8\, x_1 \;+\; 2\, [2\, x_1] \; &= 96 \\ \Rightarrow\; 8\, x_1 \;+\; 4\, x_1 \; &= 96 \\ \Rightarrow\; 12\, x_1 \; &= 96 \\ \Rightarrow\; x_1 \; &= 8 \end{aligned} \tag{43}$$

We can now obtain x_2 by substitution

$$\begin{aligned} x_2 \; &= 2\, x_1 \\ &= (2)\,(8) \;=\; 16 \end{aligned} \tag{44}$$

The utility level is obtained by substituting x_1 and x_2 in the utility function

$$\begin{aligned} u \; &= 10 x_1^{0.4}\, x_2^{0.2} \\ &= (10)\,(8)^{0.4}\,(16)^{0.2} \\ &= 40 \end{aligned} \tag{45}$$