

SINGLE VARIABLE OPTIMIZATION

1. DEFINITION OF LOCAL MAXIMA AND LOCAL MINIMA

1.1. Note on open and closed intervals.

1.1.1. *Open interval.* If a and b are two numbers with $a < b$, then the **open interval** from a to b is the collection of all numbers which are both larger than a and smaller than b . The open interval consists of all numbers between a and b . A compact way of writing this is $a < x < b$. We denote an open interval with parentheses as (a, b) .

1.1.2. *Closed interval.* If a and b are two numbers with $a < b$, then the **closed interval** from a to b is the collection of all numbers which are both greater than or equal to a and less than or equal to b . The closed interval consists of all points between a and b including a and b . A compact way of writing this is $a \leq x \leq b$. We denote a closed interval with brackets as $[a, b]$.

1.1.3. *Half-open intervals.* Intervals that are closed on one end and open on the other are called half-open intervals. We can denote these as half open on the right or the left and use a mixture of brackets and parentheses as appropriate.

1.2. Max-min theorem for continuous functions.

Theorem 1. *If f is continuous at every point of a closed interval $[a, b] \subset \mathbb{R}$, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.*

1.3. Absolute (global) extreme values.

1.3.1. *Absolute maximum.* Let f be a function with domain D . Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \text{ for all } x \in D \quad (1)$$

1.3.2. *Absolute minimum.* Let f be a function with domain D . Then f has an absolute minimum value on D at a point d if

$$f(x) \geq f(d) \text{ for all } x \in D \quad (2)$$

1.4. Local extreme values.

1.4.1. *Local maximum.* Consider a real valued function f defined with domain D . Then f is said to have a local maximum at an interior point $x^* \in D$ if there exists a real number $\delta > 0$ such that

$$f(x) \leq f(x^*) \text{ for all } x \text{ satisfying } \|x - x^*\| < \delta \quad (3)$$

1.4.2. *Local minimum.* Consider a real valued function f defined with domain D . Then f is said to have a local minimum at an interior point $\tilde{x} \in D$ if there exists a real number $\delta > 0$ such that

$$f(x) \geq f(\tilde{x}) \forall x \text{ satisfying } \|x - \tilde{x}\| < \delta \quad (4)$$

1.5. **Strict maxima and minima, optimal points, and extreme points.** If the value of f at x^* is strictly larger than at any other point in the interval, the x^* is a **strict maximum point**. Similarly \tilde{x} is a **strict minimum point** if $f(x) > f(\tilde{x})$ for all x in the interval, $x \neq \tilde{x}$. We often refer to maxima and minima as optimal points or extreme points.

2. EXTREME POINTS AND DERIVATIVES

2.1. **A first derivative test for extreme points.** Let f be a real valued function defined on a domain D . If f has a local maximum or a local minimum at an interior point $c \in D$, and if f' is defined at c , then

$$f'(c) = \left. \frac{df(\cdot)}{dx} \right|_c = 0. \quad (5)$$

Proof. To show that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that it cannot be negative. The only number that is neither positive or negative is zero, so that is what $f'(c)$ must be. Suppose that f has a local maximum value at $x = c$ so that $f(x) - f(c) \leq 0$ for all values of x sufficiently close to c . Since c is an interior point of f 's domain, $f'(c)$ is defined by the two-sided limit \square

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}. \quad (6)$$

This means that the right-hand side and left-hand side limits both exist at c and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

because

$$(x - c) > 0 \text{ and } f(x) \leq f(c)$$

Similarly,

$$f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

because

$$(x - c) < 0 \text{ and } f(x) \leq f(c)$$

Together 7 and 8 imply that $f'(c) = 0$. The proof for minimum values is similar.

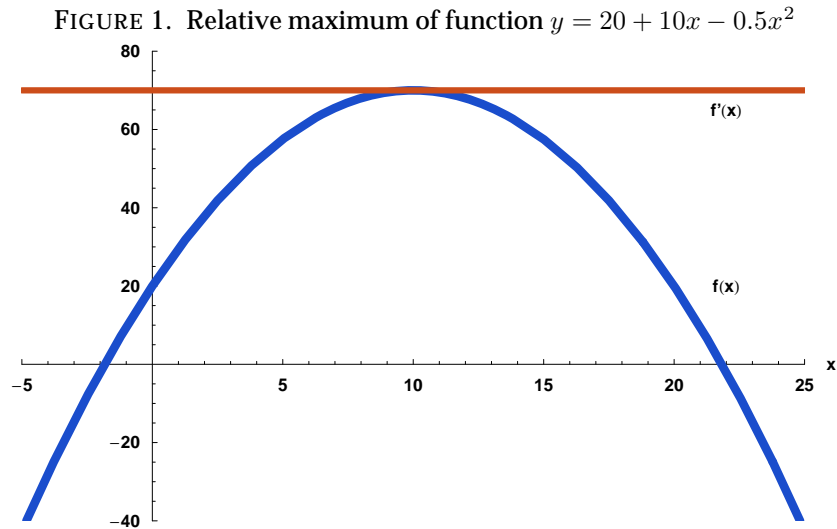
2.2. **Implication of theorem 2.** The only places where a function f can possibly have an extreme value (local or global) are

- a: interior points where $f'(\cdot) = 0$,
- b: interior points where $f'(\cdot)$ is undefined,
- c: endpoints of the domain of f .

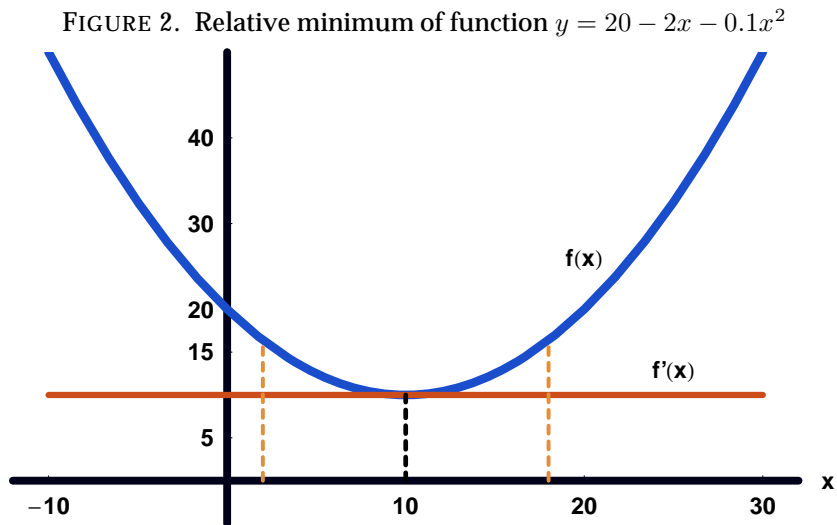
2.3. Critical (stationary) points. An interior point of the domain of a function f where f' is zero or undefined is a **critical (stationary) point** of f . The only domain points where a function f can assume extreme values are critical points and endpoints.

2.4. Graphical illustration of extreme points.

2.4.1. *A relative maximum.* Figure 1 shows the relative maximum of a function.



2.4.2. *A relative minimum.* Figure 2 shows the relative minimum of a function.

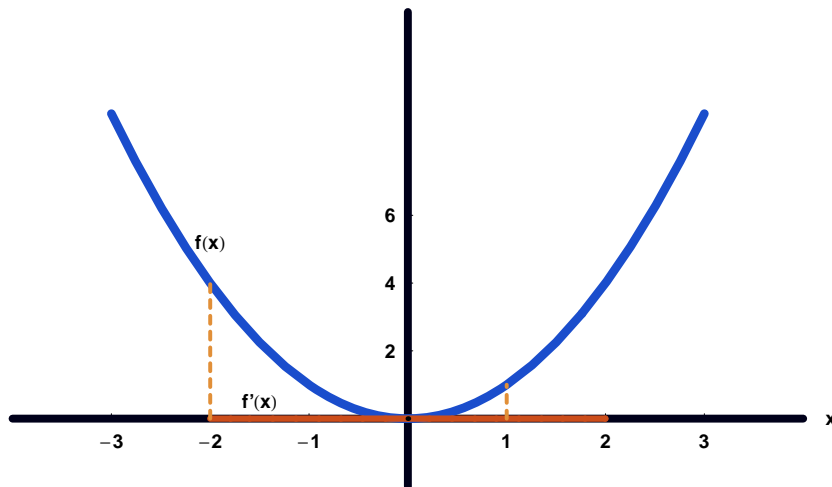


2.5. Examples of finding relative maximums and minimums.

2.5.1. *Example 1.* Find the absolute maximum and minimum values of $f(x) = x^2$, $x \in [-2, 1]$.

The only critical value of $f(x)$ is at $x = 0$ because $f'(\cdot) = 2x = 0 \Rightarrow x = 0$. At $x = 0$, $f(x) = 0$. The endpoint values are $f(-2) = 4$ and $f(1) = 1$. So the function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. Figure 3 shows the graph of example 1.

FIGURE 3. Maximum and minimum of function $y = x^2$, $x \in [-2, 1]$



2.5.2. *Example 2.* Find the absolute maximum and minimum values of $f(x) = 8x - x^4$, $x \in [-2, 1]$.

Setting the derivative equal to zero we obtain

$$\begin{aligned} 8 - 4x^3 &= 0 \\ \Rightarrow 4x^3 &= 8 \\ \Rightarrow x^3 &= 2 \\ \Rightarrow x &= 2^{\frac{1}{3}} \\ &= \sqrt[3]{2} \end{aligned}$$

This point is not in the given domain. So we check the endpoints. This yields $f(-2) = -32$ and $f(1) = 7$. So the absolute maximum is 7 at $x = 1$ and the absolute minimum is -32 at $x = -2$. Figure 4 shows the graph of example 2.

2.6. Increasing and decreasing functions.

2.6.1. *Definitions of increasing and decreasing functions.* Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I .

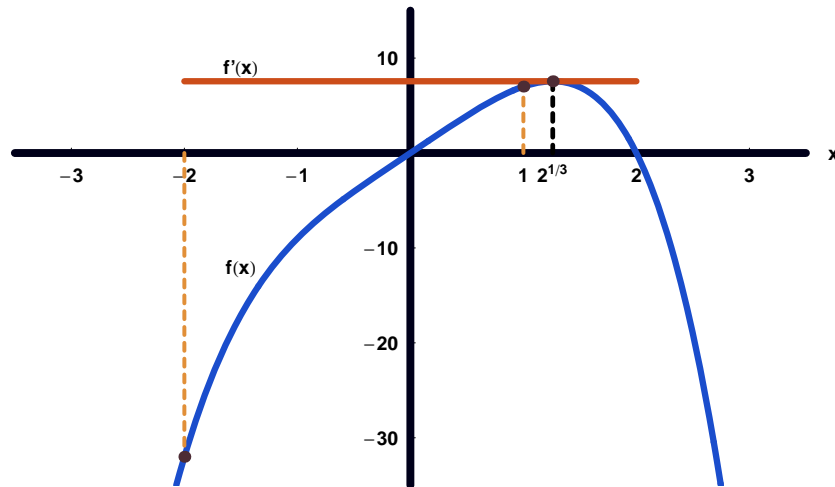
a: f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

b: f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_2) < f(x_1)$.

2.6.2. *First derivative test for increasing and decreasing functions.* Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

a: If $f'(\cdot) > 0$ at each point of (a, b) , then f increases on $[a, b]$.

b: If $f' < 0$ at each point of (a, b) , then f decreases on $[a, b]$.

FIGURE 4. Maximum and minimum of function $y = 8 - x^2$, $x \in [-2, 1]$ 

3. TESTS FOR LOCAL EXTREME VALUES

3.1. Necessary versus sufficient conditions.

While the zero derivative condition (critical value) is necessary for a maximum or minimum, it is clear that it is not sufficient. In other words, while all local extreme points are critical values, not all critical values are extreme points. Consider the three different cases below.

3.2. Types of critical points.

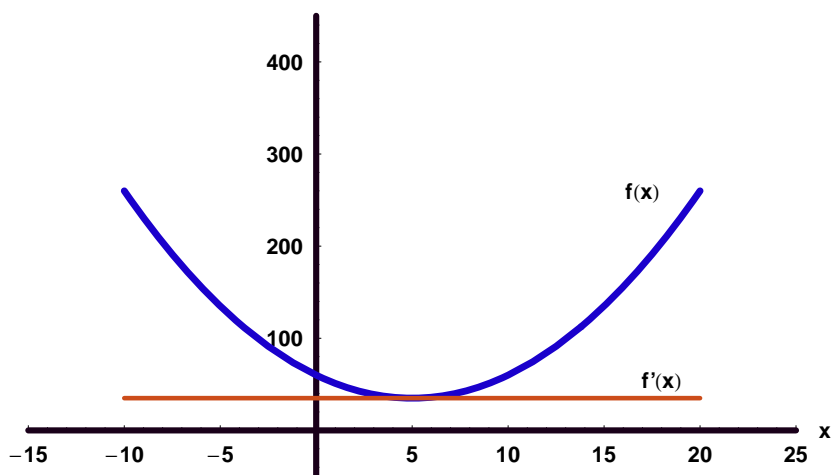
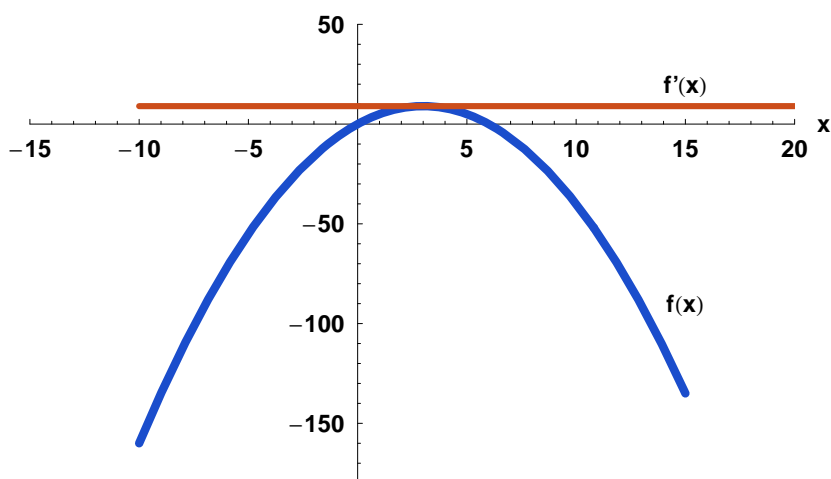
3.2.1. *Unique global minimum.* First consider a function where there is a unique global minimum, so that a relative minimum is also the global minimum. The function $y = x^2 - 10x + 60$ is a parabola that has a global minimum at $x = 5$. The graph of this function is given in figure 5.

3.2.2. *Unique global maximum.* Now consider a case where the relative maximum is a global maximum. The function $y = 6x - x^2$ is a parabola that has a global maximum at $x = 3$. The graph of this function is given in figure 6.

3.2.3. *Local maxima and minima but no global extreme values over the entire domain (in this case the real numbers).* Now consider a case where the function has local maxima and minima and there are no global maximums or minimums. Specifically consider the function $y = 50 - 6x + 1/18x^3$. The graph of this function is given in figure 7. The relative maximum is at $x = -6$ and the relative minimum is at $x = 6$.

3.2.4. *No relative maxima or minima at critical points.* Consider the case where the first derivative is equal to zero but there is no relative maximum or minimum. Such a point is called an inflection point. The second derivative is zero at this point. A graph of a function with an inflection point at a critical value is given in figure 8.

3.3. The first derivative test for local extreme values.

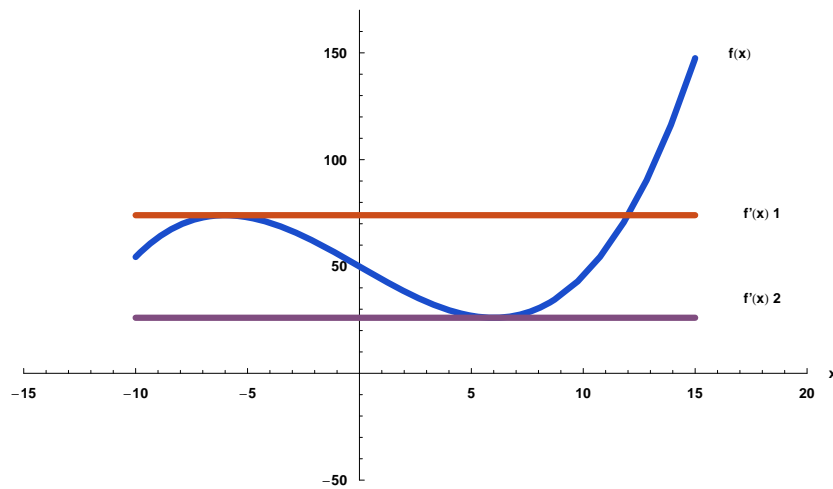
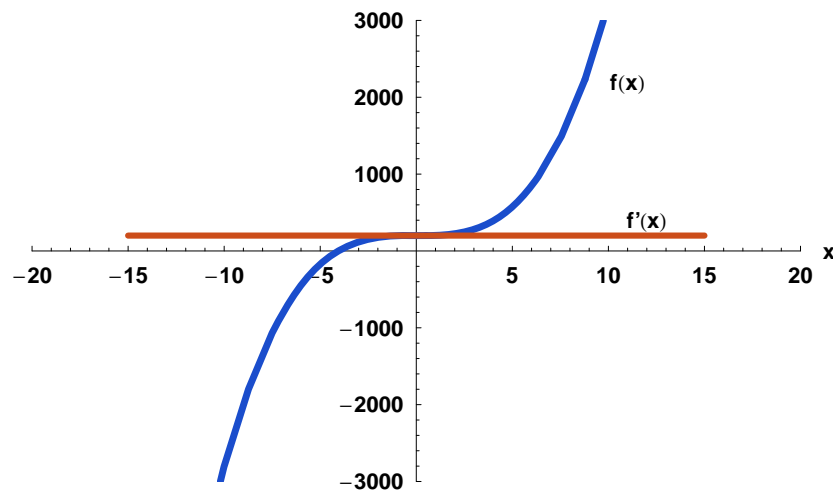
FIGURE 5. Global minimum of function $y = x^2 - 10x + 60$ FIGURE 6. Global maximum of function $y = 6x - x^2$ 

3.3.1. Local extreme values at critical points (points where $f'(c) = 0$).

- a: If $f'()$ changes from positive to negative at c ($f'(c) > 0$ for $x < c$ and $f'(c) < 0$ for $x > c$), then f has a local maximum value at c . This case is shown in figure 9.
- b: If $f'()$ changes from negative to positive at c , ($f'() < 0$ for $x < c$ and $f'() > 0$ for $x > c$), then f has a local minimum value at c . This case is shown in figure 10.
- c: If $f'()$ does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c . This case is shown in figure 11.

3.3.2. Local extreme values at endpoints.

- a: At the left endpoint a , if $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) at a . The case with the minimum or maximum at the left end point is shown in figure 12

FIGURE 7. Relative minimum and maximum of function $y = 50 - 6x + 1/18x^3$ FIGURE 8. Inflection Point of function $y = 3x^3 + 200$ 

b: At the right endpoint b , if $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) at b . This case is shown in figure 13.

3.4. The second derivative test for local extreme values.

3.4.1. The second derivative test.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

3.4.2. *Sufficient condition for a maximum or minimum.* For some integer $n \geq 1$, let f have a continuous n^{th} derivative in the open interval (a, b) . Suppose also that for some interior point $c \in (a, b)$ we have

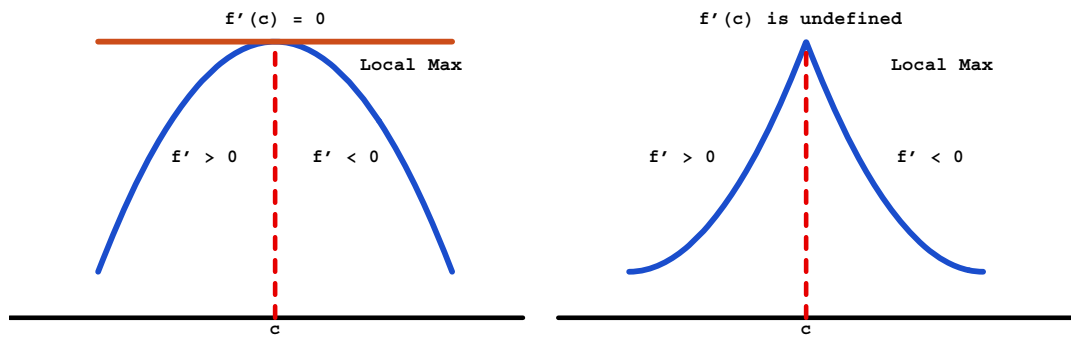
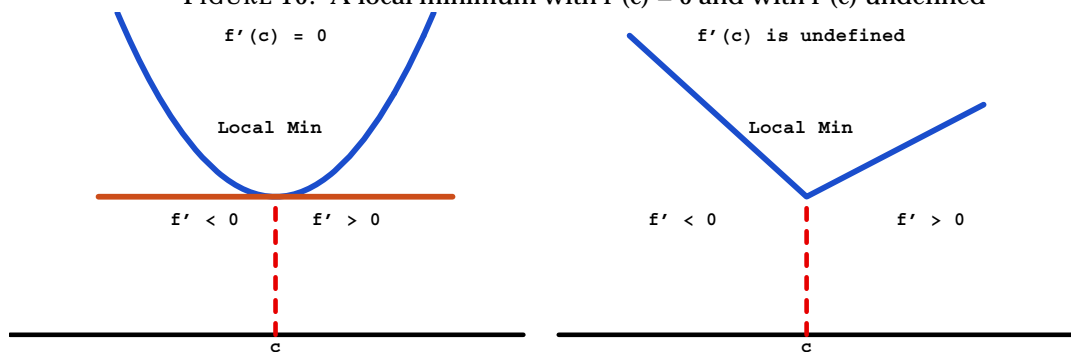
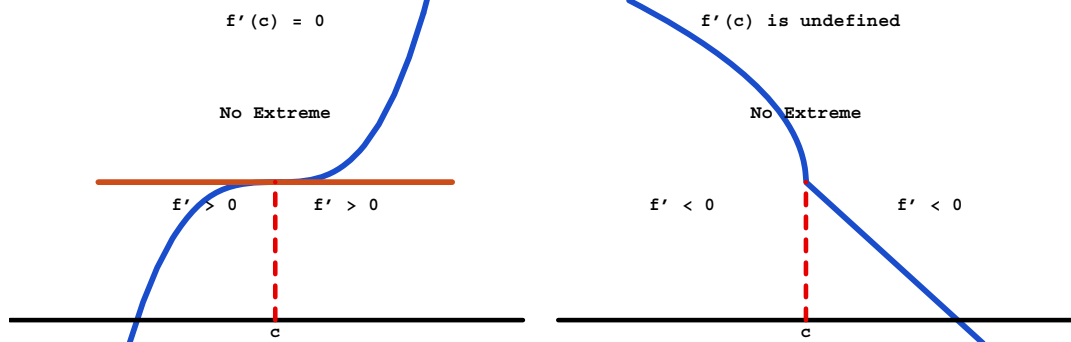
FIGURE 9. A local maximum with $f'(c) = 0$ and with $f'(c)$ undefinedFIGURE 10. A local minimum with $f'(c) = 0$ and with $f'(c)$ undefined

FIGURE 11. No Local Extreme Value



$$f'(c) = f''(c) = \dots = f^{n-1}(c) = 0, \text{ but } f^n(c) \neq 0 \quad (9)$$

Then for n even, f has a local minimum at c if $f^n(c) > 0$, and a local maximum if $f^n(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c .

3.4.3. Examples.

a: Consider a case where the function has a global minimum.

FIGURE 12. A local minimum and maximum at left endpoints

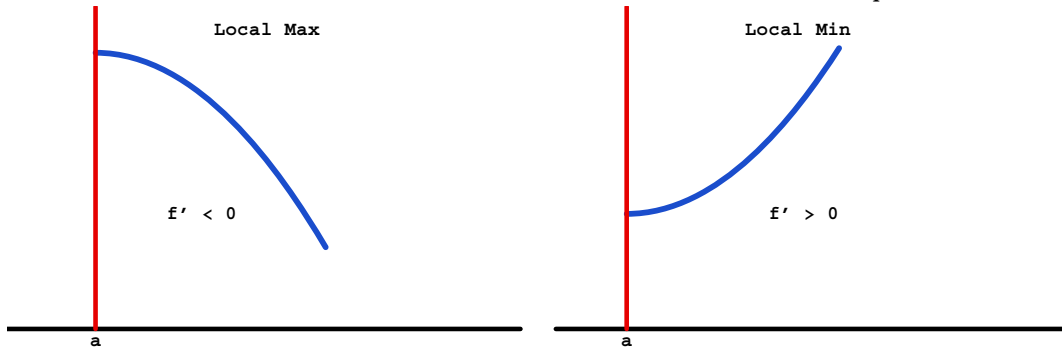
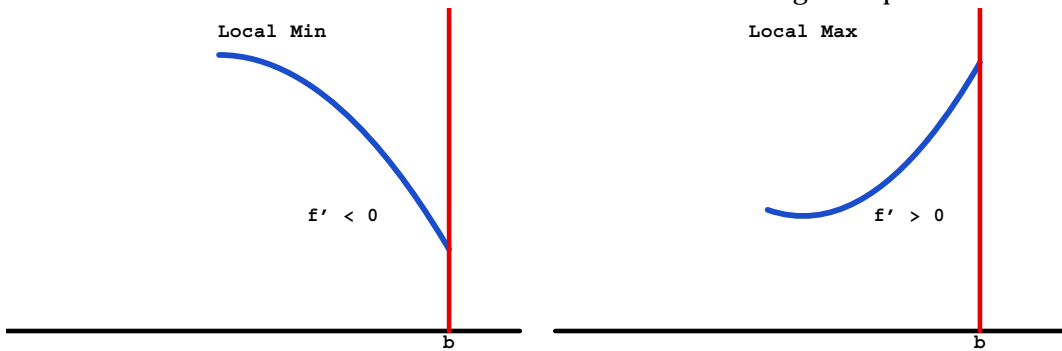


FIGURE 13. A local minimum and maximum at right endpoints



$$y = f(x) = x^2$$

$$f'(x) = 2x = 0$$

$$\Rightarrow x = 0 \text{ is an extreme point}$$

$$f''(x) = 2 \text{ which is not zero (n is even) and positive}$$

$$\Rightarrow x = 0 \text{ is a local minimum}$$

Figure 14 shows the graph for the example in part a.

b: Consider a case where the function has a global maximum.

$$y = f(x) = 6x - x^2$$

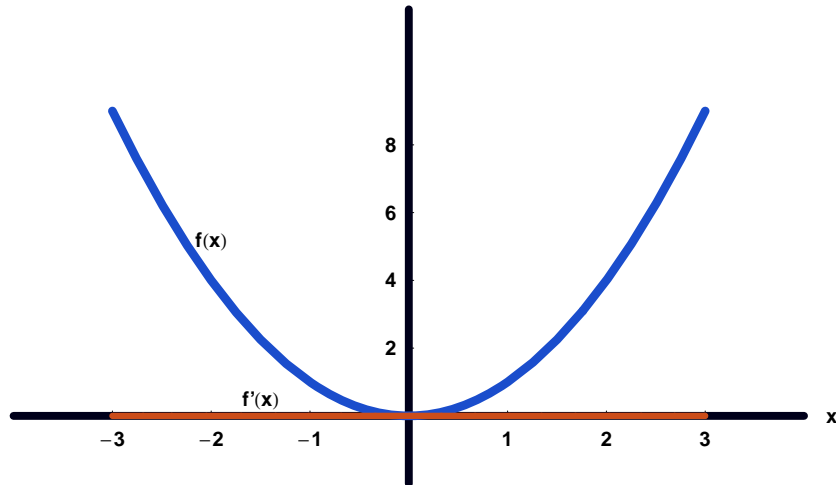
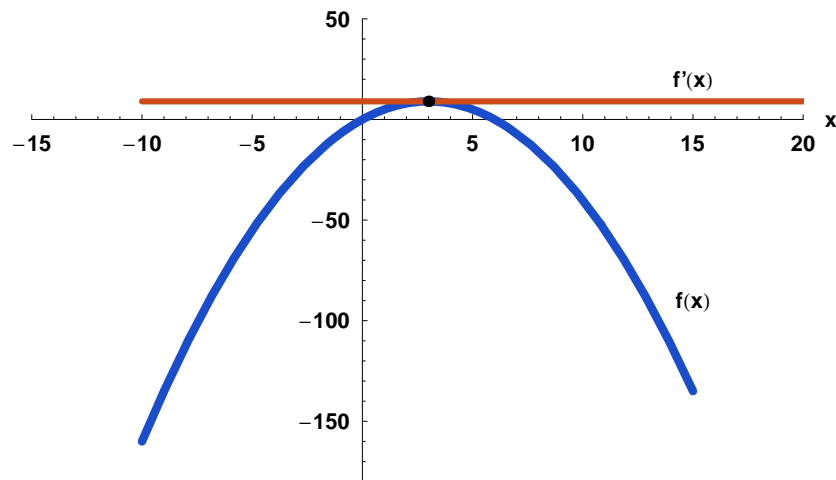
$$f'(x) = 6 - 2x = 0$$

$$\Rightarrow x = 3 \text{ is an extreme point}$$

$$f''(x) = -2 \text{ which is not zero (n is even) and negative}$$

$$\Rightarrow x = 3 \text{ is a local maximum}$$

Figure 15 shows the graph for the example in part b.

FIGURE 14. Local minimum of function $y = x^2$ FIGURE 15. Global maximum of function $y = 6x - x^2$ 

c: Consider a case where the function has no global maximum nor global minimum.

$$y = f(x) = x^3$$

$$f'(x) = 3x^2 = 0$$

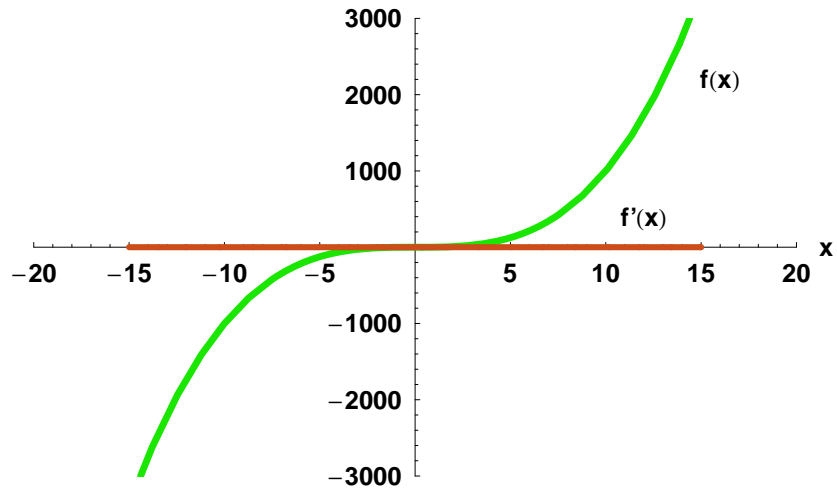
$$\Rightarrow x = 0 \text{ is an extreme point}$$

$$f''(x) = 6x \text{ which is zero at zero (n is even)}$$

$$f'''(x) = 6 \text{ which is not zero but positive (n is odd)}$$

$$\Rightarrow x = 0 \text{ is neither a local maximum nor a local minimum}$$

Figure 16 shows the graph for example c.

FIGURE 16. Function which has no global maximum nor global minimum $y = x^3$ 

3.4.4. *Partial converse of second derivative test.* Suppose $f''(a)$ exists. If f has a local minimum at a , then $f''(a) \geq 0$; if f has a local maximum at a , then $f''(a) \leq 0$.