SAMPLE MOMENTS

1. Population Moments

1.1. Moments about the origin (raw moments). The rth moment about the origin of a random variable X, denoted by μ'_r, is the expected value of X^r; symbolically,

$$\begin{aligned}\mu'_r &= E(X^r)\\ &= \sum_x x^r f(x)\end{aligned} \tag{1}$$

for r = 0, 1, 2, . . . when X is discrete and

$$\begin{aligned}\mu'_r &= E(X^r)\\ &= \int_{-\infty}^{\infty} x^r f(x)\, dx\end{aligned} \tag{2}$$

when X is continuous. The rth moment about the origin is only defined if E[X^r] exists. A moment about the origin is sometimes called a raw moment. Note that $\mu'_1 = E(X) = \mu_X$, the mean of the distribution of X, or simply the mean of X. The rth moment is sometimes written as function of θ where θ is a vector of parameters that characterize the distribution of X.

If there is a sequence of random variables, $X_1, X_2, \ldots X_n$, we will call the rth population moment of the ith random variable $\mu'_{i,\,r}$ and define it as

$$\mu'_{i,r} = E\left(X_i^r\right) \tag{3}$$

1.2. Central moments. The rth moment about the mean of a random variable X, denoted by μ_r, is the expected value of $(X - \mu_X)^r$ symbolically,

$$\begin{aligned}\mu_r &= E[\,(X - \mu_X)^r\,]\\ &= \sum_x (x - \mu_X)^r f(x)\end{aligned} \tag{4}$$

for r = 0, 1, 2, . . . when X is discrete and

$$\begin{aligned}\mu_r &= E[\,(X - \mu_X)^r\,]\\ &= \int_{-\infty}^{\infty} (x - \mu_X)^r f(x)\, dx\end{aligned} \tag{5}$$

when X is continuous. The r^{th} moment about the mean is only defined if E[(X - $\mu_X)^r$] exists. The rth moment about the mean of a random variable X is sometimes called the rth central moment of X. The rth central moment of X about a is defined as E[$(X - a)^r$]. If a = μ_X, we have the rth central moment of X about μ_X.

Note that

Date: December 7, 2005.

$$\mu_1 = E[X - \mu_X] = \int_{-\infty}^{\infty} (x - \mu_X)\, f(x)\, dx = 0$$
$$\mu_2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2\, f(x)\, dx = Var(X) = \sigma^2 \tag{6}$$

Also note that all odd moments of X around its mean are zero for symmetrical distributions, provided such moments exist.

If there is a sequence of random variables, $X_1, X_2, \ldots X_n$, we will call the r^{th} central population moment of the ith random variable $\mu_{i,r}$ and define it as

$$\mu_{i,r} = E\left(X_i^r - \mu'_{i,1}\right)^r \tag{7}$$

When the variables are identically distributed, we will drop the i subscript and write μ'_r and μ_r .

2. Sample Moments

2.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots X_n$. The first sample moment, usually called the average is defined by

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{8}$$

Corresponding to this statistic is its numerical value, $\bar{x}_n$, which is defined by

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{9}$$

where x_i represents the observed value of X_i. The rth sample moment for any t is defined by

$$\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} X_i^r \tag{10}$$

This too has a numerical counterpart given by

$$\bar{x}_n^r = \frac{1}{n} \sum_{i=1}^{n} x_i^r \tag{11}$$

2.2. **Properties of Sample Moments.**

2.2.1. *Expected value of $\bar{X}_n{}^r$.* Taking the expected value of equation 10 we obtain

$$E\left[\bar{X}_n^r\right] = E\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} E\ X_i^r = \frac{1}{n} \sum_{i=1}^{n} \mu'_{i,r} \tag{12}$$

If the X's are identically distributed, then

$$E\left[\bar{X}_n^r\right] = E\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} \mu'_r = \mu'_r \tag{13}$$

2.2.2. *Variance of $\bar{X}_n^r$.* First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

$$Var\left(\bar{X}_n^r\right) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^r\right) = \frac{1}{n^2}\, Var\left(\sum_{i=1}^{n} X_i^r\right) \tag{14}$$

If the X's are independent, then

$$Var\left(\bar{X}_n^r\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i^r\right) \tag{15}$$

If the X's are independent and identically distributed, then

$$Var\left(\bar{X}_n^r\right) = \frac{1}{n}\, Var\left(X^r\right) \tag{16}$$

where X denotes any one of the random variables (because they are all identical). In the case where r =1, we obtain

$$Var\left(\bar{X}_n\right) = \frac{1}{n}\, Var\left(X\right) = \frac{\sigma^2}{n} \tag{17}$$

3. Sample Central Moments

3.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots, X_n$. We define the sample central moments as

$$\begin{aligned} C_n^r &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^r, \; r = 1, 2, 3, \ldots, \\ \Rightarrow C_n^1 &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right) \\ \Rightarrow C_n^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^2 \end{aligned} \tag{18}$$

These are only defined if $\mu'_{i,1}$ is known.

3.2. **Properties of Sample Central Moments.**

3.2.1. *Expected value of C_n^r.* The expected value of C_n^r is given by

$$E\left(C_n^r\right) = \frac{1}{n}\sum_{i=1}^{n} E\left(X_i - \mu'_{i,1}\right)^r = \frac{1}{n}\sum_{i=1}^{n} \mu_{i,r} \tag{19}$$

The last equality follows from equation 7.

If the X_i are identically distributed, then

$$\begin{aligned} E\left(C_n^r\right) &= \mu_r \\ E\left(C_n^1\right) &= 0 \end{aligned} \tag{20}$$

3.2.2. *Variance of* C_n^r. First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

$$Var\,(\,C_n^r\,) = Var\left(\frac{1}{n}\sum_{i=1}^{n}\,(\,X_i - \mu'_{i,1}\,)^r\right) = \frac{1}{n^2}\,Var\left(\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^r\right) \tag{21}$$

If the X's are independently distributed, then

$$Var\,(\,C_n^r\,) = \frac{1}{n^2}\,\sum_{i=1}^{n} Var\left[\left(X_i - \mu'_{i,1}\right)^r\right] \tag{22}$$

If the X's are independent and identically distributed, then

$$Var\,(\,C_n^r\,) = \frac{1}{n}\,Var\left[\,(\,X - \mu'_1\,)^r\,\right] \tag{23}$$

where X denotes any one of the random variables (because they are all identical). In the case where r =1, we obtain

$$\begin{aligned} Var\,(\,C_n^1) &= \frac{1}{n}\,Var\,[\,X - \mu'_1\,] \\ &= \frac{1}{n}Var\,[\,X - \mu\,] \\ &= \frac{1}{n}\sigma^2 - 2\,Cov\,[\,X\,,\;\mu\,] + Var\,[\,\mu\,] \\ &= \frac{1}{n}\sigma^2 \end{aligned} \tag{24}$$

4. Sample About the Average

4.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots X_n$. Define the rth sample moment about the average as

$$M_n^r \;=\; \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^r, \quad r \;=\; 1, 2, 3, \ldots, \tag{25}$$

This is clearly a statistic of which we can compute a numerical value. We denote the numerical value by, m_n^r, and define it as

$$m_n^r = \frac{1}{n}\sum_{i\,=\,1}^{n}\,(\,x_i - \bar{x}_n)^r \tag{26}$$

In the special case where r = 1 we have

$$\begin{aligned} M_n^1 &= \frac{1}{n}\,\sum_{i\,=\,1}^{n}\,\left(X_i - \bar{X}_n\right) \\ &= \frac{1}{n}\,\sum_{i\,=\,1}^{n}\,X_i - \bar{X}_n \\ &= \bar{X}_n - \bar{X}_n \;=\; 0 \end{aligned} \tag{27}$$

4.2. **Properties of Sample Moments about the Average when r = 2.**

4.2.1. *Alternative ways to write M_n^r.* We can write M_n^2 in an alternative useful way by expanding the squared term and then simplifying as follows

$$
\begin{aligned}
M_n^r =& \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^r \\
\Rightarrow M_n^2 \ =& \ \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2 \\
=& \frac{1}{n} \left(\sum_{i=1}^{n} \left[X_i^{\,2} - 2\ X_i \bar{X}_n + \bar{X}_n^{\,2} \right] \right) \\
=& \frac{1}{n} \sum_{i=1}^{n} \ X_i^{\,2} - \frac{2\bar{X}_n}{n} \ \sum\nolimits_{i=1}^{n} X_i + \frac{1}{n} \ \sum_{i=1}^{n} \bar{X}_n^2 \\
=& \frac{1}{n} \sum_{i=1}^{n} \ X_i^2 - 2\bar{X}_n^2 + \bar{X}_n^2 \\
=& \frac{1}{n} \left(\sum_{i=1}^{n} \ X_i^2 \right) - \bar{X}_n^2
\end{aligned}
\tag{28}
$$

4.2.2. *Expected value of M_n^2.* The expected value of M_n^r is then given by

$$
\begin{aligned}
E\ \left(M_n^2\right) \ =& \ \frac{1}{n} E \left[\sum_{i=1}^{n} \ X_i^2 \ \right] - E\left[\bar{X}_n^2\right] \\
=& \ \frac{1}{n} \sum_{i=1}^{n} E\left[X_i^2\right] - \left(E\left[\bar{X}_n\right]\right)^2 - Var(\bar{X}_n) \\
=& \frac{1}{n} \sum_{i=1}^{n} \ \mu'_{i\,,\,2} - \left(\frac{1}{n} \sum_{i=1}^{n} \ \mu'_{i\,,\,1} \ \right)^2 - Var(\bar{X}_n\,)
\end{aligned}
\tag{29}
$$

The second line follows from the alternative definition of variance

$$
\begin{aligned}
Var\,(\ X\ ) =& E\left(\ X^{\,2}\ \right) - [\ E\,(\ X\ )\ ]^2 \\
\Rightarrow E\left(\ X^{\,2}\ \right) =& [\ E\,(\ X\ )\ ]^2 + Var\,(\ X\ ) \\
\Rightarrow E\left(\bar{X}_n^{\,2}\ \right) =& \left[\ E\left(\bar{X}_n\ \right)\ \right]^2 + Var(\bar{X}_n\ )
\end{aligned}
\tag{30}
$$

and the third line follows from equation 12. If the X_i are independent and identically distributed, then

$$
\begin{aligned}
E\ \left(M_n^2\right) = &\ \frac{1}{n}\, E\left[\sum_{i=1}^{n} X_i^2\right] - E\left[\bar{X}_n^2\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu'_{i,2}\ -\ \left(\frac{1}{n}\sum_{i=1}^{n} \mu'_{i,1}\right)^2\ -\ Var(\bar{X}_n) \\
&= \mu'_2\ -\ (\mu'_1)^2\ -\ \frac{\sigma^2}{n} \\
&= \sigma^2\ -\ \frac{1}{n}\sigma^2 \\
&= \frac{n\ -\ 1}{n}\ \sigma^2
\end{aligned}
\tag{31}
$$

where μ'_1 and μ'_2 are the first and second population moments, and μ_2 is the second central population moment for the identically distributed variables. Note that this obviously implies

$$
\begin{aligned}
E\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right] &= n\, E\left(M_n^2\right) \\
&= n\left(\frac{n\ -\ 1}{n}\right)\sigma^2 \\
&= (n-1)\,\sigma^2
\end{aligned}
\tag{32}
$$

4.2.3. *Variance of* M_n^2. By definition,

$$
Var\ \left(M_n^2\right)\ =\ E\left[\left(M_n^2\right)^2\right] - \left(E\, M_n^2\right)^2 \tag{33}
$$

The second term on the right on equation 33 is easily obtained by squaring the result in equation 31.

$$
\begin{aligned}
E\left(M_n^2\right) &= \frac{n\ -\ 1}{n}\,\sigma^2 \\
\Rightarrow \left(E\left(M_n^2\right)\right)^2 = \left(E\, M_n^2\right)^2 &= \frac{(n\ -\ 1)^2}{n^2}\,\sigma^4
\end{aligned}
\tag{34}
$$

Now consider the first term on the right hand side of equation 33. Write it as

$$
E\left[\left(M_n^2\right)^2\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)^2\right] \tag{35}
$$

Now consider writing $\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$ as follows

$$
\begin{aligned}
\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 =& \frac{1}{n} \sum_{i=1}^{n} \left((X_i - \mu) - (\bar{X} - \mu)\right)^2 \\
=& \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2 \\
\text{where } Y_i =& X_i - \mu \\
\bar{Y} =& \bar{X} - \mu
\end{aligned} \tag{36}
$$

Obviously,

$$
\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 = \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2 , \text{ where } Y_i = X_i - \mu\,, \bar{Y} = \bar{X} - \mu \tag{37}
$$

Now consider the properties of the random variable Y_i which is a transformation of X_i. First the expected value.

$$
\begin{aligned}
Y_i =& X_i - \mu \\
E\ (Y_i) =& E\,(X_i) - E\ (\mu) \\
=& \mu - \mu \\
=& 0
\end{aligned} \tag{38}
$$

The variance of Y_i is

$$
\begin{aligned}
Y_i &= X_i - \mu \\
Var\,(Y_i) &= Var\,(X_i) \\
&= \sigma^2 \text{ if } X_i \text{ are independently and identically distributed}
\end{aligned} \tag{39}
$$

Also consider $E(Y_i^4)$. We can write this as

$$
\begin{aligned}
E(\,Y^4) &= \int_{-\infty}^{\infty} y^4\, f\,(\,x\,)\ d\,x \\
&= \int_{-\infty}^{\infty} (\,x - \mu)^4\ f(x)\,d\,x \\
&= \mu_4
\end{aligned} \tag{40}
$$

Now write equation 35 as follows

$$
\begin{aligned}
E\left[\left(M_n^2\right)^2\right] &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)^2\right] && \text{(41a)}\\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right)^2\right] && \text{(41b)}\\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right] && \text{(41c)}\\
&= \frac{1}{n^2}\,E\left[\left(\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right] && \text{(41d)}
\end{aligned}
$$

Ignoring $\frac{1}{n^2}$ for now, expand equation 41 as follows

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right] &= E\left[\left(\sum_{i=1}^{n}\left(Y_i^2 - 2\,Y_i\bar{Y} + \bar{Y}^2\right)\right)^2\right] && \text{(42a)}\\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2 - 2\,\bar{Y}\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n}\bar{Y}^2\right)^2\right] && \text{(42b)}\\
&= E\left[\left(\left\{\sum_{i=1}^{n} Y_i^2\right\} - 2\,n\,\bar{Y}^2 + n\,\bar{Y}^2\right)^2\right] && \text{(42c)}\\
&= E\left[\left(\left\{\sum_{i=1}^{n} Y_i^2\right\} - n\bar{Y}^2\right)^2\right] && \text{(42d)}\\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2 - 2\,n\,\bar{Y}^2\sum_{i=1}^{n} Y_i^2 + n^2\bar{Y}^4\right] && \text{(42e)}\\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2\right] - 2\,n\,E\left[\bar{Y}^2\sum_{i=1}^{n} Y_i^2\right] + n^2\,E\left(\bar{Y}^4\right) && \text{(42f)}
\end{aligned}
$$

Now consider the first term on the right of 42 which we can write as

$$E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2\right] = E\left[\sum_{i=1}^{n} Y_i^2 \sum_{j=1}^{n} Y_j^2\right] \tag{43a}$$

$$= E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2\right] \tag{43b}$$

$$= \sum_{i=1}^{n} E\, Y_i^4 + \sum\sum\nolimits_{i \neq j} E\,Y_i^2\, E\,Y_j^2 \tag{43c}$$

$$= n\,\mu_4 + n\,(n - 1)\,\mu_2^2 \tag{43d}$$

$$= n\,\mu_4 + n\,(n - 1)\,\sigma^4 \tag{43e}$$

Now consider the second term on the right of 42 (ignoring 2n for now) which we can write as

$$E\left[\bar{Y}^2 \sum_{i=1}^{n} Y_i^2\right] = \frac{1}{n^2}\, E\left[\sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{i=1}^{n} Y_i^2\right] \tag{44a}$$

$$= \frac{1}{n^2}\, E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2 + \sum\sum\nolimits_{j \neq k} Y_j Y_k \sum_{\substack{i \neq j \\ i \neq k}} Y_i^2\right] \tag{44b}$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} E\, Y_i^4 + \sum\sum\nolimits_{i \neq j} E\,Y_i^2\, E\,Y_j^2 + \sum\sum\nolimits_{j \neq k} E\, Y_j E\,Y_k \sum_{\substack{i \neq j \\ i \neq k}} E\,Y_i^2\right] \tag{44c}$$

$$= \frac{1}{n^2}\left[n\,\mu_4 + n\,(n - 1)\,\mu_2^2 + 0\right] \tag{44d}$$

$$= \frac{1}{n}\left[\mu_4 + (n - 1)\sigma^4\right] \tag{44e}$$

The last term on the penultimate line is zero because $E(Y_j) = E(Y_k) = E(Y_i) = 0$.

Now consider the third term on the right side of 42 (ignoring n^2 for now) which we can write as

$$E\left[\bar{Y}^4\right] = \frac{1}{n^4}\, E\left[\sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{\ell=1}^{n} Y_\ell\right] \tag{45a}$$

$$= \frac{1}{n^2}\, E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum\nolimits_{i \neq k} Y_i^2\, Y_k^2 + \sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2 + \sum_{i \neq j} Y_i^2\, Y_j^2 + \cdots\right] \tag{45b}$$

where for the first double sum (i = j $\neq$ k = ℓ), for the second (i = k $\neq$ j = ℓ), and for the last (i = ℓ $\neq$ j = k) and ... indicates that all other terms include Y_i in a non-squared form, the expected value of which will be zero. Given that the Y_i are independently and identically distributed, the expected value of each of the double sums is the same, which gives

$$E\left[\bar{Y}^4\right] = \frac{1}{n^4}\, E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum\nolimits_{i \neq k} Y_i^2\, Y_k^2 + \sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2 + \sum_{i \neq j} Y_i^2\, Y_j^2 + \cdots\right] \tag{46a}$$

$$= \frac{1}{n^4}\left[\sum_{i=1}^{n} E\, Y_i^4 + 3\sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2 \;+\; \text{terms containing } EX_i\right] \tag{46b}$$

$$= \frac{1}{n^4}\left[\sum_{i=1}^{n} E\, Y_i^4 + 3\sum\sum\nolimits_{i \neq j} Y_i^2\, Y_j^2\right] \tag{46c}$$

$$= \frac{1}{n^4}\left[n\,\mu_4 + 3\,n\,(n - 1)\,(\,\mu_2\,)^2\right] \tag{46d}$$

$$= \frac{1}{n^4}\left[n\,\mu_4 + 3\,n\,(n - 1)\;\sigma^4\right] \tag{46e}$$

$$= \frac{1}{n^3}\left[\mu_4 + 3\,(n - 1)\;\sigma^4\right] \tag{46f}$$

Now combining the information in equations 44, 45, and 46 we obtain

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right] =& E\left[\left(\sum_{i=1}^{n}\left(Y_i^2-2\,Y_i\bar{Y}+\bar{Y}^2\right)\right)^2\right] && \text{(47a)}\\
=& E\left[\left(\sum_{i=1}^{n}Y_i^{\,2}\right)^2\right]-2\,n\,E\left[\bar{Y}^2\sum_{i=1}^{n}Y_i^2\right]+n^2\,E\left(\bar{Y}^{\,4}\right) && \text{(47b)}\\
=& n\mu_4+n(n-1)\mu_2^2-2n\left[\frac{1}{n}\left[\mu_4+(n-1)\,\mu_2^2\right]\right]+n^2\left[\frac{1}{n^3}\left[\mu_4+3\,(n-1)\,\mu_2^2\right]\right] && \text{(47c)}\\
=& n\,\mu_4+n\,(n-1)\,\mu_2^2-2\left[\mu_4+(n-1)\,\mu_2^2\right]+\left[\frac{1}{n}\left[\mu_4+3\,(n-1)\,\mu_2^2\right]\right] && \text{(47d)}\\
=& \frac{n^2}{n}\mu_4-\frac{2\,n}{n}\mu_4+\frac{1}{n}\mu_4+\frac{n^2(n-1)}{n}\mu_2^2-\frac{2\,n\,(n-1)}{n}\,\mu_2^2+\frac{3(n-1)}{n}\mu_2^2 && \text{(47e)}\\
=& \frac{n^2-2\,n+1}{n}\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n}\,\mu_2^2 && \text{(47f)}\\
=& \frac{n^2-2\,n+1}{n}\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n}\,\sigma^4 && \text{(47g)}
\end{aligned}
$$

Now rewrite equation 41 including $\frac{1}{n^2}$ as follows

$$
\begin{aligned}
E\left[\left(M_n^2\right)^2\right] =& \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right] && \text{(48a)}\\
=& \frac{1}{n^2}\left(\frac{n^2-2\,n+1}{n}\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n}\,\sigma_4\right) && \text{(48b)}\\
=& \frac{n^2-2\,n+1}{n^3}\,\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n^3}\,\sigma_4 && \text{(48c)}\\
=& \frac{(n-1)^2}{n^3}\,\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n^3}\,\sigma_4 && \text{(48d)}
\end{aligned}
$$

Now substitute equations 34 and 48 into equation 33 to obtain

$$
\begin{aligned}
Var\left(M_n^2\right) =& E\left[\left(M_n^2\right)^2\right]-\left(E\,M_n^2\right)^2\\
=& \frac{(n-1)^2}{n^3}\,\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n^3}\,\sigma^4-\frac{(n-1)^2}{n^2}\,\sigma^4
\end{aligned}
\qquad (49)
$$

We can simplify this as

$$
\begin{aligned}
Var\left(M_n^2\right) =& E\left[\left(M_n^2\right)^2\right] - \left(E\, M_n^2\right)^2 &\text{(50a)}\\
&= \frac{(n-1)^2}{n^3}\,\mu_4 + \frac{(n-1)\,(n^2-2n+3)}{n^3}\,\sigma^4 - \frac{n\,(n-1)^2}{n^3}\,\sigma^4 &\text{(50b)}\\
&= \frac{\mu_4\,(n-1)^2 + \left[(n-1)\,\sigma^4\right]\left(n^2 - 2n + 3 - n\,(n-1)\right)}{n^3} &\text{(50c)}\\
&= \frac{\mu_4\,(n-1)^2 + \left[(n-1)\,\sigma^4\right]\left(n^2 - 2n + 3 - n^2 + n\right)}{n^3} &\text{(50d)}\\
&= \frac{\mu_4\,(n-1)^2 + \left[(n-1)\,\sigma^4\right]\,(3-n)}{n^3}\\
&= \frac{\mu_4\,(n-1)^2 - \left[(n-1)\,\sigma^4\right]\,(n-3)}{n^3} &\text{(50e)}\\
&= \frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\sigma^4}{n^3} &\text{(50f)}
\end{aligned}
$$

5. Sample Variance

5.1. **Definition of sample variance.** The sample variance is defined as

$$
S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 \tag{51}
$$

We can write this in terms of moments about the mean as

$$
\begin{aligned}
S_n^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\\
&= \frac{n}{n-1} M_n^2 \text{ where } M_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2
\end{aligned} \tag{52}
$$

5.2. **Expected value of S^2.** We can compute the expected value of S^2 by substituting in from equation 31 as follows

$$
\begin{aligned}
E\left(S_n^2\right) &= \frac{n}{n-1}\,E\left(M_n^2\right)\\
&= \frac{n}{n-1}\,\frac{n-1}{n}\,\sigma^2\\
&= \sigma^2
\end{aligned} \tag{53}
$$

5.3. **Variance of S^2.** We can compute the variance of S^2 by substituting in from equation 50 as follows

$$\begin{aligned} Var\left(S_n^2\right) &= \frac{n^2}{(n\ -\ 1)^2}\, Var\left(M_n^2\right) \\ &= \frac{n^2}{(n\ -\ 1)^2}\left(\frac{(n\ -\ 1)^2\ \mu_4}{n^3} - \frac{(n\ -\ 1)\ (n-3)\ \sigma^4}{n^3}\right) \\ &= \frac{\mu_4}{n} - \frac{(n-3)\sigma^4}{n(n\ -\ 1)} \end{aligned} \tag{54}$$

5.4. **Definition of $\hat{\sigma}^2$.** One possible estimate of the population variance is $\hat{\sigma}^2$ which is given by

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 \\ &= M_n^2 \end{aligned} \tag{55}$$

5.5. **Expected value of $\hat{\sigma}^2$.** We can compute the expected value of $\hat{\sigma}^2$ by substituting in from equation 31 as follows

$$\begin{aligned} E\ \left(\hat{\sigma}^2\right) &= E\left(M_n^2\right) \\ &= \frac{n\ -\ 1}{n}\ \sigma^2 \end{aligned} \tag{56}$$

5.6. **Variance of $\hat{\sigma}^2$.** We can compute the variance of $\hat{\sigma}^2$ by substituting in from equation 50 as follows

$$\begin{aligned} Var\left(\hat{\sigma}^2\right) &= Var\left(M_n^2\right) \\ &= \frac{(n\ -\ 1)^2\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\sigma^4}{n^3} \\ &= \frac{\mu_4\ -\ \mu_2^2}{n} - \frac{2\,(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4\ -\ 3\mu_2^2}{n^3} \end{aligned} \tag{57}$$

We can also write this in an alternative fashion

$$\begin{aligned} Var\left(\hat{\sigma}^2\right) &= Var\ \left(M_n^2\right) \\ &= \frac{(n\ -\ 1)^2\ \mu_4}{n^3} - \frac{(n\ -\ 1)\ (n-3\ )\ \sigma^4}{n^3} \\ &= \frac{(n\ -\ 1)^2\ \mu_4}{n^3} - \frac{(n\ -\ 1)\ (n-3\ )\ \mu_2^2}{n^3} \\ &= \frac{n^2\ \mu_4 - 2\,n\,\mu_4 + \mu_4}{n^3} - \frac{n^2\,\mu_2^2\ -\ 4\,n\ \mu_2^2 + 3\,\mu_2^2}{n^3} \\ &= \frac{n^2\ (\mu_4 - \mu_2^2) - 2\,n\,(\mu_4 - 2\,\mu_2^2) + \mu_4 - 3\,\mu_2^2}{n^3} \\ &= \frac{\mu_4\ -\ \mu_2^2}{n} - \frac{2\,(\mu_4\ -\ 2\,\mu_2^2)}{n^2} + \frac{\mu_4\ -\ 3\,\mu_2^2}{n^3} \end{aligned} \tag{58}$$

6. Normal populations

6.1. **Central moments of the normal distribution.** For a normal population we can obtain the central moments by differentiating the moment generating function. The moment generating function for the central moments is as follows

$$M_X(t) = e^{\frac{t^2\,\sigma^2}{2}}\,. \tag{59}$$

The moments are then as follows. The first central moment is

$$\begin{aligned} E(X-\mu) &= \frac{d}{dt}\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,|_{t\,=\,0} \\ &= t\,\sigma^2\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,|_{t\,=\,0} \\ &= 0 \end{aligned} \tag{60}$$

The second central moment is

$$\begin{aligned} E(X\,-\,\mu)^2 =& \frac{d^2}{dt^2}\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,|_{t\,=\,0} \\ =& \frac{d}{dt}\left(t\,\sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\,|_{t\,=\,0} \\ =& \left(t^2\,\sigma^4\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+\sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\,|_{t\,=\,0} \\ =& \sigma^2 \end{aligned} \tag{61}$$

The third central moment is

$$\begin{aligned} E(X\,-\,\mu)^3 =& \frac{d^3}{dt^3}\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,|_{t\,=\,0} \\ =& \frac{d}{dt}\left(t^2\,\sigma^4\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+\sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\,|_{t\,=\,0} \\ =& \left(t^3\,\sigma^6\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+2\,t\,\sigma^4\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+t\,\sigma^4\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\,\right)\,|_{t\,=\,0} \\ =& \left(t^3\,\sigma^6\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+3\,t\,\sigma^4\,\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\,|_{t\,=\,0} \\ =& 0 \end{aligned} \tag{62}$$

The fourth central moment is

$$
\begin{aligned}
E(X - \mu)^4 &= \frac{d^4}{dt^4}\left(e^{\frac{t^2\sigma^2}{2}}\right) \; |_{t=0} \\
&= \frac{d}{dt}\left(t^3\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\,t\,\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right) \; |_{t=0} \\
&= \left(t^4\sigma^8\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\,t^2\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\,t^2\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\,\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right) \; |_{t=0} \\
&= \left(t^4\sigma^8\left(e^{\frac{t^2\sigma^2}{2}}\right) + 6\,t^2\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\,\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right) \; |_{t=0} \\
&= 3\sigma^4
\end{aligned}
\tag{63}
$$

6.2. **Variance of $\mathbf{S}^2$.** Let $X_1, X_2, \ldots X_n$ be a random sample from a normal population with mean μ and variance $\sigma^2 < \infty$.

We know from equation 54 that

$$
\begin{aligned}
Var\left(S_n^2\right) &= \frac{n^2}{(n-1)^2}\,Var\left(M_n^2\right) \\
&= \frac{n^2}{(n-1)^2}\left(\frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)(n-3)\,\sigma^4}{n^3}\right) \\
&= \frac{\mu_4}{n} - \frac{(n-3)\,\sigma^4}{n\,(n-1)}
\end{aligned}
\tag{64}
$$

If we substitute in for μ_4 from equation 63 we obtain

$$
\begin{aligned}
Var\left(S_n^2\right) &= \frac{\mu_4}{n} - \frac{(n-3)\,\sigma^4}{n\,(n-1)} \\
&= \frac{3\,\sigma^4}{n} - \frac{(n-3)\,\sigma^4}{n\,(n-1)} \\
&= \frac{(3\,(n-1) - (n-3))\,\sigma^4}{n\,(n-1)} \\
&= \frac{(3\,n - 3 - n + 3))\,\sigma^4}{n\,(n-1)} \\
&= \frac{2\,n\,\sigma^4}{n\,(n-1)} \\
&= \frac{2\,\sigma^4}{(n-1)}
\end{aligned}
\tag{65}
$$

6.3. **Variance of $\hat{\sigma}^2$.** It is easy to show that

$$
Var\left(\hat{\sigma}^2\right) = \frac{2\,\sigma^4\,(n-1)}{n^2}
\tag{66}
$$