SAMPLE MOMENTS

1. Population Moments

1.1. Moments about the origin (raw moments). The rth moment about the origin of a random variable X, denoted by $\mu'_r$, is the expected value of $X^r$; symbolically,

$$
\mu'_r = E(X^r) = \sum_x x^r f(x) \quad (1)
$$

for $r = 0, 1, 2, \ldots$ when X is discrete and

$$
\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) \, dx \quad (2)
$$

when X is continuous. The rth moment about the origin is only defined if $E[X^r]$ exists. A moment about the origin is sometimes called a raw moment. Note that $\mu'_1 = E(X) = \mu_X$, the mean of the distribution of X, or simply the mean of X. The rth moment is sometimes written as function of $\theta$ where $\theta$ is a vector of parameters that characterize the distribution of X.

If there is a sequence of random variables, $X_1, X_2, \ldots X_n$, we will call the rth population moment of the ith random variable $\mu'_{i,r}$ and define it as

$$
\mu'_{i,r} = E(X_i^r) \quad (3)
$$

1.2. Central moments. The rth moment about the mean of a random variable X, denoted by $\mu_r$, is the expected value of $(X - \mu_X)^r$ symbolically,

$$
\mu_r = E[(X - \mu_X)^r] = \sum_x (x - \mu_X)^r f(x) \quad (4)
$$

for $r = 0, 1, 2, \ldots$ when X is discrete and

$$
\mu_r = E[(X - \mu_X)^r] = \int_{-\infty}^{\infty} (x - \mu_X)^r f(x) \, dx \quad (5)
$$

when X is continuous. The rth moment about the mean is only defined if $E[(X - \mu_X)^r]$ exists. The rth moment about the mean of a random variable X is sometimes called the rth central moment of X. The rth central moment of X about a is defined as $E[(X - a)^r]$. If $a = \mu_X$, we have the rth central moment of X about $\mu_X$.

Note that

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\[ \mu_1 = E[X - \mu_X] = \int_{-\infty}^{\infty} (x - \mu_X) f(x) \, dx = 0 \quad (6) \]
\[ \mu_2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) \, dx = \text{Var}(X) = \sigma^2 \]

Also note that all odd moments of \( X \) around its mean are zero for symmetrical distributions, provided such moments exist.

If there is a sequence of random variables, \( X_1, X_2, \ldots X_n \), we will call the \( r^{th} \) central population moment of the \( i^{th} \) random variable \( \mu_{i,r} \) and define it as
\[ \mu_{i,r} = E \left( (X_i^r - \mu'_{i,1})^r \right) \quad (7) \]
When the variables are identically distributed, we will drop the \( i \) subscript and write \( \mu'_r \) and \( \mu_r \).

2. Sample Moments

2.1. Definitions. Assume there is a sequence of random variables, \( X_1, X_2, \ldots X_n \). The first sample moment, usually called the average is defined by

\[ \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad (8) \]

Corresponding to this statistic is its numerical value, \( \bar{x}_n \), which is defined by

\[ \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i \quad (9) \]

where \( x_i \) represents the observed value of \( X_i \). The \( r^{th} \) sample moment for any \( t \) is defined by

\[ \bar{X}^r_n = \frac{1}{n} \sum_{i=1}^{n} X_i^r \quad (10) \]

This too has a numerical counterpart given by

\[ \bar{x}^r_n = \frac{1}{n} \sum_{i=1}^{n} x_i^r \quad (11) \]

2.2. Properties of Sample Moments.

2.2.1. Expected value of \( \bar{X}^r_n \). Taking the expected value of equation 10 we obtain

\[ E \left[ \bar{X}^r_n \right] = E \bar{X}^r_n = \frac{1}{n} \sum_{i=1}^{n} E X_i^r = \frac{1}{n} \sum_{i=1}^{n} \mu'_{i,r} \quad (12) \]

If the \( X \)'s are identically distributed, then

\[ E \left[ \bar{X}^r_n \right] = E \bar{X}^r_n = \frac{1}{n} \sum_{i=1}^{n} \mu'_r = \mu'_r \quad (13) \]
2.2.2. Variance of $\bar{X}_n^r$. First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

\[
\text{Var} \left( \bar{X}_n^r \right) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i^r \right) = \frac{1}{n^2} \text{Var} \left( \sum_{i=1}^{n} X_i^r \right) \tag{14}
\]

If the $X$’s are independent, then

\[
\text{Var} \left( \bar{X}_n^r \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var} \left( X_i^r \right) \tag{15}
\]

If the $X$’s are independent and identically distributed, then

\[
\text{Var} \left( \bar{X}_n^r \right) = \frac{1}{n} \text{Var} \left( X^r \right) \tag{16}
\]

where $X$ denotes any one of the random variables (because they are all identical). In the case where $r = 1$, we obtain

\[
\text{Var} \left( \bar{X}_n \right) = \frac{1}{n} \text{Var} \left( X \right) = \sigma^2 / n \tag{17}
\]

3. Sample Central Moments

3.1. Definitions. Assume there is a sequence of random variables, $X_1, X_2, \ldots, X_n$. We define the sample central moments as

\[
C_n^r = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})^r, \quad r = 1, 2, 3, \ldots,
\]

⇒ $C_n^1 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})$

⇒ $C_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})^2 \tag{18}$

These are only defined if $\mu'_{i,1}$ is known.

3.2. Properties of Sample Central Moments.

3.2.1. Expected value of $C_n^r$. The expected value of $C_n^r$ is given by

\[
E \left( C_n^r \right) = \frac{1}{n} \sum_{i=1}^{n} E \left( X_i - \mu'_{i,1} \right)^r = \frac{1}{n} \sum_{i=1}^{n} \mu_{i,r} \tag{19}
\]

The last equality follows from equation 7.

If the $X_i$ are identically distributed, then

\[
E \left( C_n^r \right) = \mu_r
\]

\[
E \left( C_n^1 \right) = 0
\]

\[
E \left( C_n^2 \right) = \sigma^2
\]
3.2.2. **Variance of** $C_r^n$. First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

\[
Var (C_r^n) = Var \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})^r \right) = \frac{1}{n^2} Var \left( \sum_{i=1}^{n} (X_i - \mu'_{i,1})^r \right)
\]  

(21)

If the $X$'s are independently distributed, then

\[
Var (C_r^n) = \frac{1}{n^2} \sum_{i=1}^{n} Var \left[ (X_i - \mu'_{i,1})^r \right]
\]  

(22)

If the $X$'s are independent and identically distributed, then

\[
Var (C_r^n) = \frac{1}{n} Var \left[ (X - \mu_1) \right]
\]  

(23)

where $X$ denotes any one of the random variables (because they are all identical). In the case where $r = 1$, we obtain

\[
Var (C_1^n) = \frac{1}{n} Var \left[ X - \bar{X}_n \right] = \frac{1}{n} Var \left[ X - \mu \right] = \frac{1}{n} \sigma^2 - 2Cov[ X, \mu ] + Var[ \mu ]
\]  

(24)

4. **Sample About the Average**

4.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots, X_n$. Define the $r$th sample moment about the average as

\[
M_r^n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^r, \quad r = 1, 2, 3, \ldots,
\]  

(25)

This is clearly a statistic of which we can compute a numerical value. We denote the numerical value by, $m_r^n$, and define it as

\[
m_r^n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}_n)^r
\]  

(26)

In the special case where $r = 1$ we have

\[
M_1^n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n) = \frac{1}{n} \sum_{i=1}^{n} X_i - \bar{X}_n = \bar{X}_n - \bar{X}_n = 0
\]  

(27)

4.2. **Properties of Sample Moments about the Average when** $r = 2$. 
4.2.1. *Alternative ways to write* $M_r^n$. We can write $M_r^n$ in an alternative useful way by expanding the squared term and then simplifying as follows

$$M_r^n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^r$$

$$\Rightarrow M_r^n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \frac{1}{n} \left( \sum_{i=1}^{n} [X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2] \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \frac{2\bar{X}_n}{n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} \bar{X}_n^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\bar{X}_n^2 + \bar{X}_n^2$$

$$= \frac{1}{n} \left( \sum_{i=1}^{n} X_i^2 \right) - \bar{X}_n^2$$

(28)

4.2.2. *Expected value of* $M_r^n$. The expected value of $M_r^n$ is then given by

$$E (M_r^n) = \frac{1}{n} E \left[ \sum_{i=1}^{n} X_i^2 \right] - E [\bar{X}_n^2]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E [X_i^2] - (E [\bar{X}_n])^2 - Var(\bar{X}_n)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mu_i', 2 - \left( \frac{1}{n} \sum_{i=1}^{n} \mu_i' \right)^2 - Var(\bar{X}_n)$$

(29)

The second line follows from the alternative definition of variance

$$Var (X) = E (X^2) - [E (X)]^2$$

$$\Rightarrow E (X^2) = [E (X)]^2 + Var (X)$$

(30)

$$\Rightarrow E (\bar{X}_n^2) = [E (\bar{X}_n)]^2 + Var(\bar{X}_n)$$

and the third line follows from equation 12. If the $X_i$ are independent and identically distributed, then
where $\mu'_1$ and $\mu'_2$ are the first and second population moments, and $\mu_2$ is the second central population moment for the identically distributed variables. Note that this obviously implies

$$E \left[ \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right] = n E (M_n^2)$$

$$= n \left( \frac{n - 1}{n} \right) \sigma^2$$

$$= (n - 1) \sigma^2$$

(32)
\[
\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n} \sum_{i=1}^{n} ((X_i - \mu) - (\bar{X} - \mu))^2
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2
\]
(36)

where \( Y_i = X_i - \mu \)
\[ \bar{Y} = \bar{X} - \mu \]

Obviously,
\[
\sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} (Y_i - \bar{Y})^2, \text{ where } Y_i = X_i - \mu, \bar{Y} = \bar{X} - \mu
\]
(37)

Now consider the properties of the random variable \( Y_i \) which is a transformation of \( X_i \). First the expected value.

\[
Y_i = X_i - \mu
\]
\[
E(Y_i) = E(X_i) - E(\mu)
\]
\[
= \mu - \mu
\]
\[
= 0
\]
(38)

The variance of \( Y_i \) is

\[
Y_i = X_i - \mu
\]
\[
Var(Y_i) = Var(X_i)
\]
\[
= \sigma^2 \text{ if } X_i \text{ are independently and identically distributed}
\]
(39)

Also consider \( E(Y_i^4) \). We can write this as

\[
E(Y_i^4) = \int_{-\infty}^{\infty} y^4 f(x) \, dx
\]
\[
= \int_{-\infty}^{\infty} (x - \mu)^4 f(x) \, dx
\]
\[
= \mu_4
\]
(40)

Now write equation 35 as follows
Ignoring $\frac{1}{n^2}$ for now, expand equation 41 as follows:

\[
E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) \right] = E \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) = E \left( \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right) = \frac{1}{n^2} E \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)
\]

Now consider the first term on the right of 42 which we can write as:

\[
E \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right) \right] = E \left[ \left( \sum_{i=1}^{n} (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2) \right) \right] = E \left( \sum_{i=1}^{n} Y_i^2 - 2 \bar{Y} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \bar{Y}^2 \right) = E \left( \{ \sum_{i=1}^{n} Y_i^2 \} - 2 \bar{Y} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \bar{Y}^2 \right) = E \left( \{ \sum_{i=1}^{n} Y_i^2 \} - \bar{Y}^2 \right) = E \left( \sum_{i=1}^{n} Y_i^2 \right) - 2 n \bar{Y}^2 \sum_{i=1}^{n} Y_i^2 + n^2 \bar{Y}^4 = E \left( \sum_{i=1}^{n} Y_i^2 \right) - 2 n E \left[ \bar{Y}^2 \sum_{i=1}^{n} Y_i^2 \right] + n^2 E (\bar{Y}^4)
\]
Now consider the second term on the right of 42 (ignoring 2n for now) which we can write as

\[
E \left[ \frac{1}{n^2} \sum_{i=1}^{n} Y_i^2 \sum_{j=1}^{n} Y_j^2 \right] = \frac{1}{n^2} E \left[ \sum_{i=1}^{n} Y_i^4 + \sum_{i \neq j} Y_i^2 Y_j^2 \right] = \frac{1}{n^2} \left[ \sum_{i=1}^{n} E Y_i^4 + \sum_{i \neq j} EY_i^2 EY_j^2 \right] = \frac{1}{n^2} \left[ n \mu_4 + n(n-1) \mu_2^2 \right] = \frac{1}{n} \left[ \mu_4 + (n-1) \sigma^4 \right] \tag{44e}
\]

The last term on the penultimate line is zero because \(E(Y_j) = E(Y_k) = E(Y_i) = 0\).
Now consider the third term on the right side of 42 (ignoring \( n^2 \) for now) which we can write as

\[
E \left[ \bar{Y}^4 \right] = \frac{1}{n^4} E \left[ \sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{\ell=1}^{n} Y_{\ell} \right] = \frac{1}{n^2} E \left[ \sum_{i=1}^{n} Y_i^4 + \sum_{j \neq i} Y_i^2 Y_j^2 + \sum_{i \neq j} Y_i^2 Y_j^2 + \sum_{i \neq j} Y_i^2 Y_j^2 + \cdots \right] \tag{45a}
\]

where for the first double sum (\( i = j \neq k = \ell \)), for the second (\( i = k \neq j = \ell \)), and for the last (\( i = \ell \neq j = k \)) and ... indicates that all other terms include \( Y_i \) in a non-squared form, the expected value of which will be zero. Given that the \( Y_i \) are independently and identically distributed, the expected value of each of the double sums is the same, which gives

\[
E \left[ \bar{Y}^4 \right] = \frac{1}{n^4} E \left[ \sum_{i=1}^{n} E Y_i^4 + 3 \sum_{i \neq j} Y_i^2 Y_j^2 + \text{terms containing } E X_i \right] \tag{46a}
\]

\[
= \frac{1}{n^4} \left[ \sum_{i=1}^{n} E Y_i^4 + 3 \sum_{i \neq j} Y_i^2 Y_j^2 \right] \tag{46b}
\]

\[
= \frac{1}{n^4} \left[ \sum_{i=1}^{n} E Y_i^4 + 3 \sum_{i \neq j} Y_i^2 Y_j^2 \right] \tag{46c}
\]

\[
= \frac{1}{n^4} \left[ n\mu_4 + 3 n (n - 1) \mu_2^2 \right] \tag{46d}
\]

\[
= \frac{1}{n^4} \left[ n\mu_4 + 3 n (n - 1) \sigma^4 \right] \tag{46e}
\]

\[
= \frac{1}{n^3} \left[ \mu_4 + 3 (n - 1) \sigma^4 \right] \tag{46f}
\]

Now combining the information in equations 44, 45, and 46 we obtain
\[ E \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^2 \right] = E \left[ \left( \sum_{i=1}^{n} (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2) \right)^2 \right] \]  

(47a)

\[ = E \left[ \left( \sum_{i=1}^{n} Y_i^2 \right)^2 \right] - 2n E \left[ \bar{Y}^2 \sum_{i=1}^{n} Y_i^2 \right] + n^2 E \left( \bar{Y}^4 \right) \]  

(47b)

\[ = n\mu_4 + n(n-1)\mu_2^2 - 2n \left[ \frac{1}{n} \left( \mu_4 + (n-1)\mu_2^2 \right) \right] + n^2 \left[ \frac{1}{n^4} \left( \mu_4 + 3(n-1)\mu_2^2 \right) \right] \]  

(47c)

\[ = n\mu_4 + n(n-1)\mu_2^2 - 2 \left[ \mu_4 + (n-1)\mu_2^2 \right] + \frac{1}{n} \left[ \mu_4 + 3(n-1)\mu_2^2 \right] \]  

(47d)

\[ = \frac{n^2}{n} \mu_4 - \frac{2n}{n} \mu_4 + \frac{1}{n} \mu_4 + \frac{n^2(n-1)}{n} \mu_2^2 - \frac{2n(n-1)}{n} \mu_2^2 + \frac{3(n-1)}{n} \mu_2^2 \]  

(47e)

\[ = \frac{n^2 - 2n + 1}{n} \mu_4 + \frac{(n-1)(n^2 - 2n + 3)}{n} \mu_2^2 \]  

(47f)

\[ = \frac{n^2 - 2n + 1}{n} \mu_4 + \frac{(n-1)(n^2 - 2n + 3)}{n} \sigma^4 \]  

(47g)

Now rewrite equation 41 including \( \frac{1}{n^2} \) as follows

\[ E \left[ \left( M_n^2 \right)^2 \right] = \frac{1}{n^2} E \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^2 \right] \]  

(48a)

\[ = \frac{1}{n^2} \left( \frac{n^2 - 2n + 1}{n} \mu_4 + \frac{(n-1)(n^2 - 2n + 3)}{n^3} \sigma_4 \right) \]  

(48b)

\[ = \frac{n^2 - 2n + 1}{n^3} \mu_4 + \frac{(n-1)(n^2 - 2n + 3)}{n^3} \sigma_4 \]  

(48c)

\[ = \frac{(n-1)^2}{n^3} \mu_4 + \frac{(n-1)(n^2 - 2n + 3)}{n^3} \sigma_4 \]  

(48d)

Now substitute equations 34 and 48 into equation 33 to obtain

\[ Var \left( M_n^2 \right) = E \left[ (M_n^2)^2 \right] - (E M_n^2)^2 \]  

\[ = \frac{(n-1)^2}{n^3} \mu_4 + \frac{(n-1)(n^2 - 2n + 3)}{n^3} \sigma^4 - \frac{(n-1)^2}{n^2} \sigma^4 \]  

(49)

We can simplify this as
\[ \text{Var} \left( M_n^2 \right) = E \left[ \left( M_n^2 \right)^2 \right] - \left( E M_n^2 \right)^2 \]  
\[ = \frac{(n-1)^2}{n^3} \mu_4 + \left( \frac{n-1}{n^3} \right) (n^2 - 2n + 3) \sigma^4 \]  
\[ = \frac{\mu_4}{n^3} \left( n - 1 \right)^2 + \left[ \frac{\left( n - 1 \right) \sigma^4}{n^3} \right] \left( n^2 - 2n + 3 - n(n-1) \right) \]  
\[ = \frac{\mu_4}{n^3} \left( n - 1 \right)^2 + \left[ \frac{(n-1)\sigma^4}{n^3} \right] \left( n^2 - 2n + 3 - n^2 + n \right) \]  
\[ = \frac{\mu_4}{n^3} \left( n - 1 \right)^2 - \left[ \frac{(n-1)\sigma^4}{n^3} \right] (n-3) \]  
\[ = \frac{(n-1)^2 \mu_4}{n^3} - \frac{(n-1)(n-3)\sigma^4}{n^3} \]  
\[ = \left( n - 1 \right)^2 \frac{\mu_4}{n^3} - \frac{(n-1)(n-3)\sigma^4}{n^3} \]  

5. Sample Variance

5.1. Definition of sample variance. The sample variance is defined as

\[ S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \]  

We can write this in terms of moments about the mean as

\[ S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \]

\[ = \frac{n}{n-1} M_n^2 \quad \text{where} \quad M_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \]

5.2. Expected value of \( S^2 \). We can compute the expected value of \( S^2 \) by substituting in from equation 31 as follows

\[ E \left( S_n^2 \right) = \frac{n}{n-1} E \left( M_n^2 \right) \]

\[ = \frac{n}{n-1} \frac{n-1}{n} \sigma^2 \]

\[ = \sigma^2 \]  

5.3. Variance of \( S^2 \). We can compute the variance of \( S^2 \) by substituting in from equation 50 as follows
\[ \text{Var} (S^2_n) = \frac{n^2}{(n - 1)^2} \text{Var} (M^2_n) \]
\[ = \frac{n^2}{(n - 1)^2} \left( \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \sigma^4}{n^3} \right) \]  
\[ = \frac{\mu_4}{n} - \frac{(n - 3)\sigma^4}{n(n - 1)} \]  
\[ (54) \]

5.4. **Definition of \( \hat{\sigma}^2 \).** One possible estimate of the population variance is \( \hat{\sigma}^2 \) which is given by

\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \]
\[ = M^2_n \]  
\[ (55) \]

5.5. **Expected value of \( \hat{\sigma}^2 \).** We can compute the expected value of \( \hat{\sigma}^2 \) by substituting in from equation 31 as follows

\[ E(\hat{\sigma}^2) = E(M^2_n) \]
\[ = \frac{n}{n - 1} \sigma^2 \]  
\[ (56) \]

5.6. **Variance of \( \hat{\sigma}^2 \).** We can compute the variance of \( \hat{\sigma}^2 \) by substituting in from equation 50 as follows

\[ \text{Var}(\hat{\sigma}^2) = \text{Var}(M^2_n) \]
\[ = \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \sigma^4}{n^3} \]
\[ = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \]  
\[ (57) \]

We can also write this in an alternative fashion

\[ \text{Var}(\hat{\sigma}^2) = \text{Var}(M^2_n) \]
\[ = \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \sigma^4}{n^3} \]
\[ = \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \mu_2^2}{n^3} \]
\[ = \frac{n^2 \mu_4 - 2n \mu_4 + \mu_4}{n^3} - \frac{n^2 \mu_2^2 - 4n \mu_2^2 + 3\mu_2^2}{n^3} \]
\[ = \frac{n^2 (\mu_4 - \mu_2^2) - 2n (\mu_4 - 2\mu_2^2) + \mu_4 - 3\mu_2^2}{n^3} \]
\[ = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \]  
\[ (58) \]
6. Normal populations

6.1. Central moments of the normal distribution. For a normal population we can obtain the central moments by differentiating the moment generating function. The moment generating function for the central moments is as follows

\[ M_X(t) = e^{\frac{t^2 \sigma^2}{2}}. \] (59)

The moments are then as follows. The first central moment is

\[
E(X - \mu) = \frac{d}{dt} \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = t \sigma^2 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = 0
\] (60)

The second central moment is

\[
E(X - \mu)^2 = \frac{d^2}{dt^2} \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = \frac{d}{dt} \left( t \sigma^2 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \right) \bigg|_{t=0} = \left( t^2 \sigma^4 \left( e^{\frac{t^2 \sigma^2}{2}} \right) + \sigma^2 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \right) \bigg|_{t=0} = \sigma^2
\] (61)

The third central moment is

\[
E(X - \mu)^3 = \frac{d^3}{dt^3} \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = \frac{d}{dt} \left( t^2 \sigma^4 \left( e^{\frac{t^2 \sigma^2}{2}} \right) + \sigma^2 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \right) \bigg|_{t=0} = \left( t^3 \sigma^6 \left( e^{\frac{t^2 \sigma^2}{2}} \right) + 2 t \sigma^4 \left( e^{\frac{t^2 \sigma^2}{2}} \right) + t \sigma^4 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \right) \bigg|_{t=0} = 0
\] (62)

The fourth central moment is

\[
\]
\[ E(X - \mu)^4 = \frac{d^4}{dt^4} \left( e^{\frac{t^2}{2}} \right) |_{t=0} \]
\[ = \frac{d}{dt} \left( t^3 \sigma^6 \left( e^{\frac{t^2}{2}} \right) + 3 t^4 \sigma^4 \left( e^{\frac{t^2}{2}} \right) \right) |_{t=0} \]
\[ = \left( t^4 \sigma^8 \left( e^{\frac{t^2}{2}} \right) + 3 t^2 \sigma^6 \left( e^{\frac{t^2}{2}} \right) + 3 t^2 \sigma^6 \left( e^{\frac{t^2}{2}} \right) + 3 \sigma^4 \left( e^{\frac{t^2}{2}} \right) \right) |_{t=0} \]
\[ = \left( t^4 \sigma^8 \left( e^{\frac{t^2}{2}} \right) + 6 t^2 \sigma^6 \left( e^{\frac{t^2}{2}} \right) + 3 \sigma^4 \left( e^{\frac{t^2}{2}} \right) \right) |_{t=0} \]
\[ = 3 \sigma^4 \]

6.2. Variance of \( S^2 \). Let \( X_1, X_2, \ldots, X_n \) be a random sample from a normal population with mean \( \mu \) and variance \( \sigma^2 < \infty \).

We know from equation 54 that
\[ \text{Var} \left( S_n^2 \right) = \frac{n^2}{(n-1)^2} \text{Var} \left( M_n^2 \right) \]
\[ = \frac{n^2}{(n-1)^2} \left( \frac{(n-1)^2 \mu_4}{n^3} - \frac{(n-1)(n-3) \sigma^4}{n^3} \right) \]
\[ = \frac{\mu_4}{n} - \frac{(n-3) \sigma^4}{n(n-1)} \]  
\[ (64) \]

If we substitute in for \( \mu_4 \) from equation 63 we obtain
\[ \text{Var} \left( S_n^2 \right) = \frac{\mu_4}{n} - \frac{(n-3) \sigma^4}{n(n-1)} \]
\[ = \frac{3 \sigma^4}{n} - \frac{(n-3) \sigma^4}{n(n-1)} \]
\[ = \frac{(3(n-1) - (n-3)) \sigma^4}{n(n-1)} \]
\[ = \frac{(3n-3-n+3) \sigma^4}{n(n-1)} \]
\[ = \frac{2n \sigma^4}{n(n-1)} \]
\[ = \frac{2 \sigma^4}{n-1} \]
\[ (65) \]

6.3. Variance of \( \hat{\sigma}^2 \). It is easy to show that
\[ \text{Var} \left( \hat{\sigma}^2 \right) = \frac{2 \sigma^4 (n-1)}{n^2} \]  
\[ (66) \]