SYSTEMS OF EQUATIONS

1. Examples of systems of equations

Here are some examples of systems of equations. Each system has a number of equations and a number (not necessarily the same) of variables for which we would like to solve. The variables are typically represented by letters such as x, y, and z. The first four systems have two equations and two variables. The fifth one has three equations and three variables. The sixth one has two equations and three variables, while the last one has three equations and two variables.

a.
$$\begin{aligned} 2x + y &= 5 \\ -4x + 6y &= -2 \end{aligned}$$

b.
$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 1 \end{aligned}$$

c.
$$\begin{aligned} x + y^2 &= 5 \\ 2x + y &= 4 \end{aligned}$$

d.
$$\begin{aligned} 4x + 3y &= 24 \\ 12x + 9y &= 36 \end{aligned}$$

e.
$$\begin{aligned} x + y + z &= 6 \\ 3x - 2y + 4z &= 9 \\ x - y - z &= 0 \end{aligned}$$

f.
$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_1 - x_2 &= 2 \end{aligned}$$

g.
$$\begin{aligned} x_1 + x_2 &= 5 \\ 2x_1 - 3x_2 &= -5 \\ 4x_1 - x_2 &= 5 \end{aligned}$$

2. Solutions for systems of equations

2.1. **Definition of a solution to a statement of equation.** A **solution** of a system of equations consists of values for the variables that reduce each equation of the system to a true statement. To **solve** a system of equations means to find all solutions of the system.

2.2. **Examples of solutions.**

Date: January 31, 2005.

2.2.1. *Example a.* Consider system a in section 1 above.

$$\begin{aligned} 2x \ + \ y \ &= 5 \\ -4x \ + \ 6y \ &= -2 \end{aligned}$$

If we substitute (2,1) for (x, y) in the system we obtain:

$$\begin{aligned} 2(2) \ + \ (1) \ &= 5 \\ -4(2) \ + \ 6(1) \ &= -2 \end{aligned}$$

2.2.2. *Example b.* Consider system b in section 1 above.

$$\begin{aligned} 2x_1 \ + \ 3x_2 \ &= 7 \\ x_1 \ - \ x_2 \ &= -1 \end{aligned}$$

If we substitute (4/5, 9/5) for (x_1, x_2) in the system we obtain:

$$\begin{aligned} 2\left(\frac{4}{5}\right) \ + \ 3\left(\frac{9}{5}\right) \ &= \frac{8}{5} + \frac{27}{5} = \frac{35}{5} = 7 \\ \frac{4}{5} - \frac{9}{5} \ &= -\frac{5}{5} = -1 \end{aligned}$$

2.2.3. *Example c.* Consider system c in section 1 above.

$$\begin{aligned} x \ + \ y^2 \ &= 5 \\ 2x \ + \ y \ &= 4 \end{aligned}$$

If we substitute (1, 2) for (x, y) in the system we obtain

$$\begin{aligned} 1 \ + \ 2^2 \ &= 5 \\ 2(1) \ + \ 2 \ &= 4 \end{aligned}$$

Another solution of the system is given by (x, y) = (11/4, -3/2) as can be seen as follows:

$$\begin{aligned} \frac{11}{4} \ + \ \left(\frac{-3}{2}\right)^2 \ &= \frac{11}{4} + \frac{9}{4} = \frac{20}{4} = 5 \\ 2\left(\frac{11}{4}\right) \ + \ \left(\frac{-3}{2}\right) \ &= \frac{22}{4} + \frac{-6}{4} = \frac{16}{4} = 4 \end{aligned}$$

2.3. **Consistent systems.** When a system of equations has at least one solution, it is said to be consistent.

2.4. **Unique solutions.** When a system of equations has one and only one solution, we say that the solution is unique.

2.5. **Possible sets of solutions for a system of equations.** For a given system of equations.

1. We may have one and only one solution. In this case we say we have a **unique solution.**
2. **We may no solutions. In this case we say that the system of equations is inconsistent.**

3. **We may have more than one set of solutions but the set of solutions is not infinite. We say that the number of solutions is finite. This can occur when one of more variables appears in an equation with a power other than one and/or there is a product of variables in one of the equations. Such an equation is called a non-linear equation.**
4. **We have an infinite number of solutions to the system of equations. In this case we say the system is indeterminate.**

3. Methods of finding solutions for equations

3.1. Method of substitution.

1: Pick one of the equations and solve for one of the variables in terms of the remaining variables.
2: Substitute the result in the remaining equations.
3: If one equation in one variable results, solve this equation. Otherwise repeat step a until a single equation with a single variable remains.
4: Find the values of the remaining variables by back-substitution.
5: Check the solution found.

3.2. Examples of the method of substitution.

3.2.1. *Example 1.* Consider system b in section 1 above.

$$2x_1 + 3x_2 = 7 \tag{b.1}$$
$$x_1 - x_2 = -1 \tag{b.2}$$

Rewrite the second equation as follows

$$x_1 = x_2 - 1$$

Now substitute in the first equation, obtain one equation in one unknown and solve to obtain

$$\begin{aligned} 2(x_2 - 1) + 3x_2 &= 7 \\ \Rightarrow 2x_2 - 2 + 3x_2 &= 7 \\ \Rightarrow 5x_2 &= 9 \\ \Rightarrow x_2 &= \frac{9}{5} \end{aligned}$$

Now substitute x$_2$ = 9/5 in the second equation to obtain

$$\begin{aligned} x_1 - \frac{9}{5} &= -1 \\ \Rightarrow x_1 &= -1 + \frac{9}{5} = \frac{4}{5} \end{aligned}$$

We can verify the answer as

$$2\left(\frac{4}{5}\right) + 3\left(\frac{9}{5}\right) = \frac{8}{5} + \frac{27}{5} = \frac{35}{5} = 7$$
$$\frac{4}{5} - \frac{9}{5} = -\frac{5}{5} = -1$$

3.2.2. *Example 2.* Consider system f in section 1 above.

$$\begin{aligned} x \; + \; y \; + \; z \; &= 6 && \text{(f.1)} \\ 3x \; - \; 2y \; + \; 4z \; &= 9 && \text{(f.2)} \\ x \; - \; y \; - \; z \; &= 0 && \text{(f.3)} \end{aligned}$$

Rewrite the third equation as follows

$$x \; = \; y \; + \; z$$

Now substitute in the first and second equations to obtain

$$\begin{aligned} (y \; + \; z) \; + \; y \; + \; z \; &= 6 \\ \Rightarrow \; 2y \; + \; 2z \; &= 6 \end{aligned}$$

$$\begin{aligned} 3(y \; + \; z) \; - \; 2y \; + \; 4z \; &= 9 \\ \Rightarrow \; 3y \; + \; 3z \; - \; 2y \; + \; 4z \; &= 9 \\ \Rightarrow \; y \; + \; 7z \; &= 9 \end{aligned}$$

Now solve the new second equation for y as follows

$$\begin{aligned} y \; + \; 7z \; &= 9 \\ \Rightarrow \; y \; &= 9 \; - \; 7z \end{aligned}$$

Now substitute in the first equation to obtain

$$\begin{aligned} 2y \; + \; 2z \; &= 6 \\ \Rightarrow \; 2(9 \; - \; 7z) \; + \; 2z \; &= 6 \\ \Rightarrow \; 18 \; - \; 14z \; + \; 2z \; &= 6 \\ \Rightarrow \; 12z \; &= 12 \\ \Rightarrow \; z \; &= 1 \end{aligned}$$

Now substitute z = 1 in the revised second equation to obtain

$$\begin{aligned} y \; &= 9 \; - \; 7z \\ \Rightarrow \; y \; &= 9 \; - \; 7(1) \; = \; 2 \end{aligned}$$

Then substitute y and z in the revised third equation to obtain

$$\begin{aligned} x \; &= y \; + \; z \\ \Rightarrow \; x \; &= 2 \; + \; 1 \; = \; 3 \end{aligned}$$

So the answer is (x, y, z) = (3, 2, 1). We can verify the answer as

$$\begin{aligned} 3 \; + \; 2 \; + \; 1 \; &= 6 \\ 3(3) \; - \; 2(2) \; + \; 4(1) \; &= 9 \\ 3 \; - \; 2 \; - \; 1 \; &= 0 \end{aligned}$$

3.2.3. *Example 3.* Consider system d in section 1 above.

$$4x + 3y = 24 \tag{d.1}$$
$$12x + 9y = 36 \tag{d.2}$$

Rewrite the second equation as follows

$$\begin{aligned} 12x &= 36 - 9y \\ \Rightarrow x &= 3 - \frac{3}{4}y \end{aligned}$$

Now substitute in the first equation, obtain one equation in one unknown and solve to obtain

$$\begin{aligned} 4x + 3y &= 24 \\ \Rightarrow 4(3 - \frac{3}{4}y) + 3y &= 24 \\ \Rightarrow 12 - 3y = 3y &= 24 \\ \Rightarrow 12 &= 24 \end{aligned}$$

Since this leads to a statement which is not true, the system is inconsistent.

3.2.4. *Example 4.* Consider system c in section 1above.

$$x + y^2 = 5 \tag{c.1}$$
$$2x + y = 4 \tag{c.2}$$

Solve the second equation for y as follows

$$\begin{aligned} 2x + y &= 4 \\ \Rightarrow y &= 4 - 2x \end{aligned}$$

Now substitute in the first equation and simplify to obtain

$$\begin{aligned} x + y^2 &= 5 \\ \Rightarrow x + (4 - 2x)^2 &= 5 \\ \Rightarrow x + 16 - 16x + 4x^2 &= 5 \\ \Rightarrow 4x^2 - 15x + 11 &= 0 \\ \Rightarrow ax^2 + bx + c &= 0 \end{aligned}$$

Now solve the equation using the quadratic formula as follows

$$
\begin{aligned}
x \ &= \frac{-b \ \pm \ \sqrt{b^2 \ - \ 4ac}}{2a} \ (a \ \neq \ 0) \\
\Rightarrow \ x \ &= \frac{-(-15) \ \pm \ \sqrt{(-15)^2 \ - \ 4(4)(11)}}{2(4)} \\
&= \frac{15 \ \pm \ \sqrt{225 \ - \ 176}}{8} \\
&= \frac{15 \ \pm \ \sqrt{49}}{8} \\
&= \frac{15 \ \pm \ 7}{8} \\
\Rightarrow \ x \ &= \frac{22}{8} \ = \frac{11}{4}, \ or \ x \ = \ 1
\end{aligned}
$$

Substituting x = 1 in the modified second equation will give

$$
\begin{aligned}
y \ &= 4 \ - \ 2x \\
\Rightarrow \ y \ &= 4 \ - \ 2(1) \ = \ 2
\end{aligned}
$$

Substituting x = 11/4 in the modified second equation will give

$$
\begin{aligned}
y \ &= 4 \ - \ 2x \\
\Rightarrow \ y \ &= 4 \ - \ 2\left(\frac{11}{4}\right) \ = \ 4 \ - \frac{22}{4} \\
&= \frac{16}{4} \ - \frac{22}{4} \ = \ -\frac{6}{4} \ = \ -\frac{3}{2}
\end{aligned}
$$

3.2.5. *Example 5.* Consider system g in section 1 above.

$$
\begin{aligned}
x_1 \ + \ x_2 \ &= 5 \qquad \text{(g.1)} \\
2x_1 \ - \ 3x_2 \ &= \ -5 \qquad \text{(g.2)} \\
4x_1 \ - \ x_2 \ &= 5 \qquad \text{(g.3)}
\end{aligned}
$$

Solve the first equation for x_1 as a function of x_2 as follows

$$
\begin{aligned}
x_1 \ + \ x_2 \ &= 5 \\
\Rightarrow \ x_1 \ &= 5 - x_2
\end{aligned}
$$

Now substitute in the second and third equations to obtain

$$\begin{aligned} 2x_1 \ - \ 3x_2 \ &= \ -5 \\ \Rightarrow \ 2(5 - x_2) \ - \ 3x_2 \ &= 5 \\ \Rightarrow \ 10 \ - 2x_2 \ - \ 3x_2 \ &= 5 \\ \Rightarrow \ 15 \ - \ 5x_2 \ &= 0 \end{aligned}$$

$$\begin{aligned} 4x_1 \ - \ x_2 \ &= 5 \\ \Rightarrow \ 4(5 - x_2) \ - \ x_2 \ &= 5 \\ \Rightarrow \ 20 - 4x_2 \ - \ x_2 \ &= 5 \\ \Rightarrow \ 15 - 5x_2 \ &= 0 \end{aligned}$$

We can now solve either equation directly as follows.

$$\begin{aligned} 15 \ - \ 5x_2 \ &= 0 \\ \Rightarrow \ 5x_2 \ &= 15 \\ \Rightarrow \ x_2 \ &= 3 \\ \Rightarrow \ x_1 \ &= 2 \end{aligned}$$

We can verify that these values of x_1 and x_2 satisfy all three equations. Any one of the equations in the system is redundant, meaning we can eliminate it and still solve the system.

If we changed the third equation to $4x_1 \ - \ x_2 \ = \ -5$ we would obtain the following equation after substituting $5 \ - \ x_2$ for x_1.

$$\begin{aligned} 4x_1 \ - \ x_2 \ &= \ -5 \\ \Rightarrow \ 4(5 - x_2) \ - \ x_2 \ &= \ -5 \\ \Rightarrow \ 20 \ - \ 4x_2 \ - \ x_2 \ &= \ -5 \\ \Rightarrow \ 25 \ - \ 5x_2 \ &= 0 \end{aligned}$$

Solving the equation we obtain

$$\begin{aligned} 25 - \ 5x_2 &= 0 \\ \Rightarrow \ 5x_2 \ &= 25 \\ \Rightarrow \ x_2 \ &= 5 \\ \Rightarrow \ x_1 \ &= 0 \end{aligned}$$

We now have inconsistent solutions to the system.

3.3. **Method of elimination.**

3.3.1. *Basic idea.* The idea behind the method of elimination is to keep replacing the original equations in the system with equivalent equations until an obvious solution is reached. In effect we keep replacing equations until each equation contains just one variable.

3.3.2. *Principles for obtaining equivalent systems of equations.*

1: Changing the order in which the equations are listed produces an equivalent system.
2: Suppose that in a system of equations, we multiply both sides of a single equation by a nonzero number (leaving the other equations) unchanged. Then the resulting system of equations is equivalent to the original system.
3: Suppose that in a system of equations, we add a multiple of one equation to another equation (leaving the other equations) unchanged. Then the resulting system of equations is equivalent to the original system.

3.3.3. *Proof of principles.*

1: This principle is obvious.
2: Proof of principle 2

Proof. The key is that multiplying by a nonzero constant k is reversible, because we can also multiply by 1/k. More specifically, we can argue as follows. If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, then of course $ka_2x_2 + ka_2x_2 + \cdots + ka_nx_n = kb$. So any solution $(x_1,...,x_n)$ to the original system is also a solution to the new system. Moreover, whenever we have a solution to $ka_1x_1 + ka_2x_2 + \cdots + ka_nx_n = kb$, then multiplying the equation by 1/k shows that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Thus any solution to the new system is also a solution to the original system. □

3: Proof of principle 3

Proof. Suppose we add k times the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ to the equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = d$. If these equations are true, then the sum

$$(ka_1 + c_1)x_1 + (ka_2 + c_2)x_2 + \cdots + (ka_n + c_n)x_n = kb + d$$

is true; that is, any solution to the original system is also a solution to the new system. Could extraneous solutions have entered? Observe that the original system can be recovered from the new system by adding -k times the equation $a_1\ x_1 + a_2\ x_2 + \cdots + a_nx_n = b$ to

$$(ka_1 + c_1)x_1 + (ka_2 + c_2)x_2 + \cdots + (ka_n + c_n)x_n = kb + d.$$

So the argument of the preceding paragraph can be reapplied (with -k in place of k) to assure us that any solution to the new system was already a solution to the original system. □

3.3.4. *Steps in the case of a linear system.*

1: Take the first variable x_s (that is, the first in the order x_1, x_2, ... , x_n) having a non-zero coefficient in some equation. Then choose one of the equations in which the coefficient c of x_s is nonzero. We say this equation is the one being used. Place it just above the equations not yet used.
2: Multiply the equation being used by 1/c, so that its new coefficient for x_s is 1.
3: To each of the other equations, (both used and unused) in which x_s has a non-zero coefficient h, add -h times the equation being used. This eliminates x_s from every equation except the one being used. We now say that the variable x_s has been used.
4: Now take the next variable, x_t, having a nonzero coefficient in some unused equation. Following steps 1, 2, and 3, eliminate it from all other equations, including those already used. Again the variable x_t and the equation we have worked with in this step is called used.

Step 4 is then repeated until there is no variable with a nonzero coefficient in an unused equation.

If there are solutions to a system of linear equations, they will be obtained from this procedure. If no solutions exist, a contradiction will be obtained.

3.3.5. *Examples.*

Example 1: Consider system a in section 1.

$$2x + y = 5 \tag{a.1}$$

$$-4x + 6y = -2 \tag{a.2}$$

Let x be the variable chosen and let the first equation be the equation chosen. It is already at the top of the system. Multiply the first equation by $\frac{1}{2}$ and then rewrite the entire system as follows

$$\begin{aligned} x + \frac{y}{2} &= \frac{5}{2} \\ -4x + 6y &= -2 \end{aligned}$$

Multiply the first equation 4 and add it to the second equation. This gives

$$\begin{aligned} 4x + \frac{4y}{2} &= \frac{20}{2} \\ \Rightarrow 4x + 2y &= 10 \\ -4x + 6y &= -2 \\ ---- \quad & \quad ----- \\ 8y &= 8 \end{aligned}$$

Now rewrite the system as follows

$$\begin{aligned} x \quad + \frac{y}{2} &= \frac{5}{2} \\ 8y &= 8 \end{aligned}$$

Now choose the second equation and the second variable. Multiply it by 1/8 and then rewrite the system as

$$\begin{aligned} x \quad + \frac{y}{2} &= \frac{5}{2} \\ y &= 1 \end{aligned}$$

Multiply the second equation by -1/2 and add it to the first equation

$$\begin{aligned} x \quad + \frac{y}{2} &= \frac{5}{2} \\ -\frac{1}{2}y &= -\frac{1}{2} \\ ---- \quad & \quad ----- \\ x \qquad &= \frac{4}{2} = 2 \end{aligned}$$

The reduced system is now given by

$$\begin{aligned} x &= 2 \\ y &= 1 \end{aligned}$$

Example 2: Consider system e in section 1

$$x + y + z = 6 \tag{e.1}$$
$$3x - 2y + 4z = 9 \tag{e.2}$$
$$x - y - z = 0 \tag{e.3}$$

Choose the first variable and the first equation. Since the coefficient of x is 1, we can proceed directly. We will add (-3) times it to the second equation and (-1) times it to the third equation. We will first use (-3) as the multiplier and add the first equation to the second.

$$\begin{array}{cccc} -3x & -3y & -3z & = -18 \\ 3x & -2y & +4z & = 9 \\ --- & --- & --- & --- \\ & -5y & +z & = -9 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{cccc} x & +y & +z & = 6 \\ & -5y & +z & = -9 \\ x & -y & -z & = 0 \end{array}$$

Now write the system using (-1) as the multiplier for the first equation and add to the third equation as follows

$$\begin{array}{cccc} -x & -y & -z & = -6 \\ x & -y & -z & = 0 \\ --- & --- & --- & --- \\ & -2y & -2z & = -6 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{cccc} x+ & y & +z & = 6 \\ - & 5y & +z & = -9 \\ - & 2y & -2z & = -6 \end{array}$$

The variable x is now eliminated from all but the first equation. Now choose the second equation and y as the variable. Since its coefficient is -5, we multiply by 1/5 and add it to the first equation as follows.

$$\begin{array}{cccc} x+ & y & +z & = 6 \\ - & y & +\frac{1}{5}z & = -\frac{9}{5} \\ & --- & --- & --- \\ x & & +\frac{6}{5}z & = \frac{21}{5} \end{array}$$

The system is now given by

$$\begin{array}{rrrrl} x & & + \frac{6}{5}z & = & +\frac{21}{5} \\ - & y & + \frac{1}{5}z & = & -\frac{9}{5} \\ - & 2y & -2z & = & -6 \end{array}$$

Now multiply the second equation by -2 and add to the third equation to eliminate y from it. This will give

$$\begin{array}{rrl} 2y & -\frac{2}{5}z & = \frac{18}{5} \\ -2y & -2z & = -6 \\ ---- & ---- & --- \\ & -\frac{12}{5}z & = -\frac{12}{5} \end{array}$$

The system is now given by

$$\begin{array}{rrrrl} x & & + \frac{6}{5}z & = & \frac{21}{5} \\ - & y & + \frac{1}{5}z & = & -\frac{9}{5} \\ & & -\frac{12}{5}z & = & -\frac{12}{5} \end{array}$$

The variable y now only appears in the second equation. Now use the third equation and z. Since the coefficient of z is -12/5 we multiply by -5/12. This gives for the entire system

$$\begin{array}{rrcr} x & + \frac{6}{5}z & = & +\frac{21}{5} \\ -y & + \frac{1}{5}z & = & -\frac{9}{5} \\ & z & = & 1 \end{array}$$

First multiply the last equation by -6/5 and add to the first one. This will give

$$\begin{array}{rrrl} & x & + \frac{6}{5}z & = \frac{21}{5} \\ & & -\frac{6}{5}z & = -\frac{6}{5} \\ & & ---- & ---- \\ & x & & = \frac{15}{5} \\ \Rightarrow & x & & = 3 \end{array}$$

Now multiply the third equation by -1/5 and add to the second equation.

$$
\begin{array}{rrrcl}
 & -\ y & +\frac{1}{5}z & = & -\frac{9}{5} \\
 & & -\frac{1}{5}z & = & -\frac{1}{5} \\
 & & ---- & & ---- \\
 & -\ y & & = & -\frac{10}{5} \\
\Rightarrow & y & & = & 2
\end{array}
$$

Putting it all together we obtain

$$
\begin{aligned}
x &= 3 \\
y &= 2 \\
z &= 1
\end{aligned}
$$

Example 3: Consider system d in section 1

$$
4x + 3y = 24 \tag{d.1}
$$

$$
12x + 9y = 36 \tag{d.2}
$$

Choose the first variable and the first equation. Since the coefficient of x is 4, divide the first equation by 4 and rewrite the system as

$$
\begin{aligned}
x + \frac{3}{4}y &= 6 \\
12x + 9y &= 36
\end{aligned}
$$

Now multiply the first equation by -12 and add to the second equation.

$$
\begin{array}{rcl}
-12x - 9y & = & -72 \\
12x + 9y & = & 36 \\
----- & & ----- \\
0x + 0y & = & -36
\end{array}
$$

It is clear that we have an inconsistent system.

Example 4: Consider the following system

$$
4x + 2y = 12 \tag{q.1}
$$

$$
6x + 3y = 18 \tag{q.2}
$$

Choose the first variable and the first equation. Since the coefficient of x is 4, divide the first equation by 4 and rewrite the system as

$$
\begin{aligned}
x + \frac{1}{2}y &= 3 \\
6x + 3y &= 18
\end{aligned}
$$

Now multiply the first equation by -6 and add to the second equation.

$$\begin{array}{rcl} -6x \;-\; 3y & = & -\,18 \\ 6x \;+\; 3y & = & 18 \\ -\!-\!-\!-\!- & & -\!-\!-\!-\!- \\ 0x \;+\; 0y & = & 0 \end{array}$$

We now have an equation stating that $0x + 0y = 0$. This is a true statement, but provides no information about the values of x and y that solve the system. Solve the second equation for x to obtain

$$\begin{aligned} 6x \;+\; 3y &= 18 \\ \Rightarrow 6x &= 18 \;-\; 3y \\ \Rightarrow x &= 3 \;-\; \frac{1}{2}y \end{aligned} \tag{q$'$}$$

Any x satisfying equation (q′) will satisfy both equations in q. There are infinite solutions to system q, one for every possible value of y among the real numbers.

4. The general representation of a linear system of m equations in n unknowns

The general system of m equations in n unknowns can be written

$$\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & b_2 \\ a_{31}\,x_1 & + & a_{32}\,x_2 & + & \cdots & + & a_{3n}\,x_n & = & b_3 \\ & & & & & & \vdots & \vdots & \vdots \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & + & \cdots & + & a_{mn}\,x_n & = & b_m \end{array}$$

In this system, the a_{ij}'s and b_i's are given real numbers; a_{ij} is the coefficient of the unknown x_j in the ith equation. A **solution** of the system is an n-tuple of real numbers x_1, x_2, ... x_n which satisfies all m equations. We call the column of right-hand sides in the system a vector with m members. We call the set of all a_{ij}'s arranged in a rectangular array the coefficient matrix of the system.