TRANSFORMATIONS OF RANDOM VARIABLES

1. Introduction

1.1. **Definition.** We are often interested in the probability distributions or densities of functions of one or more random variables. Suppose we have a set of random variables, X_1, X_2, X_3, . . . X_n, with a known joint probability and/or density function. We may want to know the distribution of some function of these random variables Y = $\phi(X_1, X_2, X_3, \ldots X_n)$. Realized values of y will be related to realized values of the X's as follows

$$y = \Phi(x_1, x_2, x_3, \ldots, x_n) \tag{1}$$

A simple example might be a single random variable x with transformation

$$y = \Phi(x) = log(x) \tag{2}$$

1.2. **Techniques for finding the distribution of a transformation of random variables.**

1.2.1. *Distribution function technique.* We find the region in x_1, x_2, x_3, . . . x_n space such that $\Phi(x_1, x_2, \ldots x_n) \leq \phi$. We can then find the probability that $\Phi(x_1, x_2, \ldots x_n) \leq \phi$, i.e., P[$\Phi(x_1, x_2, \ldots x_n) \leq \phi$] by integrating the density function $f(x_1, x_2, \ldots x_n)$ over this region. Of course, $F_\Phi(\phi)$ is just P[$\Phi \leq \phi$]. Once we have $F_\Phi(\phi)$, we can find the density by integration.

1.2.2. *Method of transformations (inverse mappings).* Suppose we know the density function of x. Also suppose that the function $y = \Phi(x)$ is differentiable and monotonic for values within its range for which the density f(x) $\neq$0. This means that we can solve the equation $y = \Phi(x)$ for x as a function of y. We can then use this inverse mapping to find the density function of y. We can do a similar thing when there is more than one variable X and then there is more than one mapping Φ.

1.2.3. *Method of moment generating functions.* There is a theorem (Casella [2, p. 65]) stating that if two random variables have identical moment generating functions, then they possess the same probability distribution. The procedure is to find the moment generating function for Φ and then compare it to any and all known ones to see if there is a match. This is most commonly done to see if a distribution approaches the normal distribution as the sample size goes to infinity. The theorem is presented here for completeness.

Theorem 1. *Let $F_X(x)$ and $F_Y(y)$ be two cumulative distribution functions all of whose moments exist. Then*

- **a:** *If X and Y have bounded support, then $F_X(u) = F_Y(u)$ for all u if and only if E X^r = E Y^r for all integers r = 0,1,2, ...*
- **b:** *If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.*

For further discussion, see Billingsley [1, ch. 21-22] .

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2. Distribution Function Technique

2.1. **Procedure for using the Distribution Function Technique.** As stated earlier, we find the region in the $x_1, x_2, x_3, \ldots x_n$ space such that $\Phi(x_1, x_2, \ldots x_n) \leq \phi$. We can then find the probability that $\Phi(x_1, x_2, \ldots x_n) \leq \phi$, i.e., $P[\ \Phi(x_1, x_2, \ldots x_n) \leq \phi\ ]$ by integrating the density function $f(x_1, x_2, \ldots x_n)$ over this region. Of course, $F_\Phi(\phi)$ is just $P[\ \Phi \leq \phi\ ]$. Once we have $F_\Phi(\phi)$, we can find the density by differentiation.

2.2. **Example 1.** Let the probability density function of X be given by

$$f(x) = \begin{cases} 6\,x\,(1 - x) & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{3}$$

Now find the probability density of $Y = X^3$.

Let G(y) denote the value of the distribution function of Y at y and write

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(X^3 \leq y) \\ &= P\left(X \leq y^{1/3}\right) \\ &= \int_0^{y^{1/3}} 6\,x\,(1 - x)\,dx \\ &= \int_0^{y^{1/3}} \left(6\,x - 6\,x^2\right) dx \\ &= \left(3\,x^2 - 2\,x^3\right) \Big|_0^{y^{1/3}} \\ &= 3\,y^{2/3} - 2y \end{aligned} \tag{4}$$

Now differentiate G(y) to obtain the density function g(y)

$$\begin{aligned} g(y) &= \frac{d\,G(y)}{d\,y} \\ &= \frac{d}{d\,y}\left(3\,y^{2/3} - 2\,y\right) \\ &= 2\,y^{-1/3} - 2 \\ &= 2\,(y^{-1/3} - 1), \quad 0 < y < 1 \end{aligned} \tag{5}$$

2.3. **Example 2.** Let the probability density function of x_1 and of x_2 be given by

$$f(x_1, x_2) = \begin{cases} 2\,e^{-x_1 - 2\,x_2} & x_1 > 0\,,\ x_2 > 0 \\ 0 & otherwise \end{cases} \tag{6}$$

Now find the probability density of $Y = X_1 + X_2$ or $X_1 = Y - X_2$. Given that Y is a linear function of X_1 and X_2, we can easily find F(y) as follows.

Let $F_Y(y)$ denote the value of the distribution function of Y at y and write

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= \int_0^y \int_0^{y - x_2} 2 e^{-x_1 - 2x_2} \, dx_1 \, dx_2 \\ &= \int_0^y -2 e^{-x_1 - 2x_2} |_0^{y - x_2} \, dx_2 \\ &= \int_0^y \left[\left(-2 e^{-y + x_2 - 2x_2} \right) - \left(-2 e^{-2x2} \right) \right] dx_2 \\ &= \int_0^y -2 e^{-y - x_2} + 2 e^{-2x2} \, dx_2 \\ &= \int_0^y 2 e^{-2x2} - 2 e^{-y - x_2} \, dx_2 \end{aligned} \tag{7}$$

Now integrate with respect to x_2 as follows

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= \int_0^y 2 e^{-2x2} - 2 e^{-y - x_2} \, dx_2 \\ &= - e^{-2x2} + 2 e^{-y - x_2} |_0^y \\ &= - e^{-2y} + 2 e^{-y - y} - \left[- e^0 + 2 e^{-y} \right] \\ &= e^{-2y} - 2 e^{-y} + 1 \end{aligned} \tag{8}$$

Now differentiate $F_Y(y)$ to obtain the density function f(y)

$$\begin{aligned} f_Y(y) &= \frac{dF(y)}{dy} \\ &= \frac{d}{dy} \left(e^{-2y} - 2 e^{-y} + 1 \right) \\ &= -2 e^{-2y} + 2 e^{-y} \\ &= 2 e^{-2y} \left(-1 + e^y \right) \end{aligned} \tag{9}$$

2.4. **Example 3.** Let the probability density function of X be given by

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma \sqrt{2\pi}} \cdot e^{\frac{-1}{2} \left(\frac{x - \mu}{\sigma} \right)^2}, \quad -\infty < x < \infty \\ &= \frac{1}{\sqrt{2 \pi \sigma^2}} \cdot \exp \left[- \frac{(x - \mu)^2}{2 \sigma^2} \right], \quad -\infty < x < \infty \end{aligned} \tag{10}$$

Now let $Y = \Phi(X) = e^X$. We can then find the distribution of Y by integrating the density function of X over the appropriate area that is defined as a function of y. Let $F_Y(y)$ denote the value of the distribution function of Y at y and write

$$\begin{aligned} F_Y(y) &= P\,(Y \leq y) \\ &= P\,\left(e^X \leq y\right) = P\,(X \leq \ln y), \quad y > 0 \\ &= \int_{-\infty}^{\ln\, y} \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\,\sigma^2}\right] d\,x, \quad y > 0 \end{aligned} \tag{11}$$

Now differentiate F_Y(y) to obtain the density function f(y). In this case we will need the rules for differentiating under the integral sign. They are given by theorem 2 which we state below without proof.

Theorem 2. *Suppose that f and $\frac{\partial f}{\partial x}$ are continuous in the rectangle*

$$R = \{\,(x,\,t)\,:\, a \leq x \leq b\,,\, c \leq t \leq d\}$$

and suppose that $u_0(x)$ and $u_1(x)$ are continuously differentiable for $a \leq x \leq b$ with the range of $u_0(x)$ and $u_1(x)$ in (c, d). If ψ is given by

$$\psi\,(x) = \int_{u_0\,(x)}^{u_1\,(x)} f(x,\,t)\,d\,t \tag{12}$$

then

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{\partial}{\partial x}\int_{u_0(x)}^{u_1(x)} f(x,\,t)\,d\,t \\ &= f\,(x, u_1(x))\,\frac{du_1(x)}{dx} - f\,(x, u_0(x))\,\frac{du_0(x)}{dx} + \int_{u_0(x)}^{u_1(x)} \frac{\partial f(x,t)}{\partial x} dt \end{aligned} \tag{13}$$

If one of the bounds of integration does not depend on x, then the term involving its derivative will be zero.

For a proof of theorem 2 see (Protter [3, p. 425]). Applying this to equation 11 where y takes the role of x, $\ln y$ takes the role of $u_1(x)$, and x takes the role of t in the theorem we obtain

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx, \quad y > 0 \\ F_Y'(y) &= f_Y(y) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right]\left(\frac{1}{y}\right)\right) \\ &\quad + \int_{-\infty}^{\ln y} \frac{d}{dy}\left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) dx \\ &= \left(\frac{1}{y\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right]\right) \end{aligned} \tag{14}$$

The last line of equation 14 follows because

$$\frac{d}{dy}\left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) = 0$$

3. Method of transformations (single variable)

3.1. **Discrete examples of the method of transformations.**

3.1.1. *One-to-one function.* Find a formula for the probability distribution of the total number of heads obtained in four tosses of a coin where the probability of a head is 0.60.

The sample space, probabilities and the value of the random variable are given in table 1.

TABLE 1. **Outcomes, Probabilities and Number of Heads from Tossing a Coin Four Times.**

Element of sample space	Probability	Value of random variable X (x)
HHHH	81/625	4
HHHT	54/625	3
HHTH	54/625	3
HTHH	54/625	3
THHH	54/625	3
HHTT	36/625	2
HTHT	36/625	2
HTTH	36/625	2
THHT	36/625	2
THTH	36/625	2
TTHH	36/625	2
HTTT	24/625	1
THTT	24/625	1
TTHT	24/625	1
TTTH	24/625	1
TTTT	16/625	0

From the table we can determine the probabilities as

$$P(X = 0) = \frac{16}{625},\ P(X = 1) = \frac{96}{625},\ P(X = 2) = \frac{216}{625},\ P\,(X = 3) = \frac{216}{625},\ P\,(X = 4) = \frac{81}{625}$$

We can also compute these probabilities using counting rules. The probability of one head and then three tails is

$$\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)$$

or

$$\left(\frac{3}{5}\right)^{1}\left(\frac{2}{5}\right)^{3} = \frac{24}{625}$$

The probability of 3 heads and then one tail is

$$\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)$$

or

$$\left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^1 = \frac{54}{625}.$$

Of course there are other ways to obtain 1 head and three tails besides one head and then three tails. In particular there are $\binom{4}{1} = 4$ ways to obtain one head. And there are $\binom{4}{0} = 1$ way to obtain zero heads. Similarly, there are six ways to obtain two heads, four ways to obtain three heads and one way to obtain four heads. We can then write the probability mass function as

$$f(x) = \binom{4}{x} \left(\frac{3}{5}\right)^x \left(\frac{2}{5}\right)^{4-x} \quad for\ x = 0, 1, 2, 3, 4 \tag{15}$$

This, of course, is the binomial distribution. The probabilities of the various possible random variables are contained in table 2.

TABLE 2. **Probability of Number of Heads from Tossing a Coin Four Times**

Number of Heads x	Probability f(x)
0	16/625
1	96/625
2	216/625
3	216/625
4	81/625

Now consider a transformation of X in the form Y = $2X^2$ + X. There are five possible outcomes for Y, i.e., 0, 3, 10, 21, 36. Given that the function is one-to-one, we can make up a table describing the probability distribution for Y.

TABLE 3. **Probability of a Function of the Number of Heads from Tossing a Coin Four Times.**

Y = 2 * $(\text{\# heads})^2$ + # of heads			
Number of Heads x	Probability f(x)	**y**	g(y)
0	16/625	0	16/625
1	96/625	3	96/625
2	216/625	10	216/625
3	216/625	21	216/625
4	81/625	36	81/625

3.1.2. *Case where the transformation is not one-to-one.* Now let the transformation of X be given by Z = $(6 - 2X)^2$. The possible values for Z are 0, 4, 16, 36. When X = 2 and when X = 4, Y = 4. We can find the probability of Z by adding the probabilities for cases when X gives more than one value as shown in table 4.

TABLE 4. **Probability of a Function of the Number of Heads from Tossing a Coin Four Times (not one-to-one).**

Y = (6 - (# heads))^2		
y	Number of Heads x	g(y)
0	3	$\frac{216}{625}$
4	2, 4	$\frac{216}{625} + \frac{81}{625} = \frac{297}{625}$
16	1	$\frac{96}{625}$
36	0	$\frac{16}{625}$

3.2. **Intuitive Idea of the Method of Transformations.** The idea of a transformation is to consider the function that maps the random variable X into the random variable Y. The idea is that if we can determine the values of X that lead to any particular value of Y, we can obtain the probability of Y by summing the probabilities of those values of X that mapped into Y. In the continuous case, to find the distribution function, we want to integrate the density of X over the portion of its space that is mapped into the portion of Y in which we are interested. Suppose for example that both X and Y are defined on the real line with $0 \leq X \leq 1$ and $0 \leq Y \leq 10$. If we want to know G(5), we need to integrate the density of X over all values of x leading to a value of y less than five, where G(y) is the probability that Y is less than five.

3.3. **General formula when the random variable is discrete.** Consider a transformation defined by y = Φ(x). The function Φ defines a mapping from the sample space of the variable X, to a sample space for the random variable Y.

If X is discrete with frequency function p_X, then Φ(X) is discrete and has frequency function

$$\begin{aligned} p_{\Phi(X)}(t) &= \sum_{x:\, \Phi(x)=t} p_X(x) \\ &= \sum_{x \epsilon \Phi^{-1}(t)} p_X(x) \end{aligned} \tag{16}$$

The process is simple in this case. One identifies g^{-1}(t) for each t in the sample space of the random variable Y, and then sums the probabilities which is what we did in section 3.1.

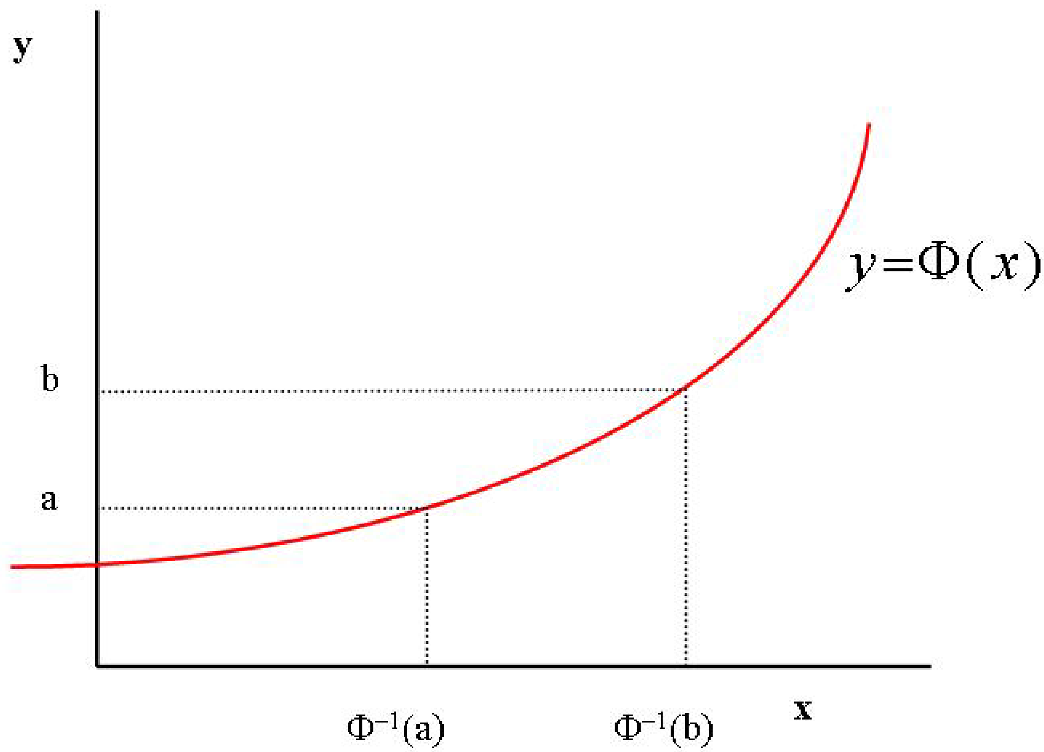
3.4. **General change of variable or transformation formula.**

Theorem 3. *Let $f_X(x)$ be the value of the probability density of the continuous random variable X at x. If the function y = Φ(x) is differentiable and either increasing or decreasing (monotonic) for all values within the range of X for which $f_X(x) \neq 0$, then for these values of x, the equation y = Φ(x) can be uniquely solved for x to give $x = \Phi^{-1}(y) = w(y)$ where $w(\cdot) = \Phi^{-1}(\cdot)$. Then for the corresponding values of y, the probability density of Y =Φ(X) is given by*

$$g(y) \;=\; f_Y(y) \;=\; \begin{cases} f_X\left[\Phi^{-1}(y)\right] \;\cdot\; \left|\dfrac{d\Phi^{-1}(y)}{d\,y}\right| & \\ f_X\,[\;w\;(y)\,] \;\cdot\; \left|\dfrac{d\;w\,(y)}{d\,y}\right| & \frac{d\Phi(x)}{dx} \neq 0 \\ f_X\;[\;w\;(y)\,] \;\cdot\; |\;w\,'(y)\;| & \\ 0 & otherwise \end{cases} \tag{17}$$

Proof. Consider the digram in figure 1.

FIGURE 1. y = $\Phi(x)$ is an increasing function.



As can be seen from in figure 1, each point on the y axis maps into a point on the x axis, that is, X must take on a value between Φ^{-1}(a) and Φ^{-1}(b) when Y takes on a value between a and b. Therefore

$$\begin{aligned} P(a\;<\;Y\;<\;b) &= P\left(\Phi^{-1}\,(a)\;<\;X\;<\;\Phi^{-1}\,(b)\right) \\ &= \int_{\Phi^{-1}\,(a)}^{\Phi^{-1}\,(b)} f_X\,(x)\,d\,x \end{aligned} \tag{18}$$

What we would like to do is replace x in the second line with y, and Φ^{-1}(a) and Φ^{-1}(b) with a and b. To do so we need to make a change of variable. Consider how we make a u substitution when we perform integration or use the chain rule for differentiation. For example if u = h(x) then du = h′(x) dx. So if x = Φ^{-1}(y), then

$$dx = \frac{d\,\Phi^{-1}(y)}{dy}\,dy.$$

Then we can write

$$\int f_X(x)\,dx = \int f_X\left(\Phi^{-1}(y)\right)\frac{d\,\Phi^{-1}(y)}{dy}\,dy \tag{19}$$

For the case of a definite integral the following lemma applies.

Lemma 1. *If the function u = h(x) has a continuous derivative on the closed interval [a, b] and f is continuous on the range of h, then*

$$\int_a^b f(h(x))\,h'(x)\,dx = \int_{h(a)}^{h(b)} f(u)\,du \tag{20}$$

Using this lemma or the intuition from equation 19 we can then rewrite equation 18 as follows

$$\begin{aligned} P(a < Y < b) &= P\left(\Phi^{-1}(a) < X < \Phi^{-1}(b)\right) \\ &= \int_{\Phi^{-1}(a)}^{\Phi^{-1}(b)} f_X(x)\,dx \\ &= \int_a^b f_X\left(\Phi^{-1}(y)\right)\frac{d\,\Phi^{-1}(y)}{dy}\,dy \end{aligned} \tag{21}$$

The probability density function, f_Y(y), of a continuous random variable Y is the function f(·) that satisfies

$$P(a < Y < b) = F(b) - F(a) = \int_a^b f_Y(t)\,dt \tag{22}$$

This then implies that the integrand in equation 21 is the density of Y, i.e., g(y), so we obtain

$$g(y) = f_Y(y) = f_X\left[\Phi^{-1}(y)\right]\cdot\frac{d\,\Phi^{-1}(y)}{dy} \tag{23}$$

as long as

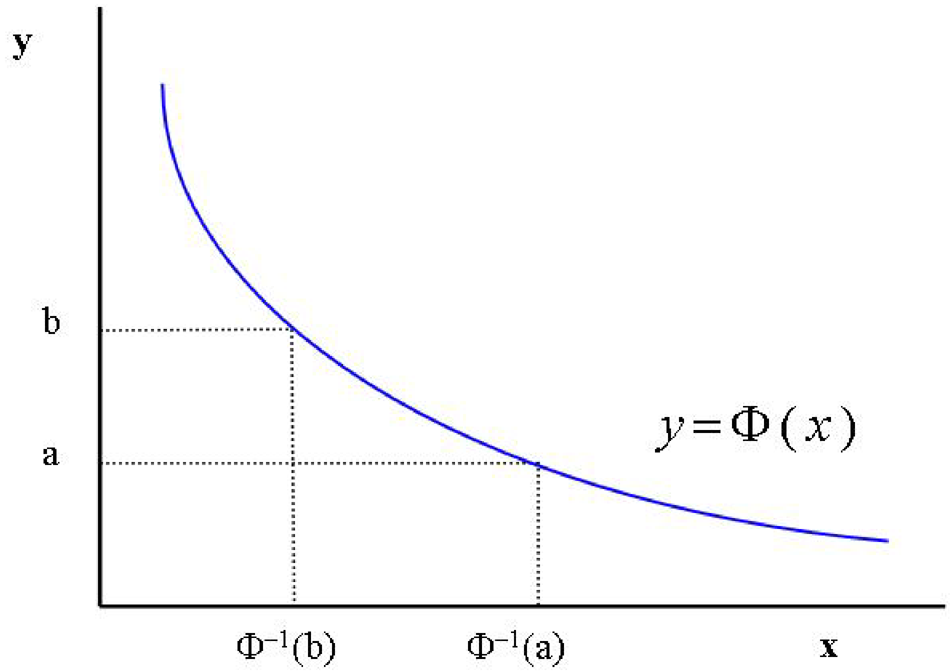
$$\frac{d\,\Phi^{-1}(y)}{dy}$$

exists. This proves the lemma if Φ is an increasing function. Now consider the case where Φ is a decreasing function as in figure 2.

As can be seen from figure 2, each point on the y axis maps into a point on the x axis, that is, X must take on a value between Φ^{-1}(a) and Φ^{-1}(b) when Y takes on a value between a and b. Therefore

$$\begin{aligned} P(a < Y < b) &= P\left(\Phi^{-1}(b) < X < \Phi^{-1}(a)\right) \\ &= \int_{\Phi^{-1}(b)}^{\Phi^{-1}(a)} f_X(x)\,dx \end{aligned} \tag{24}$$

FIGURE 2. y = $\Phi(x)$ is a decreasing function.



Making a change of variable for x = Φ^{-1}(y) as before we can write

$$\begin{aligned} P(a < Y < b) &= P\left(\Phi^{-1}(b) < X < \Phi^{-1}(a)\right) \\ &= \int_{\Phi^{-1}(b)}^{\Phi^{-1}(a)} f_X(x)\,dx \\ &= \int_b^a f_X\left(\Phi^{-1}(y)\right) \frac{d\,\Phi^{-1}(y)}{dy}\,dy \\ &= -\int_a^b f_X\left(\Phi^{-1}(y)\right) \frac{d\,\Phi^{-1}(y)}{dy}\,dy \end{aligned} \tag{25}$$

Because

$$\frac{d\,\Phi^{-1}(y)}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

when the function y = Φ(x) is increasing and

$$- \frac{d\,\Phi^{-1}(y)}{d\,y}$$

is positive when y = Φ(x) is decreasing, we can combine the two cases by writing

$$g(y) = f_Y(y) = f_X\left[\,\Phi^{-1}(y)\,\right] \cdot \left|\frac{d\,\Phi^{-1}(y)}{d\,y}\right| \tag{26}$$

□

3.5. **Examples.**

3.5.1. *Example 1.* Let X have the probability density function given by

$$f_X(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases} \tag{27}$$

Find the density function of Y = Φ(X) = 6X - 3.

Notice that $f_X(x)$ is positive for all x such that $0 \le x \le 1$. The function Φ is increasing for all X. We can then find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= 6x - 3 \\ \Rightarrow 6x &= y + 3 \\ \Rightarrow x &= \frac{y+3}{6} = \Phi^{-1}(y) \end{aligned} \tag{28}$$

We can then find the derivative of Φ^{-1} with respect to y as

$$\begin{aligned} \frac{d\Phi^{-1}}{dy} &= \frac{d}{dy}\left(\frac{y+3}{6}\right) \\ &= \frac{1}{6} \end{aligned} \tag{29}$$

The density of y is then

$$\begin{aligned} g(y) = f_Y(y) &= f_X\left[\Phi^{-1}(y)\right] \cdot \left|\frac{d\Phi^{-1}(y)}{d\,y}\right| \\ &= \left(\frac{1}{2}\right)\left(\frac{3+y}{6}\right)\left|\frac{1}{6}\right|, \quad 0 \le \frac{3+y}{6} \le 2 \end{aligned} \tag{30}$$

For all other values of y, g(y) = 0. Simplifying the density and the bounds we obtain

$$g(y) = f_Y y) = \begin{cases} \frac{3+y}{72}, & -3 \le y \le 9 \\ 0 & elsewhere \end{cases} \tag{31}$$

3.5.2. *Example 2.* Let X have the probability density function given by f_X(x). Then consider the transformation Y = Φ(X) = σX + μ, $\sigma \neq 0$. The function Φ is increasing for all X. We can then find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= \sigma\, x \,+\, \mu \\ \Rightarrow\; \sigma\, x &= y \,-\, \mu \\ \Rightarrow\; x &= \frac{y \,-\, \mu}{\sigma} \;=\; \Phi^{-1}\,(y) \end{aligned} \tag{32}$$

We can then find the derivative of Φ^{-1} with respect to y as

$$\begin{aligned} \frac{d\Phi^{-1}}{dy} &= \frac{d}{dy}\left(\frac{y \,-\, \mu}{\sigma}\right) \\ &= \frac{1}{\sigma} \end{aligned} \tag{33}$$

The density of y is then

$$\begin{aligned} f_Y(y) &= f_X\left[\,\Phi^{-1}(y)\,\right]\;\cdot\;\left|\frac{d\;\Phi^{-1}\,(y\,)}{dy}\right| \\ &= f_X\left[\,\frac{y-\mu}{\sigma}\,\right]\;\cdot\;\left|\frac{1}{\sigma}\right| \end{aligned} \tag{34}$$

3.5.3. *Example 3.* Let X have the probability density function given by

$$f_X(x) \;=\; \begin{cases} e^{-x}, & 0 \,\leq\, x \,\leq\, \infty \\ 0, & elsewhere \end{cases} \tag{35}$$

Find the density function of Y = $X^{1/2}$.

Notice that f_X(x) is positive for all x such that $0 \leq$ x $\leq \infty$. The function Φ is increasing for all X. We can then find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= x^{\frac{1}{2}} \\ \Rightarrow\; y^2 &= x \\ \Rightarrow\; x &= \Phi^{-1}(y) \;=\; y^2 \end{aligned} \tag{36}$$

We can then find the derivative of Φ^{-1} with respect to y as

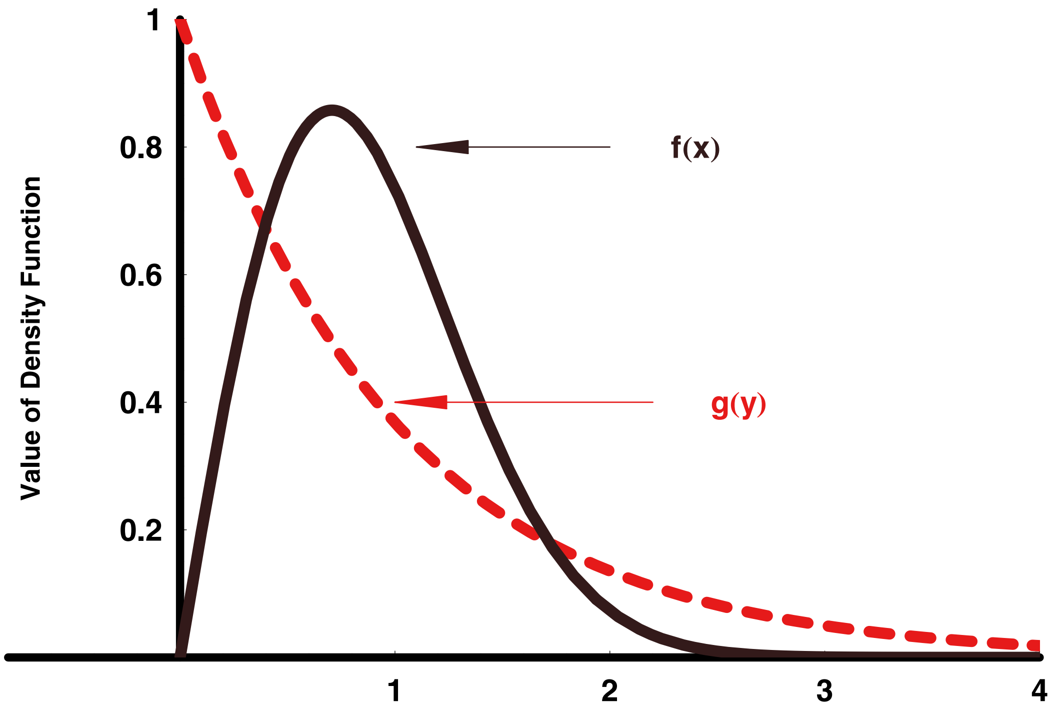
$$\begin{aligned} \frac{d\,\Phi^{-1}}{dy} &= \frac{d}{dy}\,y^2 \\ &= 2y \end{aligned} \tag{37}$$

The density of y is then

$$\begin{aligned} f_Y(y) &= f_X\left[\,\Phi^{-1}(y)\,\right]\;\cdot\;\left|\frac{d\Phi^{-1}(y)}{dy}\right| \\ &= e^{-y^2}\;\;|2y| \end{aligned} \tag{38}$$

A graph of the two density functions is shown in figure 3 .

FIGURE 3. The Two Density Functions.



4. Method of transformations (multiple variables)

4.1. **General definition of a transformation.** Let Φ be any function from R^k to R^m, k, m $\geq$ 1, such that $\Phi^{-1}(A) = \{x \epsilon R^k : \Phi(x) \epsilon A\} \epsilon ß^k$ for every A ϵ $ß^m$ where $ß^m$ is the smallest σ - field having all the open rectangles in R^m as members. If we write y = Φ(x), the function Φ defines a mapping from the sample space of the variable X (Ξ) to a sample space (Y) of the random variable Ψ. Specifically

$$\Phi(x) : \Xi \rightarrow \Psi \tag{39}$$

and

$$\Phi^{-1}(A) = \{x \epsilon \Psi : \Phi(x) \epsilon A\} \tag{40}$$

4.2. **Transformations involving multiple functions of multiple random variables.**

Theorem 4. *Let $f_{X_1 X_2}(x_1\ x_2)$ be the value of the joint probability density of the continuous random variables X_1 and X_2 at (x_1, x_2). If the functions given by $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ are partially differentiable with respect to x_1 and x_2 and represent a one-to-one transformation for all values within the range of X_1 and X_2 for which $f_{X_1 X_2}(x_1\ x_2) \neq 0$, then, for these values of x_1 and x_2, the equations $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 to give $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$ and for corresponding values of y_1 and y_2, the joint probability density of $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ is given by*

$$f_{Y_1 Y_2}(y_1\ y_2) = f_{X_1\ X_2}\ [w_1(y_1, y_2), w_2(y_1\ y_2)] \cdot |J| \tag{41}$$

where J is the Jacobian of the transformation and is defined as the determinant

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \tag{42}$$

At all other points $f_{Y_1 Y_2}(y_1\ y_2) = 0.$

4.3. **Example.** Let the probability density function of X_1 and X_2 be given by

$$f_{X_1 X_2}(x_1, x_2) = \begin{cases} e^{-(x_1 + x_2)} & x_1 \geq 0,\ x_2 \geq 0 \\ 0 & elsewhere \end{cases} \tag{43}$$

Consider two random variables Y_1 and Y_2 be defined in the following manner.

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= \frac{X_1}{X_1 + X_2} \end{aligned} \tag{44}$$

To find the joint density of Y_1 and Y_2 we first need to solve the system of equations in equation 44 for X_1 and X_2.

$$\begin{aligned}
Y_1 &= X_1 + X_2 \\
Y_2 &= \frac{X_1}{X_1 + X_2} \\
\Rightarrow X_1 &= Y_1 - X_2 \\
\Rightarrow Y_2 &= \frac{Y_1 - X_2}{Y_1 - X_2 + X_2} \\
\Rightarrow Y_2 &= \frac{Y_1 - X_2}{Y_1} \\
\Rightarrow Y_1 Y_2 &= Y_1 - X_2 \\
\Rightarrow X_2 &= Y_1 - Y_1 Y_2 = Y_1 (1 - Y_2) \\
\Rightarrow X_1 &= Y_1 - (Y_1 - Y_1 Y_2) = Y_1 Y_2
\end{aligned} \tag{45}$$

The Jacobian is given by

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \\
&= \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} \\
&= - y_2 y_1 - y_1 (1 - y_2) \\
&= - y_2 y_1 - y_1 + y_1 y_2 \\
&= - y_1
\end{aligned} \tag{46}$$

This transformation is one-to-one and maps the domain of X (Ξ) given by $x_1 > 0$ and $x_2 > 0$ in the x_1 x_2-plane into the domain of Y(Ψ) in the y_1 y_2-plane given by $y_1 > 0$ and $0 < y_2 < 1$. If we apply theorem 4 we obtain

$$\begin{aligned}
f_{Y_1 Y_2}(y_1\ y_2) &= f_{X_1\, X_2}\left[w_1\,(y_1\,,\ y_2),\ w_2(\ y_1\ y_2\ )\ \right] \cdot |\ J\ | \\
&= e^{-\,(\,y_1\, y_2\, +\, y_1\, -\, y_1\, y_2\,)}\ |-y_1\ | \\
&= e^{-\,y_1}\ |-y_1\ | \\
&= y_1\ e^{-\,y_1}
\end{aligned} \tag{47}$$

Considering all possible values of values of y_1 and y_2 we obtain

$$f_{Y_1\,Y_2}(y_1, y_2) = \begin{cases} y_1 e^{-y_1} & for\ y_1 \geq 0,\ 0 < y_2 < 1 \\ 0 & elsewhere \end{cases} \tag{48}$$

We can then find the marginal density of Y_2 by integrating over y_1 as follows

$$\begin{aligned}
f_{Y_2}(y_1,\ y_2) &= \int_0^\infty f_{Y_1\,Y_2}\,(y_1,\ y_2)\,dy_1 \\
&= \int_0^\infty y_1\ e^{-\,y_1}\ dy_1
\end{aligned} \tag{49}$$

We make a uv substitution to integrate where u, v, du, and dv are defined as

$$\begin{aligned} u &= y_1 & v &= -\,e^{-y_1} \\ du &= dy_1 & dv &= e^{-y_1}\,dy_1 \end{aligned} \tag{50}$$

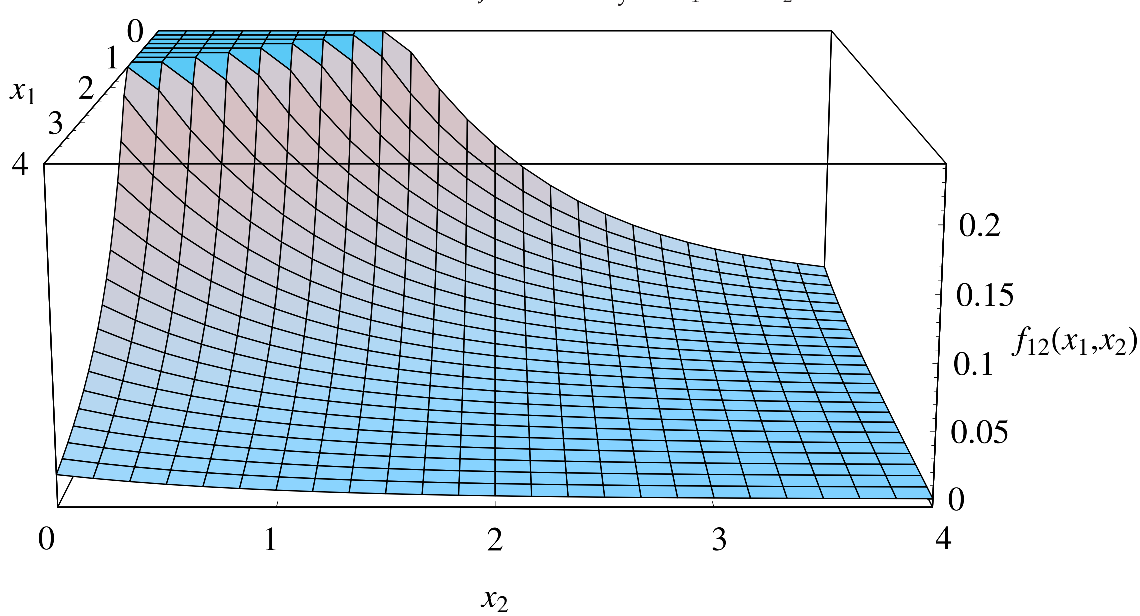
This then implies

$$\begin{aligned} f_{Y_2}(y_1,\ y_2) &= \int_0^\infty y_1\, e^{-y_1} dy_1 \\ &= -\ y_1\ e^{-\,y_1}\ |_0^\infty\ -\ \int_0^\infty -e^{-\,y_1}\ d\,y_1 \\ &= (\,0\ -\ 0\,)\ -\ \left(e^{-\,y_1}\right)\Big|_0^\infty \\ &= 0\ -\ \left(e^{-\,y_1}\right)\Big|_0^\infty \\ &= 0\ -\ \left(e^{-\,\infty}\ -\ e^0\right) \\ &= 0\ -\ 0\ +\ 1 \\ &= 1 \end{aligned} \tag{51}$$

for all y_2 such that $0 < y_2 < 1$.

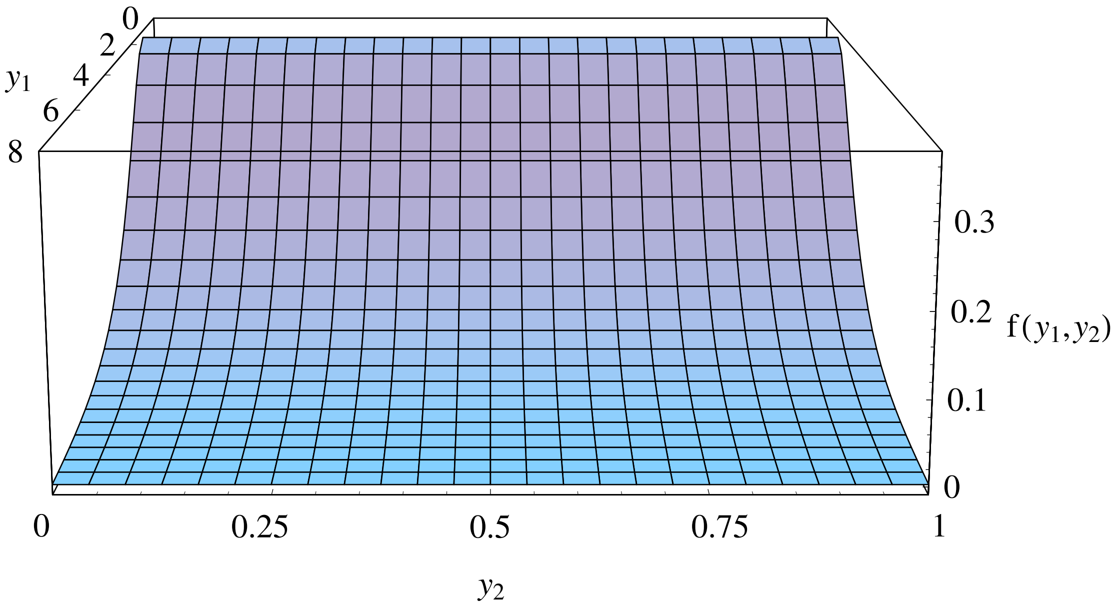
A graph of the joint densities and the marginal density follows. The joint density of (X_1, X_2) is shown in figure 4.

FIGURE 4. Joint Density of X_1 and X_2.



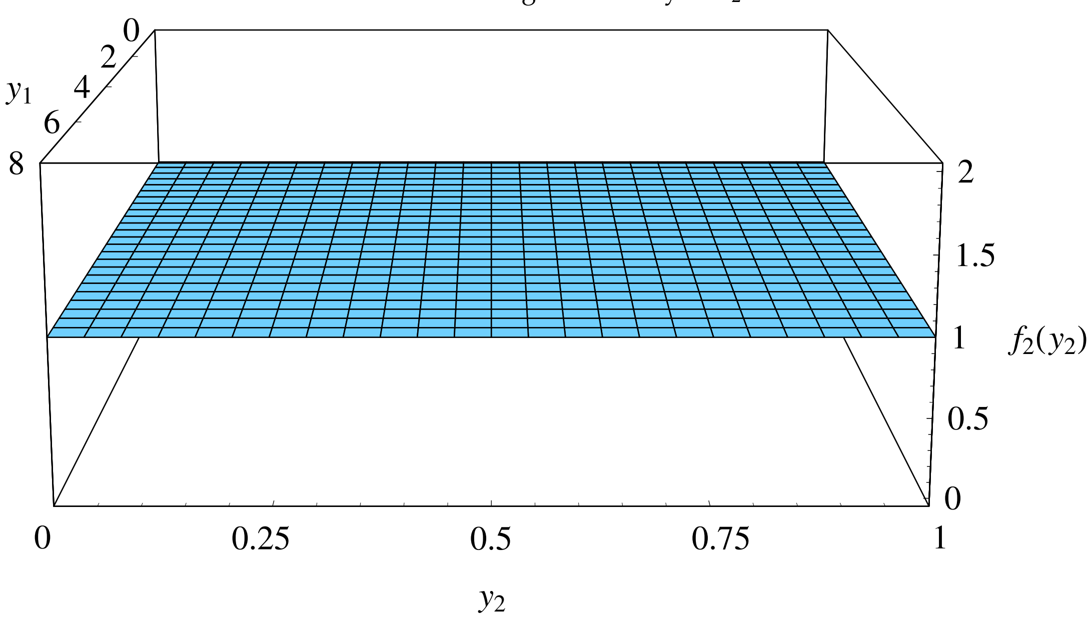
This joint density of (Y_1, Y_2) is contained in figure 5.

FIGURE 5. Joint Density of Y_1 and Y_2.



This marginal density of Y_2 is shown graphically in figure 6.

FIGURE 6. Marginal Density of Y_2.



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