GENERAL ANALYSIS OF MAXIMA/MINIMA IN CONSTRAINED OPTIMIZATION PROBLEMS

1. Statement of the Problem

Consider the problem defined by

$$\underset{x}{\text{maximize}}\; f(x)$$
$$\text{subject to } g(x) = 0$$

where $g(x) = 0$ denotes an $m \times 1$ vector of constraints, $m < n$. We can also write this as

$$\max_{x_1,\, x_2, \ldots x_n} f(x_1,\, x_2, \ldots,\, x_n)$$

subject to

$$\begin{aligned} g_1(x_1,\, x_2, \ldots,\, x_n) &= 0 \\ g_2(x_1,\, x_2, \ldots,\, x_n\,) &= 0 \\ &\vdots \\ g_m(x_1,\, x_2, \ldots,\, x_n\,) &= 0 \end{aligned} \tag{1}$$

The solution can be obtained using the Lagrangian function

$$\begin{aligned} L(x;\, \lambda) &= f(x) - \lambda' g(x) \quad \text{where } \lambda' = \; (\lambda_1,\, \lambda_2, \ldots, \lambda_m) \\ &= f(x_1,\, x_2, \ldots) - \lambda_1 g_1(x) - \lambda_2 g_2(x) - \cdots - \lambda_m g_m(x) \end{aligned} \tag{2}$$

Notice that the gradient of L will involve a set of derivatives, i.e.

$$\nabla_x L = \nabla_x f(x) - \left(\frac{\partial g}{\partial x}\right) \lambda$$

where

$$\left(\frac{\partial g}{\partial x}\right) = J_g = \begin{pmatrix} \frac{\partial g_1(x^*)}{\partial x_1} & \frac{\partial g_2(x^*)}{\partial x_1} & \cdots & \frac{\partial g_m(x^*)}{\partial x_1} \\ \frac{\partial g_1(x^*)}{\partial x_2} & \frac{\partial g_2(x^*)}{\partial x_2} & \cdots & \frac{\partial g_m(x^*)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1(x^*)}{\partial x_n} & \frac{\partial g_2(x^*)}{\partial x_n} & \cdots & \frac{\partial g_m(x^*)}{\partial x_n} \end{pmatrix} \tag{3}$$

Date: October 7, 2004.

There will be one equation for each x. There will also be equations involving the derivatives of L with respect to each λ.

2. Necessary Conditions for an Extreme Point

The necessary conditions for an extremum of f with the equality constraints $g(x) = 0$ are that

$$\nabla L(x^*, \lambda^*) = 0 \tag{4}$$

where it is implicit that the gradient in (3) is with respect to both x and λ.

3. Sufficient Conditions for an Extreme Point

3.1. **Statement of Conditions.** Let $f, g_1, \ldots, g_m$ be twice continuously differentiable real-valued functions on R^n. If there exist vectors $x^* \epsilon R^n$, $\lambda^* \epsilon R^m$ such that

$$\nabla L(x^*, \lambda^*) = 0 \tag{5}$$

and for every non-zero vector $z \epsilon R^n$ satisfying

$$z' \nabla g_i(x^*) = 0, \quad i = 1, \ldots, m \tag{6}$$

it follows that

$$z' \nabla_x^2 L(x^*, \lambda^*) z > 0, \tag{7}$$

then f has a strict local minimum at x^*, subject to $g_i(x) = 0$, $i = 1, \ldots, m$. If the inequality in (7) is reversed, then f has strict local maximum at x^*. The idea is that if equation 5 holds, then if equation 7 holds for all vectors satisfying equation 6, f will have a strict local minimum at x^*.

3.2. **Checking the Sufficient Conditions.** These conditions for a maximum or minimum can be stated in terms of the Hessian of the Lagrangian function (or bordered Hessian). Let $f, g_1, \ldots, g_m$ be twice continuously differentiable real valued functions. If there exist vectors $x^* \epsilon R^n$, $\lambda^* \epsilon R^m$, such that

$$\nabla L(x^*, \lambda^*) = 0 \tag{8}$$

and if

$$(-1)^m \det \left[\begin{array}{cccccc} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_p} & \frac{\partial g_1(x^*)}{\partial x_1} & \dots & \frac{\partial g_m(x^*)}{\partial x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_p \partial x_1} & \dots & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_p \partial x_p} & \frac{\partial g_1(x^*)}{\partial x_p} & \dots & d\frac{\partial g_m(x^*)}{\partial x_p} \\ \frac{\partial g_1(x^*)}{\partial x_1} & \dots & \frac{\partial g_1(x^*)}{\partial x_p} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_m(x^*)}{\partial x_1} & \dots & \frac{\partial g_m(x^*)}{\partial x_p} & 0 & \dots & 0 \end{array} \right) > 0 \qquad (9)$$

for $p = m + 1, \ldots, n$, then f has a strict local minimum at x^*, such that

$$g_i(x^*) = 0, \quad i = 1, \ldots, m. \qquad (10)$$

We check the determinants in (9) starting with the one that has $m + 1$ elements in each row and column of the Hessian and $m+1$ elements in each row or column of the derivative of a given constraint with respect to x. Note that m does not change as we check the various determinants so that they will all be of the same sign for a given m.

If there exist vectors $x^* \,\epsilon\, R^n$, $\lambda^* \,\epsilon\, R^m$, such that

$$\nabla L(x^*,\, \lambda^*) = 0 \qquad (11)$$

and if

$$(-1)^p \det \left[\begin{array}{cccccc} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_p} & \frac{\partial g_1(x^*)}{\partial x_1} & \dots & \frac{\partial g_m(x^*)}{\partial x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_p \partial x_1} & \dots & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_p \partial x_p} & \frac{\partial g_1(x^*)}{\partial x_p} & \dots & \frac{\partial g_m(x^*)}{\partial x_p} \\ \frac{\partial g_1(x^*)}{\partial x_1} & \dots & \frac{\partial g_1(x^*)}{\partial x_p} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_m(x^*)}{\partial x_1} & \dots & \frac{\partial g_m(x^*)}{\partial x_p} & 0 & \dots & 0 \end{array} \right) > 0 \qquad (12)$$

for $p = m + 1, \ldots, n$ then f has a strict local maximum at x^*, such that

$$g_i(x^*) = 0, \quad i = 1, \ldots, m. \tag{13}$$

We check the determinants in (12) starting with the one that has $m + 1$ elements in each row and column of the Hessian and $m + 1$ elements in each row or column of the derivative of a given constraint with respect to x. Note that p changes as we check the various determinants so that they will alternate in sign for a given m.

Consider the case where $n = 2$ and $m = 1$. Note that the first matrix we check has $p = m + 1 = 2$. Then the condition for a minimum is

$$(-1)\det \left[\begin{matrix} \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{matrix} \right) > 0 \tag{14}$$

This, of course, implies

$$\det \begin{bmatrix} \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} < 0 \tag{15}$$

The condition for a maximum is

$$(-1)^2 \ \det \left[\begin{matrix} \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{matrix} \right) > 0 \tag{16}$$

This, of course, implies

$$\det \begin{bmatrix} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} > 0 \tag{17}$$

Also consider the case where $n = 3$ and $m = 1$. We start with $p = m + 1 = 2$ and continue until $p = n$. Then the condition for a minimum is

$$(-1) \det \begin{bmatrix} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda *)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} > 0$$

$$(-1) \det \begin{bmatrix} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_3} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_3} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_3 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_3 \partial x_2} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_3 \partial x_3} & \frac{\partial g(x^*)}{\partial x_3} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & \frac{\partial g(x^*)}{\partial x_3} & 0 \end{bmatrix} > 0 \tag{18}$$

The condition for a maximum is

$$(-1)^2 \det \left[\begin{matrix} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{matrix} \right) > 0$$

$$(-1)^3 \det \left[\begin{matrix} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_3} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_3} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_3} & \frac{\partial g(x^*)}{\partial x_3} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & \frac{\partial g(x^*)}{\partial x_3} & 0 \end{matrix} \right) > 0 \qquad (19)$$

3.3. **Sufficient Condition for a Maximum and Minimum and Positive and Negative Definite Quadratic Forms.** Note that at the optimum, equation 6 is just linear in the sense that the derivatives

$$\frac{\partial g_i(x^*)}{\partial x_j}$$

are fixed numbers at the point x^* and we can write equation 6 as

$$z' J_g = 0$$

$$(z_1\, z_2 \ldots z_n) \begin{pmatrix} \frac{\partial g_1(x^*)}{\partial x_1} & \frac{\partial g_2(x^*)}{\partial x_1} & \ldots & \frac{\partial g_m(x^*)}{\partial x_1} \\ \frac{\partial g_1(x^*)}{\partial x_2} & \frac{\partial g_2(x^*)}{\partial x_2} & \ldots & \frac{\partial g_m(x^*)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1(x^*)}{\partial x_n} & \frac{\partial g_2(x^*)}{\partial x_n} & \ldots & \frac{\partial g_m(x^*)}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (20)$$

where J_g is the matrix $\left\{ \frac{\partial g_i(x^*)}{\partial x_j} \right\}$ and where there is a column of the J_g for each constraint and a row for each x variable we are considering. This then implies that the sufficient condition for a strict local maximum of the function f is that $|H_B|$ has the same sign as $(-1)^p$, that is the last $n-m$ leading principal minors of H_B ***alternate in sign*** on the constraint set denoted by equation 6. This is the same as the condition that the quadratic form $z'H_Bz$ be **negative definite** on the constraint set

$$z'\nabla g_i(x^*) = 0, \quad i = 1, \ldots, m \tag{21}$$

If $|H_B|$ and these last $n-m$ leading principal minors all have the same sign as $(-1)^m$, then $z'H_Bz$ is **positive definite** on the constraint set $z'\nabla g_i(x^*) = 0, \quad i = 1, \ldots, m$ and the function has strict local minimum at the point x^*.

If both of conditions are violated by **non-zero** leading principal minors, then $z'H_Bz$ is indefinite on the constraint set and we cannot determine whether the function has a maximum or a minimum.

3.4. **Example 1: Minimizing Cost Subject to an Output Constraint.** Consider a production function given by

$$y = 20x_1 - x_1^2 + 15x_2 - x_2^2 \tag{22}$$

Let the prices of x_1 and x_2 be 10 and 5 respectively with an output constraint of 55. Then to minimize the cost of producing 55 units of output given this prices we set up the following Lagrangian

$$L = 10x_1 + 5x_2 - \lambda(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55)$$

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= 10 - \lambda(20 - 2x_1) = 0 \\
\frac{\partial L}{\partial x_2} &= 5 - \lambda(15 - 2x_2) = 0 \\
\frac{\partial L}{\partial \lambda} &= (-1)(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55) = 0
\end{aligned} \tag{23}$$

If we take the ratio of the first two first order conditions we obtain

$$\begin{aligned}
\frac{10}{5} = 2 &= \frac{20 - 2x_1}{15 - 2x_2} \\
\Rightarrow 30 - 4x_2 &= 20 - 2x_1 \\
\Rightarrow 10 - 4x_2 &= -2x_1 \\
\Rightarrow x_1 &= 2x_2 - 5
\end{aligned} \tag{24}$$

Now plug this into the negative of the last first order condition to obtain

$$20(2x_2 - 5) - (2x_2 - 5)^2 + 15x_2 - x_2^2 - 55 = 0 \tag{25}$$

Multiplying out and solving for x_2 will give

$$
\begin{aligned}
40x_2 - 100 - (4x_2^2 - 20x_2 + 25) + 15x_2 - x_2^2 - 55 = 0 \\
\Rightarrow 40x_2 - 100 - 4x_2^2 + 20x_2 - 25 + 15x_2 - x_2^2 - 55 = 0 \\
\Rightarrow -5x_2^2 + 75x_2 - 180 = 0 \\
\Rightarrow 5x_2^2 - 75x_2 + 180 = 0 \\
\Rightarrow x_2^2 - 15x_2 + 36 = 0
\end{aligned}
\tag{26}
$$

Now solve this quadratic equation for x_2 as follows

$$
\begin{aligned}
x_2 &= \frac{15 \pm \sqrt{225 - 4(36)}}{2} \\
&= \frac{15 \pm \sqrt{81}}{2} \\
&= 12 \text{ or } 3
\end{aligned}
\tag{27}
$$

Therefore,

$$
\begin{aligned}
x_1 &= 2x_2 - 5 \\
&= 19 \text{ or } 1
\end{aligned}
\tag{28}
$$

The Lagrangian multiplier λ can be obtained by solving the first equation that was obtained by differentiating L with respect to x_1

$$
\begin{aligned}
10 - \lambda(20 - \ (19)) = 0 \\
\Rightarrow \lambda = -\frac{5}{9} \\
10 - \lambda(20 - 2(1)) = 0 \\
\Rightarrow \lambda = \frac{5}{9}
\end{aligned}
\tag{29}
$$

To check for a maximum or minimum we set up the bordered Hessian as in equations 14–17. The bordered Hessian in this case is

$$
H_B = \begin{bmatrix}
\frac{\partial^2 L(x^*, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\
\frac{\partial^2 L(x^*, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\
\frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0
\end{bmatrix}
\tag{30}
$$

We only need to compute one determinant. We compute the various elements of the bordered Hessian as follows

$$
\begin{aligned}
L &= 10x_1 + 5x_2 - \lambda(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55) \\
\frac{\partial L}{\partial x_1} &= 10 - \lambda(20 - 2x_1) \\
\frac{\partial L}{\partial x_2} &= 5 - \lambda(15 - 2x_2) \\
\frac{\partial^2 L}{\partial x_1 \partial x_1} &= 2\lambda \\
\frac{\partial^2 L}{\partial x_1 \partial x_2} &= 0 \\
\frac{\partial^2 L}{\partial x_2 \partial x_2} &= 2\lambda \\
\frac{\partial g}{\partial x_1} &= 20 - 2x_1 \\
\frac{\partial g}{\partial x_2} &= 15 - 2x_2
\end{aligned}
\tag{31}
$$

Consider first the point (19, 12, -5/9). The bordered Hessian is given by

$$
\begin{aligned}
H_B &= \begin{bmatrix} 2\lambda & 0 & 20 - 2x_1 \\ 0 & 2\lambda & 15 - 2x_2 \\ 20 - 2x_1 & 15 - 2x_2 & 0 \end{bmatrix} \\
x_1 &= 19, \quad x_2 = 12, \quad \lambda = -\frac{5}{9} \\
H_B &= \begin{bmatrix} -\frac{10}{9} & 0 & -18 \\ 0 & -\frac{10}{9} & -9 \\ -18 & -9 & 0 \end{bmatrix}
\end{aligned}
\tag{32}
$$

The determinant of the bordered Hessian is

$$
\begin{aligned}
|H_B| &= (-1)^2\left(-\frac{10}{9}\right)\begin{vmatrix} -\frac{10}{9} & -9 \\ -9 & 0 \end{vmatrix} + (-1)^3(0)\begin{vmatrix} -\frac{10}{9} & -9 \\ -9 & 0 \end{vmatrix} + (-1)^4(-18)\begin{vmatrix} 0 & -\frac{10}{9} \\ -18 & -9 \end{vmatrix} \\
&= \left(-\frac{10}{9}\right)(-81) + 0 + (-18)(-20) \\
&= 90 + 360 = 450
\end{aligned} \tag{33}
$$

Here $p = 2$ so the condition for a maximum is that $(-1)^2|H_B| > 0$, so this point is a relative maximum.

Now consider the other point, (1, 3, 5/9). The bordered Hessian is given by

$$
\begin{aligned}
H_B &= \begin{bmatrix} 2\lambda & 0 & 20 - 2x_1 \\ 0 & 2\lambda & 15 - 2x_2 \\ 20 - 2x_1 & 15 - 2x_2 & 0 \end{bmatrix} \\
x_1 &= 1, \quad x_2 = 3, \quad \lambda = \frac{5}{9} \\
H_B &= \begin{bmatrix} \frac{10}{9} & 0 & 18 \\ 0 & \frac{10}{9} & 9 \\ 18 & 9 & 0 \end{bmatrix}
\end{aligned} \tag{34}
$$

The determinant of the bordered Hessian is

$$
\begin{aligned}
|H_B| &= (-1)^2\left(\frac{10}{9}\right)\begin{vmatrix} \frac{10}{9} & 9 \\ 9 & 0 \end{vmatrix} + (-1)^3(0)\begin{vmatrix} \frac{10}{9} & 9 \\ 9 & 0 \end{vmatrix} + (-1)^4(18)\begin{vmatrix} 0 & \frac{10}{9} \\ 18 & 9 \end{vmatrix} \\
&= \left(\frac{10}{9}\right)(-81) + 0 + (18)(-20) \\
&= -90 - 360 = -450
\end{aligned} \tag{35}
$$

The condition for a minimum is that $(-1)|H_B| > 0$, so this point is a relative minimum. The minimum cost is obtained by substituting into the cost expression to obtain

$$
C = 10(1) + 5(3) = 25 \tag{36}
$$

3.5. **Example 2: Maximizing Output Subject to a Cost Constraint.** Consider a production function given by

$$y = 30x_1 + 12x_2 - x_1^2 + x_1x_2 - x_2^2 \tag{37}$$

Let the prices of x_1 and x_2 be 10 and 4 respectively with an cost constraint of \$260. Then to maximize output with a cost of \$260 given these prices we set up the following Lagrangian

$$\begin{aligned}
L &= 30x_1 + 12x_2 - x_1^2 + x_1x_2 - x_2^2 - \lambda(10x_1 + 4x_2 - 260) \\
\frac{\partial L}{\partial x_1} &= 30 - 2x_1 + x_2 - 10\lambda = 0 \\
\frac{\partial L}{\partial x_2} &= 12 + x_1 - 2x_2 - 4\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= -10x_1 - 4x_2 + 260 = 0
\end{aligned} \tag{38}$$

If we take the ratio of the first two first order conditions we obtain

$$\begin{aligned}
\frac{10}{4} = 2.5 &= \frac{30 - 2x_1 + x_2}{12 + x_1 - 2x_2} \\
\Rightarrow 30 + 2.5x_1 - 5x_2 &= 30 - 2x_1 + x_2 \\
\Rightarrow 4.5x_1 &= 6x_2 \\
\Rightarrow x_1 &= 1.3\bar{3}x_2
\end{aligned} \tag{39}$$

Now plug this value for x_1 into the negative of the last first order condition to obtain

$$\begin{aligned}
10x_1 + 4x_2 - 260 &= 0 \\
\Rightarrow (10)(1.3\bar{3}x_2) + 4x_2 - 260 &= 0 \\
\Rightarrow 13.3\bar{3}x_2 + 4x_2 &= 260 \\
\Rightarrow 17.3\bar{3}x_2 &= 260 \\
\Rightarrow x_2 &= 15 \\
\Rightarrow x_1 &= 7\left(\frac{4}{3}\right)(15) = 20
\end{aligned} \tag{40}$$

We can also find the maximum y by substituting in for x_1 and x_2.

$$\begin{aligned} y &= 30x_1 + 12x_2 - x_1^2 + x_1x_2 - x_2^2 \\ &= (30)(20) + (12)(15) - (20)^2 - (20)(15) - (15)^2 \\ &= 600 + 180 - 400 + 300 - 225 \\ &= 455 \end{aligned} \tag{41}$$

The Lagrangian multiplier λ can be obtained by solving the first equation that was obtained by differentiating L with respect to x_1

$$\begin{aligned} 30 - 2x_1 + x_2 - 10\lambda &= 0 \\ \Rightarrow 30 - 2(20) + (15) - 10\lambda &= 0 \\ \Rightarrow 30 - 40 + 15 - 10\lambda &= 0 \\ \Rightarrow 5 &= 10\lambda \\ \Rightarrow \lambda &= \frac{1}{2} \end{aligned} \tag{42}$$

To check for a maximum or minimum we set up the bordered Hessian as in equations 14–17 where $p = 2$ and $m = 1$. The bordered Hessian in this case is

$$H_B = \begin{bmatrix} \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} \tag{43}$$

We compute the various elements of the bordered Hessian as follows

$$\begin{aligned}
L &= 30x_1 + 12x_2 - x_1^2 + x_1x_2 - x_2^2 - \lambda(10x_1 + 4x_2 - 260) \\
\frac{\partial L}{\partial x_1} &= 30 - 2x_1 + x_2 - 10\lambda \\
\frac{\partial L}{\partial x_2} &= 12 + x_1 - 2x_2 - 4\lambda \\
\frac{\partial^2 L}{\partial x_1 \partial x_1} &= -2 \\
\frac{\partial^2 L}{\partial x_1 \partial x_2} &= 1 \\
\frac{\partial^2 L}{\partial x_2 \partial x_2} &= -2 \\
\frac{\partial g}{\partial x_1} &= 10 \\
\frac{\partial g}{\partial x_2} &= 4
\end{aligned} \tag{44}$$

The derivatives are all constants. The bordered Hessian is given by

$$H_B = \begin{bmatrix} -2 & 1 & 10 \\ 1 & -2 & 4 \\ 10 & 4 & 0 \end{bmatrix} \tag{45}$$

The determinant of the bordered Hessian is

$$\begin{aligned}
|H_{B|} &= (-1)^2(-2)\begin{vmatrix} -2 & 4 \\ 4 & 0 \end{vmatrix} + (-1)^3(1)\begin{vmatrix} 1 & 4 \\ 10 & 0 \end{vmatrix} + (-1)^4(10)|\begin{vmatrix} 1 & -2 \\ 10 & 4 \end{vmatrix} \\
&= (-2)(-16) - (-40) + (10)(24) \\
&= 32 + 40 + 240 = 312
\end{aligned} \tag{46}$$

The condition for a maximum is that $(-1)^2|H_B| > 0$, so this point is a relative maximum.

3.6. **Example 3: Maximizing Utility Subject to an Income Constraint.** Consider a utility function given by

$$u = x_1^{\alpha_1} x_2^{\alpha_2}$$

Now maximize this function subject to the constraint that

$$w_1x_1 + w_2x_2 = c_0$$

Set up the Lagrangian problem:

$$L = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda[w_1 x_1 + w_2 x_2 - c_0]$$

The first order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda\, w_1 \; = 0 \\
\frac{\partial L}{\partial x_2} &= \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} - \lambda w_2 \; = 0 \\
\frac{\partial L}{\partial \lambda} &= -w_1\, x_1 - w_2\, x_2 + c_0 = 0
\end{aligned}$$

Taking the ratio of the 1st and 2nd equations we obtain

$$\frac{w_1}{w_2} = \frac{\alpha_1 x_2}{\alpha_2 x_1}$$

We can now solve the equation for the 2nd quantity as a function of the 1st input quantity and the prices. Doing so we obtain

$$x_2 = \frac{\alpha_2 x_1 w_1}{\alpha_1 w_2}$$

Now substituting in the income equation we obtain

$$\begin{aligned}
w_1 x_1 + w_2 x_2 &= c_0 \\
\Rightarrow w_1 x_1 + w_2 \left[\frac{\alpha_2 x_1 w_1}{\alpha_1 w_2} \right] &= c_0 \\
\Rightarrow w_1 x_1 + \left[\frac{\alpha_2 w_1 w_2}{\alpha_1 w_2} \right] x_1 &= c_0 \\
\Rightarrow w_1 x_1 + \left[\frac{\alpha_2 w_1}{\alpha_1} \right] x_1 &= c_0 \\
\Rightarrow x_1 \left[w_1 + \frac{\alpha_2 w_1}{\alpha_1} \right] &= c_0
\end{aligned}$$

$$\Rightarrow x_1 w_1 \left[1 + \frac{\alpha_2}{\alpha_1}\right] = c_0$$

$$\Rightarrow x_1 w_1 \left[\frac{\alpha_1 + \alpha_2}{\alpha_1}\right] = c_0$$

$$\Rightarrow x_1 = \frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 + \alpha_2}\right]$$

We can now get x_2 by substitution

$$\begin{aligned} x_2 &= x_1 \left[\frac{\alpha_2 w_1}{\alpha_1 w_2}\right] \\ &= \frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 + \alpha_2}\right] \left[\frac{\alpha_2 w_1}{\alpha_1 w_2}\right] \\ &= \frac{c_0}{w_2} \left[\frac{\alpha_2}{\alpha_1 + \alpha_2}\right] \end{aligned}$$

We can find the value of the optimal u by substitution

$$\begin{aligned} u &= x_1^{\alpha_1} x_2^{\alpha_2} \\ &= \left(\frac{c_0}{w_1}\left[\frac{\alpha_1}{\alpha_1 + \alpha_2}\right]\right)^{\alpha_1} \left(\frac{c_0}{w_2}\left[\frac{\alpha_2}{\alpha_1 + \alpha_2}\right]\right)^{\alpha_2} \\ &= c_0^{\alpha_1 + \alpha_2} w_1^{-\alpha_1} w_2^{-\alpha_2} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} (\alpha_2 + \alpha_2)^{-\alpha_1 - \alpha_2} \end{aligned}$$

This can also be written

$$\begin{aligned} u &= x_1^{\alpha_1} x_2^{\alpha_2} \\ &= \left[\frac{c_0}{w_1}\left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)\right]^{\alpha_1} \left[\frac{c_0}{w_2}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)\right]^{\alpha_2} \\ &= \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)^{\alpha_2} \left(\frac{c_0}{w_1}\right)^{\alpha_1} \left(\frac{c_0}{w_2}\right)^{\alpha_2} \end{aligned}$$

For future reference note that the derivative of the optimal u with respect to c_0 is given by

$$u = c_0^{\alpha_1+\alpha_2} w_1^{-\alpha_1} w_2^{-\alpha_2} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} (\alpha_2 + \alpha_2)^{-\alpha_1-\alpha_2}$$

$$\frac{\partial u}{\partial c_0} = (\alpha_1 + \alpha_2) c_0^{\alpha_1+\alpha_2-1} w_1^{-\alpha_1} w_2^{-\alpha_2} \alpha_1^{\alpha_1} \alpha_1^{\alpha_2} (\alpha_2 + \alpha_2)^{-\alpha_1-\alpha_2}$$

$$= c_0^{\alpha_1+\alpha_2-1} w_1^{-\alpha_1} w_2^{-\alpha_2} \alpha_1^{\alpha_1} \alpha_1^{\alpha_2} (\alpha_2 + \alpha_2)^{1-\alpha_1-\alpha_2}$$

We obtain λ by substituting in either the first or second equation as follows

$$\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} - \lambda w_1 = 0$$

$$\Rightarrow \lambda = \frac{\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2}}{w_1}$$

$$\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} - \lambda w_2 = 0$$

$$\Rightarrow \lambda = \frac{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1}}{w_2}$$

If we now substitute for x_1 and x_2, we obtain

$$\lambda = \frac{\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2}}{w_1}$$

$$x_1 = \frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 + \alpha_2} \right]$$

$$x_2 = \frac{c_0}{w_2} \left[\frac{\alpha_2}{\alpha_1 + \alpha_2} \right]$$

$$\Rightarrow \lambda = \frac{\alpha_1 \left(\frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 + \alpha_2} \right] \right)^{\alpha_1 - 1} \left(\frac{c_0}{w_2} \left[\frac{\alpha_2}{\alpha_1 + \alpha_2} \right] \right)^{\alpha_2}}{w_1}$$

$$= \frac{\alpha_1 c_0^{\alpha_1+\alpha_2-1} w_1^{1-\alpha_1} w_2^{-\alpha_2} \alpha_1^{\alpha_1-1} \alpha_2^{\alpha_2} \left(\alpha_1 + \alpha_2\right)^{1-\alpha_1-\alpha_2}}{w_1}$$

$$= c_0^{\alpha_1+\alpha_2-1} w_1^{-\alpha_1} w_2^{-\alpha_2} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \left(\alpha_1 + \alpha_2\right)^{1-\alpha_1-\alpha_2}$$

Thus λ is equal to the derivative of the optimal u with respect to c_0.

To check for a maximum or minimum we set up the bordered Hessian as in equations 14–17 where $p = 2$ and $m = 1$. The bordered Hessian in this case is

$$H_B = \begin{bmatrix} \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} \tag{47}$$

We need compute the various elements of the bordered Hessian as follows

$$L = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda[w_1 x_1 + w_2 x_2 - c_0]$$

$$\frac{\partial L}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda w_1$$

$$\frac{\partial L}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} - \lambda w_2$$

$$\frac{\partial^2 L}{\partial x_1^2} = (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2}$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1}$$

$$\frac{\partial^2 L}{\partial x_2^2} = (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2}$$

$$\frac{\partial g}{\partial x_1} = w_1$$

$$\frac{\partial g}{\partial x_2} = w_2$$

The derivatives of the constraints are constants. The bordered Hessian is given by

$$H_B = \begin{bmatrix} (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2} & \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & w_1 \\ \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2} & w_2 \\ w_1 & w_2 & 0 \end{bmatrix} \tag{48}$$

To find the determinant of the bordered Hessian, expand by the third row as follows

$$|H_B| = (-1)^4 w_1 \begin{vmatrix} \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} & w_1 \\ (\alpha_2)(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2} & w_2 \end{vmatrix} + (-1)^5 w_2 \begin{vmatrix} (\alpha_1)(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} & w_1 \\ \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} & w_2 \end{vmatrix} + 0$$

$$= w_1 \begin{vmatrix} \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} & w_1 \\ (\alpha_2)(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2} & w_2 \end{vmatrix} - w_2 \begin{vmatrix} (\alpha_1)(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} & w_1 \\ \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} & w_2 \end{vmatrix}$$

$$\begin{aligned} &= w_1 w_2 \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} - w_1^2(\alpha_2)(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2} \\ &\qquad - w_2^2(\alpha_1)(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} + w_1 w_2\alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha 2-1} \\ &= 2w_1 w_2 \alpha_1 \alpha 2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} - w_1^2(\alpha_2)(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2} - w_2^2(\alpha_1)(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} \end{aligned} \tag{49}$$

For a maximum we want this expression to be positive. Rewriting it we obtain

$$2w_1 w_2 \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} - w_1^2(\alpha_2)(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2} - w_2^2(\alpha_1)(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} > 0 \tag{50}$$

We can also write it in the following convenient way

$$\begin{gathered} 2w_1 w_2 \alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} \\ +\alpha_2 w_1^2 x_1^{\alpha_1}x_2^{\alpha_2-2} - \alpha_2^2 w_1^2 x_1^{\alpha_1}x_2^{\alpha_2-2} \\ +\alpha_1 w_2^2 x_1^{\alpha_1-2}x_2^{\alpha_2} - \alpha_1^2 w_2^2 x_1^{\alpha_1-2}x_2^{\alpha_2} > 0 \end{gathered} \tag{51}$$

To eliminate the prices we can substitute from the first-order conditions.

$$w_1 = \frac{\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}}{\lambda}$$

$$w_2 = \frac{\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}$$

This then gives

$$\begin{gathered} 2\left(\frac{\alpha_1 x_1^{\alpha 1-1}x_2^{\alpha_2}}{\lambda}\right)\left(\frac{\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}\right)\alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} \\ +\alpha_2\left(\frac{\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}}{\lambda}\right)^2 x_1^{\alpha_1}x_2^{\alpha_2-2} - \alpha_2^2\left(\frac{\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}}{\lambda}\right)^2 x_1^{\alpha_1}x_2^{\alpha_2-2} \\ +\alpha_1\left(\frac{\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}\right)^2 x_1^{\alpha_1-2}x_2^{\alpha_2} - \alpha_1^2\left(\frac{\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}\right)^2 x_1^{\alpha_1-2}x_2^{\alpha_2} > 0 \end{gathered} \tag{52}$$

Multiply both sides by λ^2 and combine terms to obtain

$$\begin{gathered} 2\alpha_1^2\alpha_2^2 x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} \\ +\alpha_1^2\alpha_2 x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} - \alpha_2^2\alpha_1^2 x_1^{3\alpha1-2}x_2^{3\alpha_2-2} \\ +\alpha_1\alpha_2^2 x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} - \alpha_1^2\alpha_2^2 x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} > 0 \end{gathered} \tag{53}$$

Now factor out $x_1^{3\alpha_1-2}x_2^{3\alpha_2-2}$ to obtain

$$\begin{aligned} x_1^{3\alpha_1-2}x_2^{3\alpha_2-2}\left(2\alpha_1^2\alpha 2^2 + \alpha_1^2\alpha_2 - \alpha_2^2\alpha_1^2 + \alpha_1\alpha_2^2 - \alpha_1^2\alpha_2^2\right) &> 0 \\ \Rightarrow x_1^{3\alpha_1-2}x_2^{3\alpha_2-2}\left(\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2\right) &> 0 \end{aligned} \tag{54}$$

With positive values for x_1 and x_2 the whole expression will be positive if the last term in parentheses is positive. Then rewrite this expression as

$$\left(\alpha_1^2\alpha_2 + \alpha_1\alpha 2^2\right) > 0 \tag{55}$$

Now divide both sides by $\alpha_1^2\alpha_2^2$ (which is positive) to obtain

$$\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_1}\right) > 0 \tag{56}$$

3.7. **Some More Example Problems.**

(i) $\underset{x_1,x_2}{\text{opt}}\ [x_1x_2]$ s. t.
$x_1 + x_2 = 6$

(ii) $\underset{x_1,x_2}{\text{opt}}\ [x_1x_2 + 2x_1]$ s.t.
$4x_1 + 2x_2 = 60$

(iii) $\underset{x_1,x_2}{\text{opt}}\ [x_1^2 + x_2^2]$ s.t
$x_1 + 2x_2 = 20$

(iv) $\underset{x_1,x_2}{\text{opt}}\ [x_1x_2]$ s.t.
$x_1^2 + 4x_2^2 = 1$

(v) $\underset{x_1,x_2}{\text{opt}}\ [x_1^{\frac{1}{4}}x_2^{\frac{1}{2}}]$ s.t.
$2x_1 + 8x_2 = 60$

4. The Implicit Function Theorem

4.1. **Statement of Theorem.** We are often interested in solving implicit systems of equations for m variables, say $x_1, x_2, \ldots, x_m$ in terms of m+p variables where there are a minimum of m equations in the system. We typically label the variables $x_{m+1}, x_{m+2}, \ldots, x_{m+p}$, $y_1, y_2, \ldots, y_p$. We are frequently interested in the derivatives $\frac{\partial x_i}{\partial x_j}$ where it is implicit that all other x_k and all y_ℓ are held constant. The conditions guaranteeing that we can solve for m of the variables in terms of p variables along with a formula for computing derivatives is given by the implicit function theorem.

Theorem 1 (Implicit Function Theorem)**.** *Suppose that ϕ_i are real-valued functions defined on a domain D and continuously differentiable on an open set $D^1 \subset D \subset R^{m+p}$, where $p > 0$ and*

$$\begin{aligned} \phi(x_1^0,\, x_2^0,\, \ldots,\, x_m^0,\, y_1^0,\, y_2^0,\, \ldots,\, y_p^0) \;=\phi_i(x^0,\, y^0) \;=\; 0, \\ i \;=\; 1,\, 2,\, \ldots,\, m, \text{ and } (x^0,\, y^0) \in D^1. \end{aligned} \tag{57}$$

Assume the Jacobian matrix $[\frac{\partial \phi_i(x^0,\, y^0)}{\partial x_j}]$ has rank m. Then there exists a neighborhood $N_\delta(x^0, y^0) \subset D^1$, an open set $D^2 \subset R^p$ containing y^0 and real valued functions ψ_k, k = 1, 2, ..., m, continuously differentiable on D^2, such that the following conditions are satisfied:

$$x_k^0 \;=\; \psi_k(y^0),\; k \;=\; 1,\, 2,\, \ldots,\, m. \tag{58}$$

For every $y \in D^2$, we have

$$\begin{aligned} &\phi_i(\psi_1(y),\, \psi_2(y),\, \ldots,\, \psi_m(y),\, y_1,\, y_2,\, \ldots,\, y_p\,) \;\equiv\; 0, \quad i \;=\; 1,\, 2,\, \ldots,\, m. \\ &\quad or \\ &\phi_i(\psi(y), y) \;\equiv\; 0, \quad i \;=\; 1,\, 2,\, \ldots,\, m. \end{aligned} \tag{59}$$

We also have that for all (x,y) $\in N_\delta(x^0, y^0)$, the Jacobian matrix $[\frac{\partial \phi_i(x,\, y)}{\partial x_j}]$ has rank m. Furthermore for $y \in D^2$, the partial derivatives of $\psi(y)$ are the solutions of the set of linear equations

$$\sum_{k=1}^{m} \frac{\partial \phi_i(\psi(y), y)}{\partial x_k}\, \frac{\partial \psi_k(y)}{\partial y_j} \;=\; \frac{-\partial \phi_i(\psi(y), y}{\partial y_j} \qquad i = 1,\, 2,\, \ldots,\, m \tag{60}$$

4.2. **Example with one equation and three variables.** Consider one implicit equation with three variables.

$$\phi(x_1^0,\, x_2^0,\, y^0) \;=\; 0 \tag{61}$$

The implicit function theorem says that we can solve equation 61 for x_1^0 as a function of x_2^0 and y^0, i.e.,

$$x_1^0 \;=\; \psi_1(x_2^0,\, y^0) \tag{62}$$

and that

$$\phi(\psi_1(x_2,\, y),\, x_2,\, y) \;=\; 0 \tag{63}$$

The theorem then says that

$$\begin{aligned}
\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1} \frac{\partial \psi_1}{\partial x_2} &= \frac{-\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2} \\
\Rightarrow \; \frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1} \; \frac{\partial x_1(x_2,\, y)}{\partial x_2} &= - \; \frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2} \\
\Rightarrow \; \frac{\partial x_1(x_2,\, y)}{\partial x_2} &= \frac{-\, \frac{\partial \phi(\psi_1(x_2,y),\, x_2,\, y)}{\partial x_2}}{\frac{\partial \phi(\psi_1(x_2,y),\, x_2,\, y)}{\partial x_1}}
\end{aligned} \tag{64}$$

Consider the following example.

$$\begin{aligned}
\phi(x_1^0,\, x_2^0,\, y^0) \;&=\; 0 \\
y^0 \;-\; f(x_1^0,\, x_2^0) \;&=\; 0
\end{aligned} \tag{65}$$

The theorem says that we can solve the equation for x_1^0.

$$x_1^0 \;=\; \psi_1(x_2^0,\, y^0) \tag{66}$$

It is also true that

$$\begin{aligned}
\phi(\psi_1(x_2,\, y),\, x_2,\, y) \; &= 0 \\
y \;-\; f(\psi_1(x_2,\, y), x_2) \; &= 0
\end{aligned} \tag{67}$$

Now compute the relevant derivatives

$$\begin{aligned}
\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_1} \; &= \; - \; \frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_1} \\
\frac{\partial \phi(\psi_1(x_2,\, y),\, x_2,\, y)}{\partial x_2} \; &= \; - \; \frac{\partial f(\psi_1(x_2,\, y), x_2)}{\partial x_2}
\end{aligned} \tag{68}$$

The theorem then says that

$$\begin{aligned}
\frac{\partial x_1(x_2,\, y)}{\partial x_2} \; &= \; - \; \left[\frac{\frac{\partial \phi(\psi_1(x_2,y),\, x_2,\, y)}{\partial x_2}}{\frac{\partial \phi(\psi_1(x_2,y),\, x_2,\, y)}{\partial x_1}} \right] \\
&= \; - \; \left[\frac{-\, \frac{\partial f(\psi_1(x_2,y), x_2)}{\partial x_2}}{-\, \frac{\partial f(\psi_1(x_2,y), x_2)}{\partial x_1}} \right] \\
&= \; - \; \frac{\frac{\partial f(\psi_1(x_2,y), x_2)}{\partial x_2}}{\frac{\partial f(\psi_1(x_2,y), x_2)}{\partial x_1}}
\end{aligned} \tag{69}$$

4.3. **Example with two equations and three variables.** Consider the following system of equations

$$\begin{aligned}
\phi_1(x_1,\, x_2,\, y) \; &= 3x_1 \;+\; 2x_2 \;+\; 4y \;=\; 0 \\
\phi_2\,(x_1,\, x_2,\, y) \; &= 4x_1 \;+\; \; x_2 \;+\; \; y \;=\; 0
\end{aligned} \tag{70}$$

The Jacobian is given by

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \tag{71}$$

We can solve system 70 for x_1 and x_2 as functions of y. Move y to the right hand side in each equation.

$$3x_1 + 2x_2 = -4y \tag{72a}$$

$$4x_1 + x_2 = -y \tag{72b}$$

Now solve equation 72b for x_2

$$x_2 = -y - 4x_1 \tag{73}$$

Substitute the solution to equation 73 into equation 72a and simplify

$$\begin{aligned} 3x_1 + 2(-y - 4x_1) &= -4y \\ \Rightarrow 3x_1 - 2y - 8x_1 &= -4y \\ \Rightarrow -5x_1 &= -2y \\ \Rightarrow x_1 = \frac{2}{5}y &= \psi_1(y) \end{aligned} \tag{74}$$

Substitute the solution to equation 74 into equation 73 and simplify

$$\begin{aligned} x_2 = -y - 4\left[\frac{2}{5}y\right] & \\ \Rightarrow x_2 = -\frac{5}{5}y - \frac{8}{5}y & \\ = -\frac{13}{5}y = \psi_2(y) & \end{aligned} \tag{75}$$

If we substitute these expressions for x_1 ad x_2 into equation 70 we obtain

$$\begin{aligned} \phi_1\left(\frac{2}{5}y\,,\,-\frac{13}{5}y,\,y\right)) &= 3\left[\frac{2}{5}y\right] + 2\left[-\frac{13}{5}y\right] + 4y \\ &= \frac{6}{5}y - \frac{26}{5}y + \frac{20}{5}y \\ &= -\frac{20}{5}y + \frac{20}{5}y = 0 \end{aligned} \tag{76}$$

and

$$\begin{aligned} \phi_2\left(\frac{2}{5}y\ ,\ -\ \frac{13}{5}y,\ y\right)) &= 4\left[\frac{2}{5}y\right]\ +\ \left[-\ \frac{13}{5}y\right]\ +\ y \\ &= \frac{8}{5}y\ -\ \frac{13}{5}y\ +\ \frac{5}{5}y \\ &= \frac{13}{5}y\ -\ \frac{13}{5}y\ =\ 0 \end{aligned} \tag{77}$$

Furthermore

$$\begin{aligned} \frac{\partial\psi_1)}{\partial y} &= \frac{2}{5} \\ \frac{\partial\psi_2)}{\partial y} &= -\ \frac{13}{5} \end{aligned} \tag{78}$$

We can solve for these partial derivatives using equation 60 as follows

$$\frac{\partial\phi_1}{\partial x_1}\,\frac{\partial\psi_1}{\partial y}\ +\ \frac{\partial\phi_1}{\partial x_2}\,\frac{\partial\psi_2}{\partial y}\ =\frac{-\partial\phi_1}{\partial y} \tag{79a}$$

$$\frac{\partial\phi_2}{\partial x_1}\,\frac{\partial\psi_1}{\partial y}\ +\ \frac{\partial\phi_2}{\partial x_2}\,\frac{\partial\psi_2}{\partial y}\ =\frac{-\partial\phi_2}{\partial y} \tag{79b}$$

Now substitute in the derivatives of ϕ_1 and ϕ_2 with respect to x_1, x_2, and y.

$$3\,\frac{\partial\psi_1}{\partial y}\ +\ 2\,\frac{\partial\psi_2}{\partial y}\ =\ -\ 4 \tag{80a}$$

$$4\,\frac{\partial\psi_1}{\partial y}\ +\ 1\,\frac{\partial\psi_2}{\partial y}\ =\ -\ 1 \tag{80b}$$

Solve equation 80b for $\frac{\partial\psi_2}{\partial y}$

$$\frac{\partial\psi_2}{\partial y}\ =\ -1\ -\ 4\,\frac{\partial\psi_1}{\partial y} \tag{81}$$

Now substitute the answer from equation 81 into equation 80a

$$\begin{aligned} 3\,\frac{\partial\psi_1}{\partial y}\ +\ 2\left(-1\ -\ 4\,\frac{\partial\psi_1}{\partial y}\right)\ &=\ -\ 4 \\ \Rightarrow\ 3\,\frac{\partial\psi_1}{\partial y}\ -\ 2\ -\ 8\,\frac{\partial\psi_1}{\partial y}\ &=\ -\ 4 \\ \Rightarrow\ -\ 5\,\frac{\partial\psi_1}{\partial y}\ &=\ -\ 2 \\ \Rightarrow\ \frac{\partial\psi_1}{\partial y}\ &= \frac{2}{5} \end{aligned} \tag{82}$$

If we substitute equation 82 into equation 81 we obtain

$$\begin{aligned} \frac{\partial \psi_2}{\partial y} &= -\,1\,-\,4\,\frac{\partial \psi_1}{\partial y} \\ \Rightarrow\ \frac{\partial \psi_2}{\partial y} &= -\,1\,-\,4\left(\frac{2}{5}\right) \\ &= \frac{-5}{5}\,-\,\frac{8}{5}\,=\,-\frac{13}{5} \end{aligned} \tag{83}$$

5. Formal Analysis of Lagrangian Multipliers and Equality Constrained Problems

5.1. **Definition of the Lagrangian.** Consider a function on n variables denoted $f(x) = f(x_1,\, x_2, \ldots,\, x_n)$. Suppose x^* minimizes $f(x)$ for all $x \,\epsilon\, N_\delta(x^*)$ that satisfy

$$g_i(x) = 0 \qquad i = 1, \ldots, m$$

Assume the Jacobian matrix (J) of the constraint equations $g_i(x^*)$ has rank m. Then:

$$\nabla f(x^*) = \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) \tag{84}$$

In other words the gradient of f at x^* is a linear combination of the gradients of g_i at x^* with weights λ_i^*. For later reference note that the Jacobian can be written

$$J_g = \begin{pmatrix} \frac{\partial g_1(x^*)}{\partial x_1} & \frac{\partial g_2(x^*)}{\partial x_1} & \cdots & \frac{\partial g_m(x^*)}{\partial x_1} \\ \frac{\partial g_1(x^*)}{\partial x_2} & \frac{\partial g_2(x^*)}{\partial x_2} & \cdots & \frac{\partial g_m(x^*)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1(x^*)}{\partial x_n} & \frac{\partial g_2(x^*)}{\partial x_n} & \cdots & \frac{\partial g_m(x^*)}{\partial x_n} \end{pmatrix} \tag{85}$$

Proof:

By suitable rearrangement of the rows we can always assume the $m \times m$ matrix formed from the first m rows of the Jacobian $\left(\frac{\partial g_i(x^*)}{\partial x_j}\right)$ is non-singular. Therefore the set of linear equations:

$$\sum_{i=1}^{m} \frac{\partial g_i(x^*)}{\partial x_j} \lambda_j = \frac{\partial f(x^*)}{\partial x_j} \quad j = 1, \ldots,\, m \tag{86}$$

will have a unique solution λ^*. In matrix notation we can write equation 86 as

$$J\lambda = \nabla f$$

If J is invertible, we can solve the system for λ. Therefore (84) is true for the first m elements of $\nabla f(x^*)$.

We must show (84) is also true for the last $n-m$ elements. Let $\tilde{x} = (x_{m+1},\, x_{m+2}, \ldots,\, x_n)$. Then by using the implicit function theorem we can solve for the first m xs in terms of the remaining xs or $\tilde{x}$.

$$x_j^* = h_j(\tilde{x}^*) \qquad j = 1, \ldots, m \tag{87}$$

We can define $f(x^*)$ as

$$f(x^*) = f(h_1(\tilde{x}^*),\, h_2(\tilde{x}^*) \ldots h_m(\tilde{x}^*),\, x_{m+1}^* \ldots x_n^*) \tag{88}$$

Since we are at a minimum, we know that the first partial derivatives of f with respect to $x_{m+1},\, x_{m+2}, \ldots,\, x_n$ must vanish at x^*, i.e.

$$\frac{\partial f(x^*)}{\partial x_j} = 0 \qquad j = m+1, \ldots, n$$

Totally differentiating (88) we obtain

$$\frac{\partial f(x^*)}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial f(x^*)}{\partial x_k} \frac{\partial h_k(\tilde{x}^*)}{\partial x_j} + \frac{\partial f(x^*)}{\partial x_j} = 0 \tag{89}$$

$$j = m+1, \ldots, n$$

by the implicit function theorem. We can also use the implicit function theorem to find the derivative of the ith constraint with respect to the jth variable where the jth variable goes from $m+1$ to n. Applying the theorem to

$$g_i(x^*) = g_i(h_1(\tilde{x}^*),\, h_2(\tilde{x}^*) \ldots h_m(\tilde{x}^*),\, x_{m+1}^* \ldots x_n^*) = 0$$

we obtain

$$\sum_{k=1}^{m} \frac{\partial g_i(x^*)}{\partial x_k} \frac{\partial h_k(\tilde{x}^*)}{\partial x_j} = \frac{-\partial g_i(x^*)}{\partial x_j} \qquad i = 1, \ldots, m \tag{90}$$

Now multiply each side of (90) by λ_i^* and add them up.

$$\sum_{i=1}^{m} \sum_{k=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_k} \frac{\partial h_k(\tilde{x}^*)}{\partial x_j} + \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0 \tag{91}$$

$$j = m+1, \ldots, n$$

Now subtract (91) from (89) to obtain:

$$\sum_{k=1}^{m} \left[\frac{\partial f(x^*)}{\partial x_k} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_k} \right] + \frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0 \tag{92}$$

$$j = m+1, \ldots, n$$

The bracket term is zero from (86) so that

$$\frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0 \qquad j = m+1, \ldots, n \tag{93}$$

Since (86) implies this is true, for $j = 1, \ldots, m$ we know it is true for $j = 1, 2, \ldots, n$ and we are finished.

The λ_i are called Lagrange multipliers and the expression

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) \tag{94}$$

is called the Lagrangian function.

5.2. **Proof of Necessary Conditions.** The necessary conditions for an extreme point are

$$\begin{aligned} &\nabla L(x^*, \lambda^*) = \nabla f(x^*) - J_g(x^*)\lambda = 0 \\ &\Rightarrow \frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0 \quad j = m+1, \ldots, n \end{aligned} \tag{95}$$

This is obvious from (84) and (94).

5.3. **Proof of Sufficient Conditions.** The sufficient conditions are repeated here for convenience

Let $f, g_1, \ldots, g_m$ be twice continuously differentiable real-valued functions on R^n. If there exist vectors $x^* \,\epsilon\, R^n$, $\lambda^* \,\epsilon R^m$ such that

$$\nabla L(x^* \lambda) = 0 \tag{5}$$

and for every non-zero vector $z \,\epsilon\, R^n$ satisfying

$$z' \nabla g_i(x^*) = 0, \ldots i = 1, \ldots, m \tag{6}$$

it follows that

$$z' \nabla_x^2 L(x^*, \lambda^*) z > 0 \tag{7}$$

then f has a strict local minimum at x^*, subject to $g_i(x) = 0$, $i = 1, \ldots, , m$. If the inequality in (7) is reversed , then f has strict local maximum at x^*.

Proof:

Assume x^* is not a strict local minimum. Then there exists a neighborhood $N_\delta(x^*)$ and a sequence $\{z^k\}$, $z_k \,\epsilon\, N_\delta(x^*)$, $z^k \neq x^*$ converging to x^* such that for every $\mathrm{z^k} \,\epsilon\, \{\mathrm{z^k}\}$.

$$g_i(z^k) = 0 \qquad i = l, \ldots, m \tag{96}$$

$$f(x^*) \geq f(z^k) \tag{97}$$

This simply says that since x^* is not the minimum value subject to the constraints there exists a sequence of values in the neighborhood of x^* that satisfies the constraints and has an objective function value less than or equal to $f(^*)$.

The proof will require the mean value theorem which is repeated here for completeness.

Mean Value Theorem

Theorem 2. *Let f be defined on an open subset (Ω) of R^n and have values in R^1. Suppose the set Ω contains the points a,b and the line segment S joining them, and that f is differentiable at every point of this segment. Then there exists a point c on S such that*

$$\begin{aligned} f(b) - f(a) &= \nabla f(c)'(b-a) \\ &= \frac{\partial f(c)}{\partial x_1}(b_1 - a_1) + \frac{\partial f(c)}{\partial x_2}(b_2 - a_2) + \cdots + \frac{\partial f(c)}{\partial x_n}(b_n - a_n) \end{aligned} \tag{98}$$

where b is the vector $(b_1,\, b_2, \ldots,\, b^n)$ and a is the vector $(a_1,\, a_2, \ldots,\, a^n)$.

Now let y^k and z^k be vectors in R^n and let $z^k = x^* + \theta^k y^k$ where $\theta^k > 0$ and $\| y^k \|= 1$ so that $z^k - x* = \theta^k y^k$. The sequence $\{\theta^k,\, y^k\}$ has a subsequence that converges to $(0,\, \bar{y})$ where $\| y \|= 1$. Now if we use the mean value theorem we obtain for each k in this subsequence

$$g_i(z^k) - g_i(x^*) = \theta^k y^{k\prime} \nabla g_i(x^* + \gamma_i^k \theta^k y^k) = 0, \quad i = 1, \ldots, m \tag{99}$$

where γ_i^k is a number between 0 and 1 and g_i is the ith constraint. The expression is equal to zero because we assume that the constraint is satisfied at the optimal point and at the point z^k by equation 98.

Expression 99 follows from the mean value theorem because $z^k - x^* = \theta^k y^k$ and with γ_i^k between zero and one, $\gamma_i^k \theta^k y^k$ is between $z^k = x^* + \theta^k$ and x^*

If we use the mean value theorem to evaluate $f(z_k)$ we obtain

$$f(z^k) - f(x^*) = \theta^k y^{k\prime} \nabla f(x^* + \eta^k \theta^k y^k) \leq 0 \tag{100}$$

where $0 < \eta_k < 1$. This is less than zero by our assumption in equation 97.

If we divide (99) and (100) by θ^k and take the limit as k $\rightarrow \infty$ we obtain

$$\lim_{k \to \infty} \left[y^{k\prime} \nabla g_i(x^* + \eta^k \theta^k y^k) \right] = \bar{y}' \nabla g_i(x^*) = 0 \quad i = 1,\, 2,\, \ldots,\, m \tag{101}$$

$$\lim_{k \to \infty} \left[y^{k\prime} \nabla f(x^* + \eta^k \theta^k y^k) \right] = \bar{y}' \nabla f_i(x^*) \leq 0 \tag{102}$$

Now remember from Taylor's theorem that we can write the Lagrangian in (95) as

$$\begin{aligned} L(z^k,\, \lambda^*) &= L(x^*,\, \lambda^*) + (z^k - x^*)' \nabla_x L(x^*,\, \lambda^*) \\ &\qquad + \frac{1}{2} \theta^{k^2} (z^k - x^*)' \nabla_x^2 L(x^* + \beta^k \theta^k y^k, \lambda^*)(z^k - x^*) \\ &= L(x^*, \lambda^*) + \theta^k y^{k\prime} \nabla_x L(x^*,\, \lambda^*) + \frac{1}{2} \theta^{k^2} y^{k\prime} \nabla_x^2 L(x^* + \beta^k \theta^k y^k,\, \lambda^*) y^k \end{aligned} \tag{103}$$

where $0 < \beta^k < 1$.

Now note that

$$L(z^k,\, \lambda^*) = f(z^k) - \sum_{i=1}^{m} \lambda^i g_i(z^k)$$
$$L(x^*,\, \lambda^*) = f(x^*) - \sum_{i=1}^{m} \lambda_i g_i(x^*)$$

and that at the optimum or at the assumed point z^k, $g_i(\cdot) = 0$.

Also $\nabla L(x^*,\, \lambda^*) = 0$ at the optimum so the second term on the right hand side of (103) is zero. Move the first term to the left hand side to obtain

$$L(z^k,\, \lambda^*) - L(x^*,\, \lambda^*) = \frac{1}{2}\theta^{k^2} y^{k\prime} \nabla_x^2 L(x^* + \beta^k \theta^k y^k,\, \lambda^*) y^k \tag{104}$$

Because we assumed $f(x^*) \geq f(z^k)$ in (97) and that $g(\cdot)$ is zero at either x^* or z^k, it is clear that

$$L(z^k,\, \lambda^*) - L(x^*,\, \lambda^*) \leq 0 \tag{105}$$

Therefore,

$$\frac{1}{2}\theta^{k^2} y^{k\prime} \nabla_x^2 L(x^* + \beta^k \theta^k y^k, \lambda^*) y^k \leq 0 \tag{106}$$

Divide both sides by $\frac{1}{2}\theta^{k^2}$ to obtain

$$y^{k\prime} \nabla_x^2 L(x^* + \beta^k \theta^k y^k,\, \lambda^*) y^k \leq 0 \tag{107}$$

Now take the limit as $k \rightarrow \infty$ to obtain

$$\bar{y}' \nabla_x^2 L(x^*,\, \lambda^*) \bar{y} \leq 0 \tag{108}$$

We are finished since $\bar{y} \neq 0$, and by equation 101,

$$\bar{y}' \nabla g_i(x^*) = 0, \quad i = 1,\, 2, \ldots,\, m$$

that is, if x^* is not a minimum then we have a non-zero vector y satisfying

$$\bar{y}' \nabla g_i(x^*) = 0, \quad i = 1,\, 2, \ldots,\, m \tag{109}$$

where $\bar{y}' \nabla_x^2 L(x^*,\, \lambda^*) \bar{y} \leq 0$. But if x^* is a minimum then equation 6 rather than (108) will hold.