Final Exam
(Do all five problems. Weights: #1 - 15%, #2 - 20%, #3 - 15%, #4 - 20%, #5 - 30%)

1. Given the following probabilities of events:

\[ P(A) = 0.3, \quad P(B) = 0.2, \quad P(C) = 0.4, \quad P(A \cap B) = 0.06, \quad P(B \cap C) = 0.08 \]
\[ P(A \cap C) = 0.13, \quad P(A \cap B \cap C) = 0.04, \]

find the following conditional probabilities:

a. \( P(A \mid B) \).  
b. \( P(A \mid B^C) \).  
c. \( P(B^C \mid A \cap C) \).

(Hint: Draw a Venn diagram.)

2. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. with probability density function given by

\[
f(x; \theta) = \begin{cases} 2 e^{-2x} & \text{for } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}
\]

where \( \theta \) is a positive parameter. Find the maximum likelihood estimator of \( \theta \), based on this random sample of size \( n \).

3. Let \( X_1 \) and \( X_2 \) be random variables with:

\[
E[X_1] = \mu_1, \quad E[X_2] = \mu_2, \quad \text{Var}(X_1) = \sigma_1^2, \quad \text{Var}(X_2) = \sigma_2^2, \quad \text{and} \quad \text{Cov}(X_1, X_2) = \rho_{12}.
\]

Define random variables \( Y_1 \) and \( Y_2 \) as follows: \( Y_1 = X_1 + X_2 \) and \( Y_2 = X_1 - X_2 \).

Find \( \text{Cov}(Y_1, Y_2) \).

4. Let \( X \) be a continuous random variable with probability density function

\[
f(x) = \begin{cases} 6 x (1 - x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}
\]

a. Find \( E[X^{0.5}] \).

b. Find \( (E[X])^{0.5} \).

c. Interpret the results for parts a and b in terms of Jensen's inequality.
5. Consider a random sample of size 25 from a normal population with unknown mean, \( \mu \), and known variance, \( \sigma^2 = 100 \):

\[ X_1, X_2, \ldots, X_{25} \text{ i.i.d. } \mathcal{N}(\mu, \sigma^2 = 100). \]

a. Carefully and completely explain how you would carry out the following hypothesis test:

Test \( H_0: \mu = 10 \)

vs. \( H_1: \mu > 10 \) at the 5% significance level.

b. Calculate the power of the test in part a against the alternative \( H_1: \mu = 14 \).

c. Suppose that, for a given random sample of size 25, the realization of the sample mean,

\[ \bar{X} = \frac{1}{25} \sum_{i=1}^{25} X_i, \]

equals 23.71. Calculate a 95% (two-sided) confidence interval for \( \mu \) based on this sample.

d. Briefly explain how the test procedure designed in part a and the confidence interval calculated in part c would have to be modified if the population variance were unknown but the realization of the sample variance,

\[ S^2 = \frac{1}{24} \sum_{i=1}^{25} (X_i - \bar{X})^2, \]

equals 100.